Annales Univ. Sci. Budapest., Sect. Comp. 49 (2019) 425–427

UNSOLVED PROBLEMS

THREE NEW CONJECTURES RELATED TO THE VALUES OF ARITHMETIC FUNCTIONS AT CONSECUTIVE INTEGERS

Jean-Marie De Koninck[1] (Québec, Canada)
Imre Kátai and Bui Minh Phong (Budapest, Hungary)

1. Introduction

Let $\mathcal{M}_1^*$ stand for the set of completely multiplicative functions f such that $|f(n)| = 1$ for all integers $n \geq 1$ and let $\mathcal{A}^*$ be the set of completely additive functions. Given $f \in \mathcal{M}_1^*$, we set $\delta_f(n) := f(n+1)\overline{f(n)}$ for each integer $n \geq 1$, whereas given $f \in \mathcal{A}^*$, we set $\Delta_f(n) := f(n+1) - f(n)$ for each integer $n \geq 1$.

Given $f \in \mathcal{M}_1^*$, we say that $w \in \mathbb{C}$ is a *strong limit point* of the sequence $(\delta_f(n))_{n\geq 1}$ if there exists an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{j\to\infty} \delta_f(n_j) = w$ and $\liminf\limits_{x\to\infty} \dfrac{1}{x}\#\{n_j < x\} = c$ for some constant positive c. Similarly, given $f \in \mathcal{A}^*$, we say that $w \in \mathbb{C}$ is a *strong logarithmic limit point* of the sequence $(\delta_f(n))_{n\geq 1}$ if there exists an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{j\to\infty} \delta_f(n_j) = w$ and such that $\liminf\limits_{x\to\infty} \dfrac{1}{\log x} \sum\limits_{n_j<x} \dfrac{1}{n_j} = c$ for some constant positive c.

Here, letting $\mathcal{H}$ (resp. $\mathcal{H}_{\log}$) stand for the set of those $f \in \mathcal{M}_1^*$ which have at least one strong limit point (resp. at least one strong logarithmic limit point), we conjecture that all functions in $\mathcal{H}$ are necessarily of a certain particular form and we also conjecture that the set $\mathcal{H}_{\log}$ is the same as the set $\mathcal{H}$.

Key words and phrases: Arithmetic functions.
2010 Mathematics Subject Classification: 11N37, 11N64.
[1] Research supported in part by a grant from NSERC.

Similarly, we let $\mathcal{K}$ be the set of those $f \in \mathcal{A}^*$ for which there exists some real number $\lambda \in [0,1)$ and an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{j\to\infty} \|\Delta_f(n_j) - \lambda\| = 0$ and $\liminf\limits_{x\to\infty} \dfrac{1}{x}\#\{n_j < x\} = c$ for some positive constant c. Here, we conjecture that all functions in $f \in \mathcal{K}$ can be written as $f(n) = d\log n + u(n) + v(n)$ for some constant $d > 0$ and where $u(n)$ and $v(n)$ are some basic functions belonging to the set $\mathcal{K}$.

2. Characterisation of those functions belonging to $\mathcal{H}$

Consider the following three categories of functions.

(A) Those functions f of the form $f(n) = n^{it}$ for some real number t. Clearly, all these functions belong to $\mathcal{H}$.

(B) Those functions $f \in \mathcal{M}_1^*$ such that for some $k \in \mathbb{N}$, we have $f^k(n) = 1$ for all integers $n \geq 1$. Clearly, all these functions also belong to $\mathcal{H}$.

(C) Let $\mathcal{B}$ a set of primes such that $\sum_{p\in\mathcal{B}} 1/p < \infty$. Moreover, let $\mathcal{N}(\mathcal{B})$ be the multiplicative semigroup generated by $\mathcal{B}$. We construct a particular function $f \in \mathcal{M}_1^*$ as follows. Let ξ be an arbitrary point on the unit circle. We then define f on the primes p by

$$f(p) = \begin{cases} \xi & \text{if} \quad p \in \mathcal{B}, \\ 1 & \text{if} \quad p \notin \mathcal{B}. \end{cases}$$

One can prove that such functions f belong to $\mathcal{H}$. Indeed, let f be such a function and let $b_1, b_2 \in \mathcal{N}(\mathcal{B})$ be such that $(b_1, b_2) = 1$. Since the set of those positive integers n of the form $n = b_1\nu$ and for which $b_2\mu - b_1\nu = 1$ with $(\mu\nu, \mathcal{B}) = 1$ is of positive density, we may therefore conclude that $f \in \mathcal{H}$.

Given three arithmetic functions f_A, f_B, f_C belonging to the categories A, B, C, respectively, consider the arithmetic function $f(n) := f_A(n) \cdot f_B(n) \cdot f_C(n)$. One can easily prove that $f \in \mathcal{H}$.

Conjecture 1. *If $f \in \mathcal{H}$, then $f(n) = f_A(n) \cdot f_B(n) \cdot f_C(n)$ for some functions f_A, f_B, f_C belonging to the categories A, B, C, respectively.*

Conjecture 2. *The set $\mathcal{H}_{\log}$ is the same as the set $\mathcal{H}$.*

3. Characterisation of those functions belonging to $\mathcal{K}$

Consider the following three categories of functions.

(A1) Those functions f of the form $f(n) = d \log n$ for some real number d. Clearly, all these functions belong to $\mathcal{K}$.

(B1) Let $u \in \mathcal{A}^*$ be such that $ku(n) \equiv 0 \pmod 1$ for some $k \in \mathbb{N}$. One can easily see that all such functions u belong to $\mathcal{K}$.

(C1) Let $\mathcal{B}$ a set of primes such that $\sum_{p\in\mathcal{B}} 1/p < \infty$. We construct a particular function $v \in \mathcal{A}^*$ as follows. Let ξ be an arbitrary point on the unit circle. We then define v on the primes p by

$$v(p) = \begin{cases} \xi & \text{if} \quad p \in \mathcal{B}, \\ 0 \pmod 1 & \text{if} \quad p \notin \mathcal{B}. \end{cases}$$

One can prove that such functions v belong to $\mathcal{K}$.

Given any real number d and two arithmetic functions u and v belonging respectively to the categories B1 and C1, consider the arithmetic function $f(n) := d \log n + u(n) + v(n)$. One can easily prove that $f \in \mathcal{K}$.

Conjecture 3. *If $f \in \mathcal{K}$, then $f(n) = d \log n + u(n) + v(n)$ for some real number d and some functions u and v belonging respectively to the categories B1 and C1.*

J.-M. De Koninck
Départment de mathématiques
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

I. Kátai and B.M. Phong
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
Pázmány Péter sétány 1/C
H-1117 Budapest, Hungary
katai@inf.elte.hu
bui@inf.inf.elte.hu