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TO THE MEMORY OF PROFESSOR JÁNOS GALAMBOS

(1 September 1940 – 19 December 2019)

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Abstract. This paper provides a summary of the life and contributions of János Galambos, from Temple University. It describes the papers, conferences, books and all his contributions to Probability and Statistics, mainly to extreme value analysis. It also includes a list of his publications.

1. The beginning. His Hungarian roots

János Galambos was born on September 1, 1940, in Hungary, at Zirc, a town in Veszprém county, famous by its beautiful Cistercian Abbey.

Figure 1. The Cistercian Abbey in Zirc, Hungary

Key words and phrases: Number Theory, Extremes, Probability.

In 1958, he entered Loránd Eötvös University in Budapest, and graduated with a Ph.D. in 1963, under the supervision of Alfréd Rényi, an outstanding Mathematician Professor, who influenced János future work.

Eötvös Loránd University (ELTE) is a Hungarian public research university based in Budapest and founded in 1635. ELTE is one of the largest and most prestigious public higher education institutions in Hungary. The 28 000 students at ELTE are organized into eight faculties, and into research institutes located throughout Budapest and on the scenic banks of the Danube. ELTE is affiliated with 5 Nobel laureates, as well as winners of the Wolf Prize, Fulkerson Prize and Abel Prize, the latest of which was Abel Prize winner Endre Szemerédi in 2012 (Wikipedia).

The predecessor of Eötvös Loránd University was founded in 1635 by Cardinal Péter Pázmány in Nagyszombat, Kingdom of Hungary, (today Trnava, Slovakia) as a Catholic university for teaching theology and philosophy. In 1770, the University was transferred to Buda. It was named Royal University of Pest until 1873, then University of Budapest until 1921, when it was renamed Royal Hungarian Pámány Péter University after its founder Péter Pázmány. The Faculty of Science started its autonomous life in 1949 when the Faculty of Theology was separated from the university. The university received its current name in 1950, after one of its most well-known physicists, Baron Loránd Eötvös (Wikipedia).

In 1964, Galambos was married to Éva, who also graduated from L. Eötvös University with majors in mathematics and physics.

János Galambos served as Assistant Professor at Eötvös University, from 1964-1965.

Galambos's dissertation dealt with probabilistic inequalities, and he continued to work on this topic throughout the years. He mainly worked on Bonferroni-type inequalities, inequalities which are valid in any probability space, and used them as a unifying tool to solve problems in different branches of mathematics.

His first paper on this topic dealt with what is known as the graph-sieve inequalities. These types of results originated in number theory, and they were later applied to probability. These results continue to be the most general results of this type and have turned out to be very important in the derivation of limit theorems in combinatorics, extremes, random subsets, etc. [10, 128]. This approach was later expanded in [26].

Early in his career, János had strong ties with distinguished Hungarian mathematicians such as Alfréd Rényi, Lajos Takács, and Paul Erdős, with whom he later had joint works.

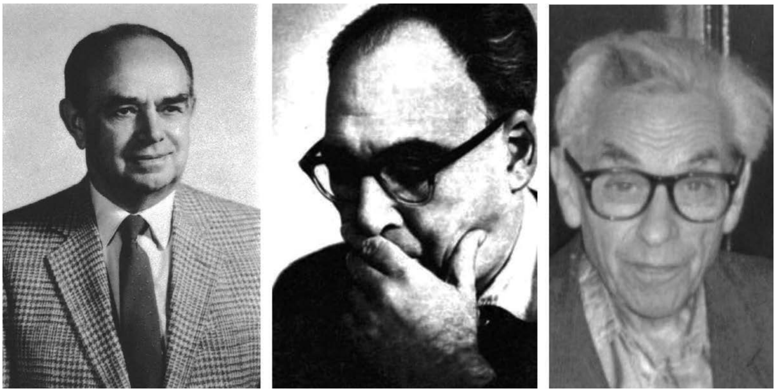

Figure 2. From left to right, Lajos Takács, Alfréd Rényi and Paul Erdős

In [118] with Gani, he recognizes this influence and admiration for Lajos Takács.

> *“This book is dedicated by its contributors to an exceptional man - Lajos Takács: A probabilist of distinction, a pioneer of queuing theory, a teacher of inspiration, and a man of kindness and decency whom it is a privilege to call a friend.”*

2. János Galambos African experience

Later, he moved as a Lecturer to the University of Ghana, Legon, from 1965-69. This University is the oldest and largest of all Ghanaian universities and tertiary institutions. It was founded in 1948 as the University College of the Gold Coast, and was originally an affiliate college of the University of London, which supervised its academic programmes and awarded degrees. It gained full university status in 1961, and now has nearly 40 000 students. The university is mainly based at Legon, about twelve kilometres northeast of the centre of Accra (Wikipedia).

In Ghana he published his first important papers in Probability, one with Rényi, [5], and others in very relevant journals, such as the Journal of the London Mathematical Society, [7], the Annales de l'Institute Henri Poincare, [8], or the Quarterly Journal of Mathematics, [9]. One can assume that this stage was not easy, because of his life change and the lack of resources, but on the other hand a nice human experience.

Figure 3. The University of Ghana, Legon

Later he spent one year at the University of Ibadan, Nigeria, from 1969–70. The origins of this university are in the University of London. Established in 1948, the University of Ibadan, UI as it is fondly referred to, is the first University in Nigeria. Until 1962 when it became a full-fledged independent University, it was a College of the University of London in a special relationship scheme. The University, which took off with academic programmes in Arts, Science and Medicine, is now a comprehensive citadel of learning with academic programmes in sixteen Faculties, including Sciences. It was the first degree awarding institution in Nigeria. It is located inland about 70 miles from Lagos (Wikipedia).

Figure 4. The University of Ibadan, Nigeria

3. Temple University first period

In 1970 he joined the faculty at Temple University, Philadelphia, and has remained there ever since.

Galambos's initial work dealt with problems in number theory, and in particular, with the investigation of the limiting distributions of arithmetical functions. Galambos fully exploited this connection by developing a probabilistic framework from which results on arithmetical functions can be obtained. He also developed a general framework in which arbitrary arithmetical functions can be approximated by additive ones [31], leading to limit theorems with and without normalization constants [31, 39].

Galambos started his long sequel of papers on extremes by characterizing the limit distribution for the maximum (under proper normalization) of a wide class of dependent random variables [21].

At Temple, he continued with important works in Probability theory, [12, 18, 57, 58], related to limit [11, 28, 32, 33], and asymptotic distributions [24], and maximum of random variables [21, 22]. For example, Bakštys, [153], derived necessary and sufficient conditions for the existence and symmetry of a limiting distribution continuous at zero, and Galambos extended these results to the case when the above sum diverges [16].

He also extended a result of Mogyoródi, [178], to a random number of dependent random variables [30, 113]. This work and its generalizations [30, 88, 99] constitute an important chapter on the theory of extremes for dependent random variables.

Galambos recognized and stressed the importance of exchangeability in the study of order statistics [58, 65, 88]. He proved that in studying the distribution of the number of occurrences in a given finite sequence of events, one can always assume that the events are exchangeable [32]; this led to the derivation of the distribution of order statistics. In the case where this property extends to infinite sequences, he obtained Poisson limits for sequences of dependent random variables, limit laws for mixtures [47], and derivations of the limit distributions of order statistics [32, 35, 47].

Specially relevant to the future work of János were the works on order statistics, [19, 35, 43, 44, 46] and record times,[45]. Galambos was also interested in other aspects of statistics. His joint work with H.A. David [40] is considered to be one of the fundamental papers on the theory of concomitants. These papers were published in journal such as Bulletin of the London Mathematical Society, Journal of Applied Probability, Annals of Probability, Annals of Mathematical Statistics, Proceedings of the American Mathematical Society and the Journal of the American Statistical Association.

At that time the importance of his work was already recognized throughout the world and he was a frequent speaker at international conferences and traveled widely as a guest of universities and scientific institutions.

In 1974 he published one paper with Paul Erdős, [39], which makes János an Erdős 1 fellow and other co-authors, including myself, Erdős 2 fellows (to my knowledge, only 4 Spanish mathematicians have this honor).

Galambos's interest in number theory was not limited to the topics described above. For example, he investigated asymptotic properties of intermediate prime divisors and the asymptotic extremal property of prime divisors. Galambos applied this result to investigate extremal properties associated with prime divisors [54], and he extended results of Erdős, [172], into Poisson limits.

4. Appearance of his extreme order statistics book and Lecture Notes in Statistics Monograph

A very relevant milestone in his career was the publication of his seemingly book: "The Asymptotic Theory of Extreme Order Statistics", by Wiley in 1978.

The field of extremes, maxima and minima of random variables, has attracted the attention of engineers, scientists, probabilists, and statisticians for many years. The fact that engineering works need to be designed for extreme conditions forces us to pay special attention to singular values more than to regular (or mean) values. Since the statistical theory for dealing with mean values is very different from the statistical theory required for extremes, one cannot solve the above indicated problems without a specialized knowledge of statistical theory for extremes. Applications of extremes include all fields of research, for example, ocean, structural and hydraulics engineering, meteorology, and the study of material strength, traffic, corrosion, pollution, and so on.

After finishing my Civil Engineering studies at the University of Cantabria and later my Ph. D. degrees, one awarded by Northwestern University and another by the Polytechnical University of Madrid, I soon realized that an engineer cannot survive without knowledge of extremes, because engineering design must be based on extraordinary values, such as large loads, winds, waves, earthquakes, etc. or smaller values, such as strengths, water supplies, etc. Consequently, I was motivated to study and analyze the problem of extremes.

One of my first works, [156], after joining the University of Cantabria consisted of developing a model to reproduce the random wave behavior with the aim of designing maritime works. The model reproduced the occurrence of Poissonian storms having each a random number of sea states with a Poissonian random number of waves, which were assumed to follow a Rayleigh distribution. The aim of this model was to derive the statistical distribution of the largest waves.

Later I was concerned with approximating distribution functions in their tails, motivated by the fact that in the engineering practice you are forced to extrapolate the existing data. More precisely, based on a reduced set of data you are asked to predict the probability of occurrence of very large (larger than the observed) waves.

During that period I encountered for the first time to János, not physically, but throughout his first extremes book, [60], in which I discovered many interesting results on the problem of extremes. I must recognize here with gratitude that having the opportunity to read his book changed my life.

Another important János' milestone took place at that time, because he published a Springer Monograph, with Samuel Kotz, in the Lecture Notes in Statistics Series, a very prestigious publication, titled "Characterizations of probability distributions: A Unified Approach with an Emphasis on Exponential and Related Models".

In fact, in [66], János points out some important problems associated with models based on arbitrary assumptions and with classical methods to fit models and test statistical hypotheses and suggest functional equations as the alternative. This is a brilliant idea that should be recognized. In fact several papers of myself have been directed to this aim. In this work, he includes the following relevant statement:

> *"The application of probability theory and mathematical statistics to real life situations involves two steps. First, a model is set up, and then probabilistic or statistical methods are applied within the adopted model. In many instances the first step is arbitrary. For example, in statistical analysis one frequently assumes that the source of the data is a normal population. Although an assumption of this kind may be "confirmed" by a goodness-of-fit test, it is well known that such tests only confirm that the population distribution could be normal, without excluding the possibility of other population distributions. In fact, there are many studies which indicate that standard statistical methods cannot distinguish population distributions (such as the normal, lognormal, logistic and gamma distributions) which are uniformly close to each other for certain ranges of their parameters. For some problems (e.g., estimating the mean), an arbitrary distributional assumption does not make much difference. For others, however, a slight deviation from the true population distribution may result in an absolutely wrong decision, which can be very costly (either in human life or in financial terms).*

and also:

> *The way of avoiding an arbitrary distributional assumption is to develop a characterization theorem from some basic assumptions of the real life situation we face. In such characterization theorems, Solutions of functional equations play a central role. The aim of the present paper is to bring together a large variety of problems of probability theory in which the original problem reduces to finding the solution or Solutions of a functional equation. Since most of these equations are well known to the prospective readers, only one actual method of solution will be included in this paper. This is the so-called method of limit laws, whose inclusión serves to bring out the probabilistic nature and flavors of the topics discussed. However, it should be noted that though the method of limit laws appears at present to be specific to probabilistic situations, it may be possible to use it, as a technique with no direct interpretation, to solve functional equations unrelated to probability theory."*

An example of application of the method of limit laws is in [89], where a practically relevant functional equation is solved.

Due to the great recognition of János, he was asked to participate very actively in the Encyclopedia of Statistical Sciences, published by Wiley, with entries in: *"Characterizations of distributions", "Exponentiual Distribution", "Exchangeability", "Multivariate order statistics", "Multivariate stable distributions", "Probabilistic number theory", "Raikov's theorem", "Truncation methods in Probability theory", and "Two-series theorem"* and also in the Handbook of Statistics, with entries in *"Order statistics" and "Univariate extreme value theory and applications"*.

In 1984 his book on "Introductory Probability Theory" is published by Marcel Dekker and his book on extremes, [60] is translated into the Russian language, [73], which was also translated into the Chinese language in 2001, [138].

He developed nonparametric tests for extreme value distributions, [69, 102, 136], and was instrumental in the development of a methodology for the proper use of extreme value theory in applications, [122, 137].

5. The Vimeiro NATO ISI Advanced Study Institute on extremes

In 1983 the NATO ISI Advanced Study Institute on extremes was held in Vimeiro (Portugal), where, among other important researchers, participated János Galambos from Temple University, Barry C. Arnold, from the University of California (Riverside) and Richard Smith, from the Imperial College (London). There I met them for the first time and the possibility of spending a sabbatical year at Temple University arose. Later János and Éva Galambos facilitated me to expend my sabbatical year at Temple, where I moved with my wife and four children. János was kindly involved in finding a house for my family.

During my sabbatical leave at Temple, August 1985 to June 1986, we were supposed to write a joint book on extremes, but soon Wiley, the editorial company of the János book, claimed about the possibility of the new book competing with his old book, and János, who already had initiated writing the book with me, had to abandon the project. This is explained in the prolog of my book, [157]) in which it is indicated that some pages are due to János.

Two years after my stay at Temple, János' book became out of print, and he published the new edition, [60].

During that year I worked very hard in the book, that was finished at the end of my sabbatical stay and published a little later by Academic Press in 1988.

In the book I tried to bring to the engineers and applied people the already existing theory on extremes and complementing it with new methods and ideas in order to make the important advances useful to people not expert in Statistics. I also was concerned with including practical examples with real data to illustrate some possible applications and motivate researchers.

One of the main contributions included in the book, and discussed with János, is an important theorem, that permits deciding the domain of attraction of a given distribution and provides sets of sequences that allow the convergence to the limit distributions. Before, three theorems were required, one for the Weibull, one for the Gumbel and one for the Frechet types domains of attraction. If instead of the theoretical distribution you have data, which is the usual case in practice, the previous method is useless and you must use the curvature method, that we developed together with J. M. Sarabia and presented at Oberwolfach in 1987, [102]. We summarize this method below.

In 1987 János published the second edition of his book, [88] and in 1988 his book on "Advanced Probability Theory" by Marcel Dekker.

6. Some joint work with János Galambos

In this section I discuss some joint works with János.

6.1. Bivariate distributions with normal conditionals

I still had some time to work with János in some papers, some of them were finished during this year, and some later as, [81], and [82] .

My first contact with conditional specification took place at Temple University (Philadelphia) in 1985 when I was working with János Galambos. One Thursday he invited me to write a paper to be presented at a Conference to start 10 days later. He said:

> "I have been invited to deliver one of the main talks at a Conference on Weighted Distributions at Penn State University in Pittsburgh. Since I am going to Pittsburgh by car, you can come with me, and you can finance your hotel expenses with some small financial support you will receive because of presenting the paper."

Then, I asked János "what are weighted distributions?", and joking he answered me: "I don't know". So, I had ten days to write a paper on a topic I did not know.

During that weekend I decided what paper to present and I was working very hard to be able to show János some work the next Monday. Since I knew that all conditionals of a bivariate normal were normal, I asked myself about the possibility of existing other bivariate distributions with the same property. Fortunately, I had some experience on functional equations, and I was able to solve the problem producing the first draft. From this draft, we generated the joint paper, which in summary was as follows.

Assume a bivariate absolutely continuous distribution whose marginal and conditional probability density functions, respectively, are $f_{(X,Y)}(x,y)$, $g(x)$, $h(y)$, $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

It is evident that

$$f_{(X,Y)}(x,y) = f_{X|Y}(x|y)h(y) = f_{Y|X}(y|x)g(x). \tag{6.1}$$

If we assume that all conditional distributions are normal, we must have

$$\frac{\exp\left\{-\frac{1}{2}\left[\frac{y-a(x)}{b(x)}\right]^2\right\}}{b(x)}g(x) = \frac{\exp\left\{-\frac{1}{2}\left[\frac{x-d(y)}{c(y)}\right]^2\right\}h(y)}{c(y)}, \tag{6.2}$$

where $b(x) > 0, c(y) > 0$ and $a(x)$ and $d(y)$ define the regression lines and $b(x)$ and $c(y)$ are the corresponding standard deviations. Solving this equation, we have:

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{G}{2}\right\} \times$$

$$\times \exp\left\{-\frac{1}{2}[2Hx+2Ay+Jx^2+Dy^2+2Bxy+2Cx^2y+2Exy^2+Fx^2y^2]\right\}.$$

For the function $f(x,y)$ to be a density function, constants A, B, C, D, E, F, G, H and J must satisfy the following conditions:

$$\begin{array}{l} (i)\;\; F=E=C=0,\;\; D>0,\;\; J>0,\;\; B^2<DJ, \\ (ii)\;\; F>0,\;\; FD>E^2,\;\; JF>C^2. \end{array}$$

Model (i) is the bivariate normal model, and Model (ii) has the following properties:

- Regression lines are not straight lines
- Marginal distributions are not normal.
- The mode or the modes are at the intersection of the regression lines.

The surprising result is that there exists another family with normal conditionals. Figure 5 shows three examples of distributions with normal conditionals, where the upper figure corresponds to the normal case and the other two are not normal, one being unimodal and the other an interesting bimodal example.

These results were published at that conference, [81], Professor Barry Arnold from the University of California (Riverside) was at that conference and two months later wrote János to ask for permision to publish a paper inspired in ours but dealing with exponential conditionals instead of normals, [152]. Our results were also published in a congress in Cairo, [89], and one extension to characterize normal distributions in [103]. A little later we solved the problem of Weibull conditionals in [105], gamma conditionals in [108] and beta kind two in [167].

I must recognize here the importance of being familiar with functional equations and the important role played by János Aczél's book, [148], on making this possible. I cannot understand how functional equations, which are as important and powerful as differential equations (see, [162], [161]) are not studied in regular courses at the graduate level.

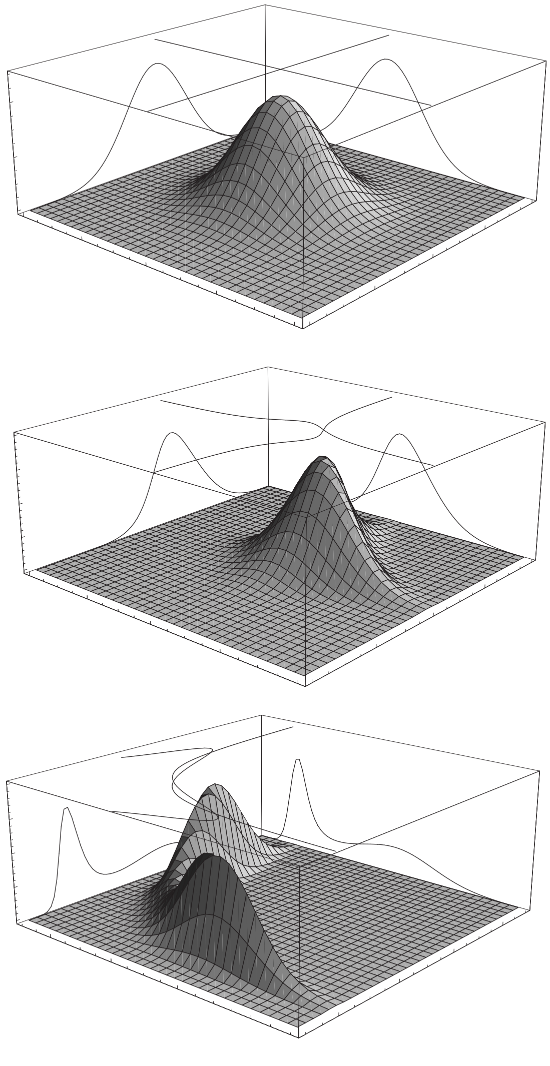

Figure 5. Examples of distributions with normal conditionals showing the joint densities, the marginals as horizontal projections, and the regression lines on a top projection. The upper figure corresponds to the normal case. The other two are not normal, one being unimodal and the other an interesting bimodal example

6.2. Solving a functional equation arising in fatigue models

In 1984, working with Alfonso Fernández Canteli at the ETH in Zürich, on a fatigue problem we arrive at the following functional equation (see [163], [166], [164]):

$$1-\exp\left\{-\left[a(x)y+b(x)\right]^{c(x)}\right\}=1-\exp\left\{-\left[d(y)x+e(y)\right]^{f(y)}\right\};\quad \forall x,y \tag{6.3}$$

which states a compatibility condition on two Weibull distributions (see Figure 6).

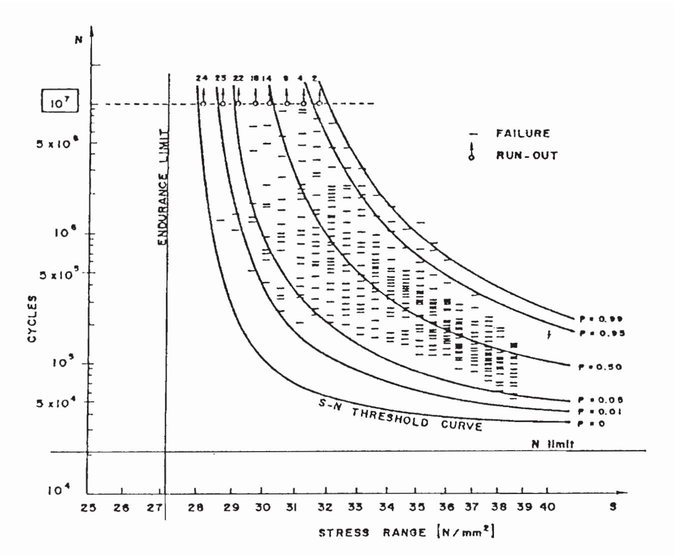

Figure 6. Illustration of the S-N field with the percentile curves associated with Weibull distributions.

At that time, using a theorem of [148], Alfonso and I were able to solve it for the particular case of $c(x) = constant$ and $f(y) = constant$ and obtained very interesting results for modeling the fatigue Wöhler fields.

Later with János, and using asymptotic theory, we were able to solve this functional equation in general, [89] and [82], for the case in which

$$1-\exp\left\{-\left[a(x)y+b(x)\right]^{c(x)}\right\}=1-\exp\left\{-\left[d(y)x+e(y)\right]^{f(y)}\right\};\quad \forall x,y \tag{6.4}$$

is a cdf as a function of y for all x and, at the same time, a cdf as a function of x for all y. The main difficulty arises when $c(x)$ and $f(y)$ are not constant or even identical. The final result is that the cdf in (6.4) must belong to one of the two families:

$$1 - \exp\left[-D(Ex+F)^{C\log(Ay+B)}\right] \tag{6.5}$$

or

$$1 - \exp\left\{-\left[C(x-A)(y-B)+D\right]^{E}\right\}, \tag{6.6}$$

where A, B, C, D, E and F are constants.

One application of the first model is given in [164]. The second model is being studied at the present time.

Recently, we have discovered that Equation (6.4) has interest and arises in crack and damage growth models. This allows us to conclude that for large size of long pieces of material it is justified the use of straight lines or hyperboles as crack or damage growth curves.

6.3. Some characterizations of the normal distribution

In 1989 we published a joint work, [103], in which we characterized the normal distribution based on conditionals and other properties. In particular we stated the following theorem.

Theorem 6.1. *Assume that $f(x,y)$ is a bivariate density whose conditionals $f_x(x)$ and $g_y(y)$ are univariate normal densities. Denote the expectation and variance parameters as $m_1(y)$ and $\sigma_1^2(y)$ for $g_y(x)$ and by $m_2(x)$ and $\sigma_2^2(x)$ for $f_x(y)$. Then, $f(x,y)$ is normal if, and only if, any of the following properties holds:*

1. *$\sigma_1(y)$ or $\sigma_2(x)$ is constant.*
2. *as $y \to +\infty$, $y^2\sigma_1(y) \to +\infty$ (or similar property for $\sigma_2(x)$ holds),*
3. *as $y \to +\infty$, $\liminf \sigma_1(y) \neq 0$ (or a similar property for $\sigma_2(x)$).*

Under any one of these conditions, $m_1(y)$ and $m_2(x)$ are linear.

In the case of independent identically distributed random variables (i.i.d.), Galambos and A. Obretenov [94] obtained necessary and sufficient conditions for the existence of a limit distribution of the maximum of the X_i's (under proper normalization). These conditions are expressed in terms of the hazard rate and expected residual life of the underlying population.

6.4. The Curvature Method

The problem of checking whether a sample comes from a maximal Gumbel family of distributions, and the problem of determining whether the domain of attraction of a given sample is maximal Gumbel, are different. In the former case, the whole sample is expected to follow a linear trend when plotted on a maximal Gumbel probability paper, while only the upper tail is expected to exhibit that property for the latter.

Consequently, it is incorrect to reject the assumption that a maximal Gumbel domain of attraction does not fit the data only because the data do not show a linear trend when plotted on a maximal Gumbel probability paper. This is because two distributions with the same tail, even though the rest be completely different, lead to exactly the same domain of attraction, and should be approximated by the same model.

Since only the right (left) tail governs the domain of attraction for maxima (minima), then one should focus only on the behavior of the high- (low-) order statistics. The remaining data values are not needed. In fact, they can be more of a hinderance than of a help in solving the problem. The problem is then to determine how many data points should be considered in the analysis or to decide which weights must be assigned to each data point.

When selecting the weights for different data points, the following considerations should be taken into account:

1. The tail quantiles estimates have larger variance than those not in the tail.

2. Data on the tail of interest have more information on the limit distribution than those not in the tail.

These are contradicting considerations and should be balanced. However, on the other tail the weights must be very small if not zero.

With the aim of illustrating graphically the role of the tails in the extreme behavior, Figures 7 and 8 show well-known distributions on maximal and minimal Gumbel probability papers, respectively. Note that those who are not Gumbel exhibit nonlinear trends. Figure 7 shows an almost linear trend for the right tail of the normal distribution, confirming that the normal belongs to the maximal Gumbel domain of attraction. Note also the curvatures in the right tail of the uniform (positive curvature and vertical asymptote) and Cauchy distributions (negative curvature and horizontal trend) showing Weibull and Fréchet domains of attractions for maxima, respectively.

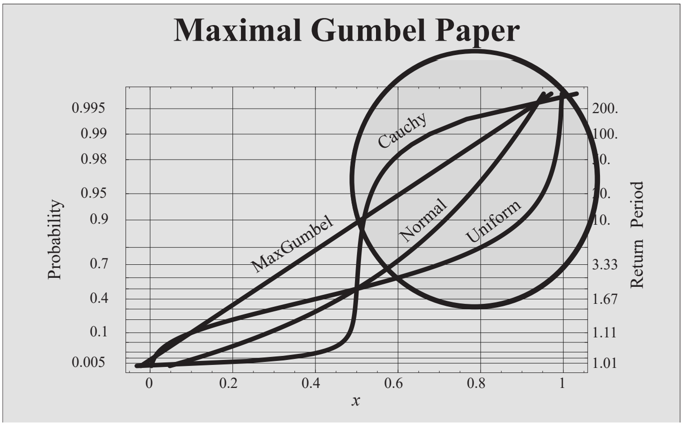

Figure 7. The uniform, normal, Cauchy, and maximal Gumbel distributions plotted on a maximal Gumbel probability paper

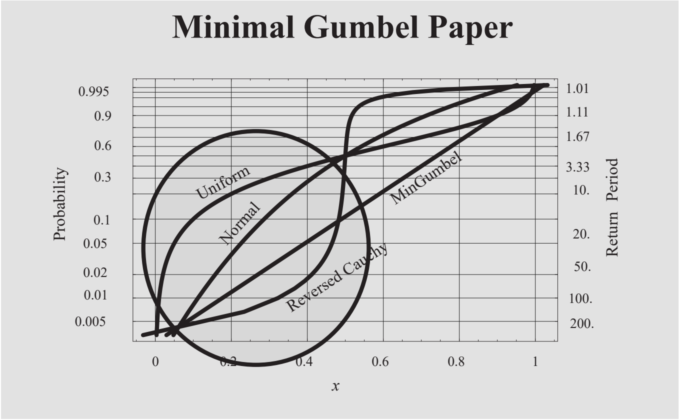

Figure 8. The uniform, normal reversed Cauchy, and minimal Gumbel distributions plotted on a minimal Gumbel probability paper

Figure 8 shows an almost linear trend for the left tail of the normal distribution, confirming that it belongs to the minimal Gumbel domain of attraction. Note also the curvatures in the left tail of the uniform (negative curvature and vertical asymptote) and the reversed Cauchy distributions (positive curvature and horizontal trend) showing minimal Weibull and Fréchet domains of attractions, respectively.

The method to be described below was done with János Galambos and has the same appealing geometrical property of the basic idea that is used for the probability paper method, i.e., the statistics upon which a decision will be made is based on the tail curvature (see [102]). This curvature can be measured in different ways. For example, by the difference or the quotient of slopes at two points. In addition, any of these two slopes can be measured by utilizing two or more data points. The latter option seems to be better in order to reduce variances. Here we propose to fit two straight lines, by least-squares, to two tail intervals and to use the quotient of their slopes to measure the curvature. More precisely, we use the statistic

$$S = \frac{S_{n_1,n_2}}{S_{n_3,n_4}}, \tag{6.7}$$

where $S_{i,j}$ is the slope of the least-squares straight line fitted on Gumbel probability paper, to the rth order statistics with $i \le r \le j$. Thus, we can write

$$S_{ij} = \frac{m\Sigma_{11} - \Sigma_{10}\Sigma_{01}}{m\Sigma_{20} - \Sigma_{10}\Sigma_{01}}, \tag{6.8}$$

where $m = n_j - n_i + 1$ and

$$\Sigma_{10} = \sum_{k=n_i}^{n_j} \left\{ -\log\left[-\log\left(\frac{k-0.5}{n} \right) \right] \right\}, \quad \Sigma_{01} = \sum_{k=n_i}^{n_j} x_k, \tag{6.9}$$

$$\Sigma_{11} = \sum_{k=n_i}^{n_j} \left\{ -x_k \log\left[-\log\left(\frac{k-0.5}{n} \right) \right] \right\}, \tag{6.10}$$

$$\Sigma_{20} = \sum_{k=n_i}^{n_j} \left\{ -\log\left[-\log\left(\frac{k-0.5}{n} \right) \right] \right\}^2, \tag{6.11}$$

and n is the sample size.

An important property of the least squares slope S_{ij} is that it is a linear combination of order statistics with coefficients which add up to zero. This property makes the statistic S location and scale invariant.

The selection of n_1, n_2, n_3 and n_4 must be based on the sample size and the speed of convergence to the asymptotic distribution, which sometimes can be inferred from the sample. Apart from speed of convergence considerations, we have selected the following values when the right tail is of interest

$$n_1 = n + 1 - [2\sqrt{n}], \qquad n_4 = n, \tag{6.12}$$

$$n_2 = n_3 = n + 1 - \frac{[2\sqrt{n}]}{2}, \tag{6.13}$$

where $[x]$ means the integer part of x. The $\sqrt{n}$ is selected to ensure using only high order statistics.

According to the above theory and with the values in (6.13), if the statistic S is well above 1 we can decide that the domain of attraction is Weibull type. And, if it is well below 1, the decision is in favor of a Fréchet type. However, in order to be able to give significance levels of the test, we need to know the cdf of S. The asymptotic properties of this method have been studied by [102].

If we are interested in the left tail, we can make the change of variables $X = -X$ and use the previous method for the resulting right tail.

I have already mentioned that he was critical with classical probability and statistical methods and that he look for properties to be reproduced by functional equations in order to derive adequate physical and engineering models.

A clear and important example is his work [135] with N. Macri, in which he demonstrates that statisticians and engineers must be aware about the assumptions implied by theoretical models and shows how extreme models must be used.

I recommend the reading of this work, in which, in only three pages, he discusses:

1. The need to consider the independence or not of the random sample.
2. The importance of the bounded or unbounded character of the tail of interest in extremes.
3. The possible existence of contradictions in the model.
4. The limitations of asymptotic models and the determination of reasonable sample sizes for them to be applicable.
5. The lack of sense of justifying divergences in the model based on outliers in the case of extreme value problems.
6. The fact that the peaks over threshold method involves two sensitive passages to the limit, which makes the estimator impractical for not very large data sizes.
7. The risk of interpreting data from a Gumbel type domain of attraction as a Weibull type, which is on the unsafe side.
8. The importance of using tail data for analyzing extremes, that is, the k largest (smallest) sample values in a sample of size n, but such that $k/n \to 0$.
9. The practical value of graphical methods.

10. How to use the curvature method based on two groups of consecutive tail data, to test domains of attractions.
11. The value of Monte Carlo methods to derive critical values for tests of significance.
12. The relevance of location and scale invariant estimators.

I end this comment with the final paragraph of this paper:

> *"Finally, in extreme value-related statistical evaluations, one should not try to modify a conclusion by reference to outliers. Extremes that do not fit into an assumed model are clear indications that the assumption is wrong and a different model is to be selected. This remark is based not on the present calculations but on long experience in extreme value evaluations."*

6.5. Weibull conditionals family

Another joint work with János Galambos was the characterization of all bivariate distributions with Weibull conditionals, [105].

Definition 6.1. We say that X has a Weibull distribution if

$$P(X > x) = \exp[-(x/\sigma)^{\gamma}], \quad x > 0, \tag{6.14}$$

where $\sigma > 0$ and $\gamma > 0$.

If (6.14) holds we write $X \sim \text{Weibull}(\sigma, \gamma)$. Our goal is to characterize all bivariate distributions with Weibull conditionals, a problem solved in [105], that is,

$$X|Y = y \sim \text{Weibull}(\sigma_1(y), \gamma_1(y)), \quad y > 0, \tag{6.15}$$

and

$$Y|X = x \sim \text{Weibull}(\sigma_2(x), \gamma_2(x)), \quad x > 0. \tag{6.16}$$

Writing the joint density corresponding to (6.15) and (6.16) as products of marginals and conditionals yields the following functional equation, valid for $x, y > 0$:

$$\begin{aligned} &f_Y(y)\frac{\gamma_1(y)}{\sigma_1(y)}\left[\frac{x}{\sigma_1(y)}\right]^{\gamma_1(y)-1} \exp\left[-\left(\frac{x}{\sigma_1(y)}\right)^{\gamma_1(y)}\right] = \\ &= f_X(x)\frac{\gamma_2(x)}{\sigma_2(x)}\left[\frac{y}{\sigma_2(x)}\right]^{\gamma_2(x)-1} \exp\left[-\left(\frac{y}{\sigma_2(x)}\right)^{\gamma_2(x)}\right]. \end{aligned} \tag{6.17}$$

For suitably defined functions $\phi_1(y), \phi_2(y), \psi_1(x)$ and $\psi_2(x)$, (6.17) can be written in the form

$$\phi_1(y)x^{\gamma_1(y)}\exp\left[-\phi_2(y)x^{\gamma_1(y)}\right] = \psi_1(x)y^{\gamma_2(x)}\exp\left[-\psi_2(x)y^{\gamma_2(x)}\right]. \tag{6.18}$$

Castillo and Galambos, [105], indicate how to solve this functional equation and discuss the corresonding solutions.

6.6. Gamma conditionals family

When returning to Spain, we invited János to deliver a course at the University of Cantabria in Spain, and the Estadística Española Journal invited János, Sarabia and I to write a paper, [108], in which we dealt with the characterization of bivariate models with gamma conditionals.

The family of gamma distributions with scale and shape parameters forms a two-parameter exponential family of the form

$$f(x;\theta_1,\theta_2) = x^{-1}e^{\theta_1\log x-\theta_2 x}\theta_2^{\theta_1}\left[\Gamma(\theta_1)\right]^{-1}I(x>0). \tag{6.19}$$

If X has its density of the form (6.19) then we write $X \sim \Gamma(\theta_1,\theta_2)$. If we require all conditionals to be in the family (6.19) then the general gamma conditionals class of densities is given by

$$f(x,y) = (xy)^{-1}\exp\left\{(1 \quad -x \quad \log x)M\begin{pmatrix}1\\ -y\\ \log y\end{pmatrix}\right\},\quad x>0,\quad y>0, \tag{6.20}$$

where M is a matrix of parameters.

It remains only to determine appropriate values of the parameters M to ensure integrability of this joint density. Such conditions were provided by Castillo, Galambos, and Sarabia (1990) using a different parametrization.

The conditional distribution $X|Y=y$ is:

$$X|Y=y \sim \Gamma(m_{20}+m_{22}\log y - m_{21}y, m_{10}-m_{11}y+m_{12}\log y), \tag{6.21}$$

the marginal density of X is:

$$f_X(x) = x^{-1}\frac{\Gamma(m_{02}+m_{22}\log x - m_{12}x)e^{m_{00}-m_{10}x+m_{20}\log x}}{(m_{01}-m_{11}x+m_{21}\log x)^{m_{02}+m_{22}\log x-m_{12}x}},\quad x>0, \tag{6.22}$$

and the regression of X on Y becomes:

$$E(X|Y=y) = \frac{m_{20}+m_{22}\log y - m_{21}y}{m_{10}+m_{12}\log y - m_{11}y}. \tag{6.23}$$

Analogous expressions are obtained for $Y|X=x$, $f_Y(y)$ and $E(Y|X=x)$.

The requisite conditions for a proper density (positive and integrable) $f(x,y)$ in (6.20) lead to six different families. The datails can be seen in [151].

7. Some works with Imre Kátai

Later, I met János in Budapest at the Loránd Eötvös University, where he started before his stays in two African universities just before arriving in Philadelphia (Temple University). There I contacted Imre Kátai, the organizer of this special issue dedicated to János, and worked with some colleagues in [149].

He has published several papers with Imre Kátai, during the period 1986-1992, [83, 90, 91, 95, 101, 106, 113, 114], on random walks, random sample sizes, characterizations and Bonferroni-type inequalities, among other topics. In particular, Imre Kátai organized and published a volume in honor of János in 2011 in celebration of his 70th birthday, in which Italo Simonelli, [182][*], publishes a laudation, and I published our joint work, [159]. Imre Kátai is organizing this one too.

Figure 9. Imre Kátai

Galambos derived a method for proving Bonferroni-type inequalities [41], later refined in [62], which had significant implications in the derivation of inequalities for multivariate extreme distributions. Extensions to the bivariate case were obtained by Galambos and Y. Xu [126], and to $P(v_n \geq k)$ by Galambos and Simonelli [131]. These results provide a method of proof as well as a method to derive new inequalities in closed form, needed when S_k is known for several $k's$, or in proving limit theorems [104].

[*]I acknowledge the invitation of I. Simonelli to incorporate some of his work to this paper.

8. The NIST Extreme Value Theory and Applications Conference

Though not very well known, János mantained a consultant experience with the National Institute of Standards and Technology (NIST) at Gaithersburg, Maryland, which is one of the most important institutes dedicated to safety. This activity allowed him to have a special practical sense in modelling. He was able to separate theory from practice and being aware of the fact that mathematical and statistical assumptions have in no few cases nothing to do with the physical or engineering reality.

In fact, János, with James Lechner and Emil Simiu, organized the very important Extreme Value Theory and Applications Conference at NIST in 1993 and they also edited the Conference proceedings in 1994, where János published his papers [122] and [123].

János Galambos, who was a consultant at NIST (National Institute of Standards and Technology in Gaithersburg, Maryland (US) organized a Conference on Extremes, and he invited me to teach a course on extremes, to deliver one of the main invited talks, [158], and to participate in the panel discussion on the future of extreme value theory and its applications, together with Laurens de Haan, Lucien La Cam and Richard L. Smith. At the conference I also presented one expert system for extreme value analysis, [160], in which all the steps to be followed to select the adequate distribution for engineering design based on extremes are explained and the user is guided to avoid common errors.

Soon, I was invited to write a paper on extreme wave heights by the Journal of Waterways, Ports, Coastal, and Ocean Engineering, [168], and later another in the Journal of Research of the National Institute of Standards and Technology, [169],

Later, I worked on extremes with Barry Arnold and José María Sarabia in [150] and several years later I published the second edition of my book, [165], with Ali S. Hadi (Cornell University), N. Balakrishnan (Mc Master University) and J. M. Sarabia (University of Cantabria).

9. His last stage at Temple University

In addition to writing over 130 papers and 8 books, two of which have been translated into Russian and Chinese, he has edited several books. The importance of his work has been recognized throughout the world: he has been a frequent speaker at international conferences and has traveled widely as a guest of universities and scientific institutions. Indeed he has worked/lectured on every continent except Antarctica. He is an elected member of the Hungarian Academy of Science, the International Statistical Institute, the Spanish Royal Academy of Engineers, and a Fellow of the Institute of Mathematical Statistics.

He has had 12 Ph.D. students, who now work in the USA, China, and Korea.

Galambos started working on products dealing with multiplicative functions [7, 8, 16], and realized the need to develop a unified theory. He suggested that Simonelli investigate limit theorems for products of independent random variables with and without normalization. Following Simonelli's work in [181] Galambos and Simonelli extended and applied these results in a sequel of papers which culminated in a book [146].

Characterization results based on finite and infinite products were developed in [142, 145, 128]. The theory of products was also applied to study the limit distribution of multiplicative functions; Galambos and Simonelli provided a pure probabilistic proof of Bakštys's theorem [24], and consequently, of Erdős-Whitney's theorem.

Finally, I want to recognize the very important role played by Éva Galambos, who has always supported János throughout his professional life. It is fair to recognize Éva's relevant contribution in this document.

10. Epilogue

It has been a pleasure to participate in this special issue of the Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae Sectio Computatorica to honor Prof. János Galambos. The scientific community must recognize the role played by János on extremes and in other areas of Probability, Statistics and Mathematics (see [125, 128, 146]). I must congratulate Imre Kátai for having the nice idea of this special issue, and thank all participants to make this possible.

The last time a saw János was in 2007 on the occasion of a trip to the University of Maryland. From there I traveled to Philadelphia to see János. Let this paragraph an occassion to transmit him personally my gratitude. Some of my successes have been possible because of you, and thus, they are yours too.

Figure 10. A photo of János Galambos

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