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# DIGIT PATTERNS IN REAL NUMBERS CREATED FROM PERMUTATIONS

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*Dedicated to the memory of Dr. László Dringó*

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**Abstract.** Given a positive integer $k$, we construct a binary number $0.a_1a_2a_3\ldots$ having the property that any sequence $a_{m+1}\ldots a_{m+k}$ of $k$ consecutive digits from its binary expansion appears with a frequency directly related to the various permutations of the set $\{1, 2, \ldots, k+1\}$.

## 1. Introduction

Given a positive integer $k$, let $\Pi_k$ be the set of the permutations of the set $\{1, 2, \ldots, k+1\}$. Various interesting aspects of this set $\Pi_k$ can be studied; see for instance the book of Pemmaraju [1]. Here, we use this set to construct real numbers with an interesting property, as follows. Given $\pi \in \Pi_k$, let $j_1, j_2, \ldots, j_{k+1}$ be defined by $\pi(i) = j_i$. Further set, for each $h = 1, 2, \ldots, k$,

$$\rho(j_h, j_{h+1}) = \begin{cases} 1 & \text{if} \quad j_{h+1} > j_h, \\ 0 & \text{if} \quad j_{h+1} < j_h. \end{cases}$$

Moreover, given $(\delta_1, \delta_2, \ldots, \delta_k) \in \{0,1\}^k$, set

$$D(\delta_1, \delta_2, \ldots, \delta_k) := \#\{\pi \in \Pi_k : \rho(\pi(i), \pi(i+1)) = \delta_i \text{ for } i = 1, 2, \ldots, k\}$$

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and

$$\kappa(\delta_1,\delta_2,\ldots,\delta_k) := \frac{D(\delta_1,\delta_2,\ldots,\delta_k)}{(k+1)!}.$$

As we will see in Section 4,

$$\kappa(\delta_1,\delta_2,\ldots,\delta_k) \geq \frac{1}{(k+1)!} \qquad (k=1,2,\ldots).$$

To illustrate the function $\kappa(\delta_1,\delta_2,\ldots,\delta_k)$, if we choose the case $k=4$, we obtain the following table.

| $(\delta_1,\delta_2,\delta_3,\delta_4)$ | $D(\delta_1,\delta_2,\delta_3,\delta_4)$ | $\kappa(\delta_1,\delta_2,\delta_3,\delta_4)$ |
|---|---|---|
| (0,0,0,0) | 1 | 1/120 |
| (0,0,0,1) | 4 | 1/30 |
| (0,0,1,0) | 9 | 3/40 |
| (0,0,1,1) | 6 | 1/20 |
| (0,1,0,0) | 9 | 3/40 |
| (0,1,0,1) | 16 | 2/15 |
| (0,1,1,0) | 11 | 11/120 |
| (0,1,1,1) | 4 | 1/30 |
| (1,0,0,0) | 4 | 1/30 |
| (1,0,0,1) | 11 | 11/120 |
| (1,0,1,0) | 16 | 2/15 |
| (1,0,1,1) | 9 | 3/40 |
| (1,1,0,0) | 6 | 1/20 |
| (1,1,0,1) | 9 | 3/40 |
| (1,1,1,0) | 4 | 1/30 |
| (1,1,1,1) | 1 | 1/120 |

Our purpose in this short paper is to construct some binary number

$$\alpha = 0.a_1a_2a_3\ldots,$$

that is, where each digit $a_i \in \{0,1\}$, and such that

$$\lim_{N\to\infty}\frac{1}{N}\#\{m\leq N : a_{m+1}\ldots a_{m+k} = \delta_1\ldots\delta_k\} = \kappa(\delta_1,\ldots,\delta_k).$$

To construct $\alpha$, we proceed as follows. First we set

$$\mathcal{F}_N = [e^N, e^{N+1}) \quad \text{and} \quad \mathcal{L}_N = [\log N, N] \qquad (N=1,2,\ldots).$$

Let $p(n) = p_N(n)$ stand for the smallest prime divisor of $n$ which is located in the interval $\mathcal{L}_N$. Observe that the number of those $n \in \mathcal{F}_N$ which do not contain any prime divisors in $\mathcal{L}_N$ is bounded by

$$ce^N \prod_{\substack{p \in \mathcal{L}_N \\ p \in \wp}} \left(1 - \frac{1}{p}\right) \le ce^N \frac{\log\log N}{\log N}.$$

To each number $n \in \mathcal{F}_N$, we associate the number

$$\epsilon_n = \begin{cases} 1 & \text{if } p(n+1) > p(n) \text{ and } n+1 \in \mathcal{F}_N, \\ 0 & \text{otherwise} \end{cases}$$

for some absolute constant $c > 0$, where $\wp$ stands for the set of all primes. Thus, $\epsilon_n = 0$ if $p(n+1) < p(n)$ or if $n < e^{N+1} < n+1$ or if either $p(n)$ or $p(n+1)$ does not exist. Then, to each $N \in \mathbb{N}$, we associate the number

$$\xi_N = \text{Concat}(\epsilon_n : n \in \mathcal{F}_N),$$

and we then define

$$\alpha = 0.\xi_2\xi_3\xi_4\ldots. \tag{1.1}$$

## 2. The distribution function of $(\{2^n\alpha\})_{n\ge1}$

With $\alpha$ as in (1.1), let $0 < u < 1$ written as

$$u = \frac{t_1}{2} + \frac{t_2}{2^2} + \frac{t_3}{2^3} + \cdots \tag{2.1}$$

Here, we may assume that $t_n = 0$ for infinitely many $n \in \mathbb{N}$. We can prove that

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N : \{2^n\alpha\} \le u\} = F(u) \quad \text{exists.} \tag{2.2}$$

To see this, we proceed as follows. Let $r_1 < r_2 < \cdots$ be a sequence of integers such that $t_{r_j} = 0$ for some $j \in \mathbb{N}$ and then set $u_j := \sum_{\nu=1}^{r_j - 1} \frac{t_\nu}{2^\nu}$ and further define $\widetilde{u_j} := \frac{1}{2^{r_j}} + u_j$. It is clear that

$$u_j \le u < \widetilde{u_j} \qquad (j \in \mathbb{N}).$$

We then introduce the two functions

$$F_1(u) = \liminf_{N\to\infty} \frac{1}{N}\#\{n \le N : \{2^n\alpha\} < u\},$$
$$F_2(u) = \limsup_{N\to\infty} \frac{1}{N}\#\{n \le N : \{2^n\alpha\} < u\}.$$

With these definitions, we easily see that

$$F(u_j) = \sum_{\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{r_j}}{2^{r_j}} \le u_j} \kappa(a_1,\ldots,a_{r_j}) \le F_1(u), \tag{2.3}$$

$$F(\widetilde{u_j}) = \sum_{\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{r_j}}{2^{r_j}} \le \widetilde{u_j}} \kappa(a_1,\ldots,a_{r_j}) \ge F_2(u). \tag{2.4}$$

Moreover,

$$F(\widetilde{u_j}) - F(u_j) = \kappa(t_1,\ldots,t_{r_{j-1}},1). \tag{2.5}$$

Also, observe that it will follow from Theorem 4.1 below that

$$\lim_{m\to\infty} \max_{\delta_1,\ldots,\delta_m\in\{0,1\}^m} \kappa(\delta_1,\ldots,\delta_m) = 0. \tag{2.6}$$

It then follows from (2.3), (2.4), (2.5) and (2.6) that $F_1(u) = F_2(u)$ and therefore that $F(u)$ exists, as claimed.

We can even prove that $F(u)$ is a continuous function. To show this, we first fix $u$ and choose two sequences of numbers $(u_M)_{M\ge1}$ and $(v_M)_{M\ge1}$ such that $u_M < u < v_M$ for each $M \ge 1$, and such that $u_M \to u$ and $v_M \to u$ as $M \to \infty$. Then, let $s$ be an integer such that $\lfloor u\cdot 2^M\rfloor = s$ and choose $u_M = \dfrac{s}{2^M}$ and $v_M = \dfrac{s+1}{2^M}$. We then have

$$F(u_M) = \sum_{\frac{a_1}{2}+\cdots+\frac{a_M}{2^M} \le s/2^M} \kappa(a_1,\ldots,a_M),$$
$$F(v_M) = \sum_{\frac{a_1}{2}+\cdots+\frac{a_M}{2^M} \le (s+1)/2^M} \kappa(a_1,\ldots,a_M),$$

with

$$\frac{s+1}{2^M} = \frac{b_1}{2} + \cdots + \frac{b_M}{2^M}.$$

Since $F(v_M) - F(u_M) = \kappa(b_1,\ldots,b_M) \to 0$ as $M \to \infty$, we therefore have that $\lim_{M\to\infty} F(u_M) = F(u)$ and $\lim_{M\to\infty} F(v_M) = F(u)$. Thus we proved that $F(u)$ is continuous at the points $u \in \mathbb{R}\setminus\mathbb{Q}$ and also continuous from the right at those points $u \in \mathbb{Q}$ of the form $u = s/(2^R)$, where $s$ is an odd integer. Moreover, assuming that $K$ is an integer larger than $R$ and setting $u_K = u - 1/2^K$, we have $F(u) - F(u_K) = \kappa(t_1, t_2, \ldots, t_{R-1}, 0, 1, 1, \ldots, 1)$ which tends to 0 as $K$ tends to infinity, thereby establishing that $F(u)$ is continuous from the left, as well.

## 3. Main theorem

**Theorem 3.1.** *Given an integer $k \geq 2$ and an arbitrary $k$-tuple $(\delta_1, \ldots, \delta_k) \in \in \{0,1\}^k$, we have*

$$\lim_{M\to\infty} \frac{1}{M} \# \left\{ m \leq M : \{2^m \alpha\} \in \left[ \frac{\delta_1}{2} + \cdots + \frac{\delta_k}{2^k}, \frac{\delta_1}{2} + \cdots + \frac{\delta_k}{2^k} + \frac{1}{2^k} \right) \right\} =$$

$$= \kappa(\delta_1, \ldots, \delta_k).$$

**Proof.** Let $p_1, p_2, \ldots, p_{k+1}$ be distinct primes located in the interval $\mathcal{L}_N$. Let us count those $n \in \mathcal{F}_N$ for which $p(n+j) = p_j$. Also, let $\{i_1, \ldots, i_{k+1}\}$ be that particular permutation of the set $\{1, 2, \ldots, k+1\}$ for which $p_{i_1} < p_{i_2} < \cdots < < p_{i_{k+1}}$. Then, set

$$Q_U := \prod_{\substack{\log N < p < U \\ p \in \wp}} p.$$

Since $n + j \equiv 0 \pmod{p_j}$ for $j = 1, 2, \ldots, k+1$, it follows that $n = m \cdot p_1 p_2 \cdots p_{k+1} + r$ with $(r, p_1 p_2 \cdots p_{k+1}) = 1$. Moreover, $(n + i_1, Q_{p_{i_1}}) = 1$, $(n + i_2, Q_{p_{i_2}}) = 1$, …, $(n + i_{k+1}, Q_{p_{i_{k+1}}}) = 1$. Using standard asymptotic sieve techniques, we can write these conditions in the form

$$\prod_{\ell=1}^{k+1} (n + i_\ell, Q_{p_{i_1}}) = 1, \ \prod_{\ell=2}^{k+1} \left( n + i_\ell, \frac{Q_{p_{i_2}}}{Q_{p_{i_1}}} \right) = 1, \ldots, \left( n + i_{k+1}, \frac{Q_{p_{i_{k+1}}}}{Q_{p_{i_k}}} \right) = 1,$$

$$n \equiv r \pmod{p_1 p_2 \cdots p_{k+1}}.$$

Thus the number of such numbers $n \in \mathcal{F}_N$ is, as $N \to \infty$,

$$(1 + o(1)) \frac{\# \mathcal{F}_N}{p_1 p_2 \cdots p_{k+1}} \cdot \prod_{\substack{\log N < q < p_{i_1} \\ q \in \wp}} \left( 1 - \frac{k+1}{q} \right) \cdot$$

$$\cdot \prod_{\substack{p_{i_1} < q < p_{i_2} \\ q \in \wp}} \left( 1 - \frac{k}{q} \right) \cdots \prod_{\substack{p_{i_k} < q < p_{i_{k+1}} \\ q \in \wp}} \left( 1 - \frac{1}{q} \right) =$$

$$= \ (1 + o(1)) \# \mathcal{F}_N \cdot \log\log\log N \cdot \prod_{i=1}^{k+1} \frac{1}{p_i \log\log p_i}.$$

The important observation here is that this asymptotic behavior does not depend on the particular permutation of the primes $p_1, p_2, \ldots, p_{k+1}$ we choose. We may therefore conclude that, as $N \to \infty$,

$$\frac{1}{\#\mathcal{F}_N}\#\left\{m \in \mathcal{F}_N : \{2^m\alpha\} \in \left[\frac{\delta_1}{2} + \cdots + \frac{\delta_k}{2^k}, \frac{\delta_1}{2} + \cdots + \frac{\delta_k}{2^k} + \frac{1}{2^k}\right)\right\} =$$
$$= (1 + o(1))\kappa(\delta_1, \ldots, \delta_k).$$

Now, we need to count those $\{2^m\alpha\}$ $(m = 1, 2, \ldots, \lfloor x \rfloor)$ not only for the particular values $x = e^N$, but also for the more general values $x \in (e^N, e^{N+1})$.

So, let $\varepsilon > 0$ be an arbitrarily small number and set $x = e^{N+\theta}$ with $0 < \theta < 1$. We now examine two separate cases. If $\theta < \varepsilon$, then

$$\#\{n : e^N \le n < x\} < e^N(e^\varepsilon - 1) < 2\varepsilon e^N.$$

On the other hand, if $\theta > \varepsilon$, setting $S := [e^N, e^{N+\theta})$, we may then repeat the above argument for the interval $S$ instead of $\mathcal{F}_N$ and obtain the same result. Therefore, in both cases, the proof is complete. ∎

## 4. The size of $\boldsymbol{\kappa(\delta_1, \ldots, \delta_k)}$

**Theorem 4.1.** *Let $k \ge 2$ and let $a_1, a_2, \ldots, a_k \in \{0, 1\}$ be given. Then,*

$$\frac{1}{(k+1)!} \le \kappa(a_1, a_2, \ldots, a_k) < \frac{1}{2^{\lfloor k/2 \rfloor}}.$$

**Proof.** First, we prove the first inequality, namely

$$\kappa(a_1, a_2, \ldots, a_k) \ge \frac{1}{(k+1)!}. \tag{4.1}$$

To do so, we let $j_1, \ldots, j_r$ be the indices of those $a_i$'s for which $a_{j_\nu} = 0$ $(\nu = 1, \ldots, r)$ and let $t_1, \ldots, t_s$ be the indices of those $a_i$'s for which $a_{t_\mu} = 1$ $(\mu = 1, \ldots, s)$. The case where one of the two sets $\{j_1, \ldots, j_r\}$ or $\{t_1, \ldots, t_s\}$ is empty is much more simple. So, let $S = \{1, \ldots, r\}$ and $M = \{r+1, \ldots, k+1\}$. Now, let $\{u(1), \ldots, u(k+1)\}$ be a permutation of $\{1, 2, \ldots, k+1\}$ satisfying

1. $\{u(j_1 + 1), u(j_2 + 1), \ldots, u(j_r + 1)\}$ is a permutation of $S$ satisfying the condition

   If $j_{\ell+1} = j_\ell + 1$ for some $\ell \in \{1, \ldots, r\}$, then $u(j_\ell + 1) > u(j_{\ell+1} + 1)$.

   Observe that such a permutation clearly exists.

2. $\{u(t_1+1), u(t_2+1), \ldots, u(t_{k-r}+1)\}$ is a permutation of $M$ satisfying the condition

   $$\text{If } t_{\nu+1} = t_\nu + 1 \text{ for some } \nu \in \{1, \ldots, s\}, \quad \text{then} \quad u(t_\nu + 1) < u(t_{\nu+1} + 1).$$

   Such a permutation also clearly exists.

For such a permutation $\{u(1), \ldots, u(k+1)\}$, we have that $\rho(u(\ell), u(\ell+1)) = a_\ell$ for $\ell = 1, \ldots, k$. The special case $S = \emptyset$ is very simple, because in this case, $u(j) = j$ for each $j = 1, \ldots, k+1$. On the other hand if $M = \emptyset$, then

$$u(1) = k+1,\ u(2) = k,\ \ldots,\ u(k+1) = 1.$$

This completes the proof of (4.1).

We will now prove the second inequality in Theorem 4.1, namely

$$\kappa(a_1, a_2, \ldots, a_k) < \frac{1}{2^{\lfloor k/2 \rfloor}}. \tag{4.2}$$

Assume that $\{j_1, j_2, \ldots, j_{k+1}\}$ is a permutation of $\{1, 2, \ldots, k+1\}$ satisfying $\rho(j_\ell, j_{\ell+1}) = a_\ell$ for $\ell = 1, \ldots, k$. Assume first that $k+1$ is even and consider the pairs

$$(j_1, j_2), \quad (j_3, j_4), \quad \ldots, \quad (j_k, j_{k+1}).$$

If $a_1 = 1$, then $j_1 < j_2$; if $a_1 = 0$, then $j_2 > j_1$; if $a_3 = 1$, then $j_3 < j_4$; if $a_3 = 0$, then $j_3 > j_4$, and so on, up to if $a_k = 1$, then $j_k < j_{k+1}$; if $a_k = 0$, then $j_{k+1} < j_k$.

To sum up, this means that the number of associated permutations is no larger than $\dfrac{(k+1)!}{2^{(k+1)/2}}$.

The case where $k+1$ is odd can be treated in a similar manner, since we then have that $k$ is even, in which case we consider the $k/2+1$ numbers

$$(j_1, j_2), \quad (j_3, j_4), \quad \ldots, \quad (j_{k-1}, j_k), \quad j_{k+1},$$

which allows us in the end to conclude that the number of associated permutations is no larger than $\dfrac{(k+1)!}{2^{k/2}}$.

In both cases, we have proved (4.2) and the proof of Theorem 4.1 is complete. ∎

## References

[1] **Pemmaraju, S.,** *Computational Discrete Mathematics*, Cambridge University Press, 2003, xiv+480 pages.

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