UNSOLVED PROBLEMS

THREE NEW CONJECTURES RELATED TO THE VALUES OF ARITHMETIC FUNCTIONS AT CONSECUTIVE INTEGERS

Jean-Marie De Koninck\textsuperscript{1} (Québec, Canada)
Imre Kátai and Bui Minh Phong (Budapest, Hungary)

1. Introduction

Let $M_1^\ast$ stand for the set of completely multiplicative functions $f$ such that $|f(n)| = 1$ for all integers $n \geq 1$ and let $A^\ast$ be the set of completely additive functions. Given $f \in M_1^\ast$, we set $\delta_f(n) := f(n+1)f(n)$ for each integer $n \geq 1$, whereas given $f \in A^\ast$, we set $\Delta_f(n) := f(n+1) - f(n)$ for each integer $n \geq 1$.

Given $f \in M_1^\ast$, we say that $w \in \mathbb{C}$ is a strong limit point of the sequence $(\delta_f(n))_{n \geq 1}$ if there exists an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{j \to \infty} \delta_f(n_j) = w$ and $\liminf_{x \to \infty} \frac{1}{x} \# \{ n_j < x \} = c$ for some constant positive $c$. Similarly, given $f \in A^\ast$, we say that $w \in \mathbb{C}$ is a strong logarithmic limit point of the sequence $(\delta_f(n))_{n \geq 1}$ if there exists an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{j \to \infty} \delta_f(n_j) = w$ and such that $\liminf_{x \to \infty} \frac{1}{\log x} \sum_{n_j < x} \frac{1}{n_j} = c$ for some constant positive $c$.

Here, letting $\mathcal{H}$ (resp. $\mathcal{H}_{\log}$) stand for the set of those $f \in M_1^\ast$ which have at least one strong limit point (resp. at least one strong logarithmic limit point), we conjecture that all functions in $\mathcal{H}$ are necessarily of a certain particular form and we also conjecture that the set $\mathcal{H}_{\log}$ is the same as the set $\mathcal{H}$.

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Similarly, we let $K$ be the set of those $f \in A^*$ for which there exists some real number $\lambda \in [0, 1)$ and an infinite sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{j \to \infty} \|\Delta f(n_j) - \lambda\| = 0$ and $\liminf_{x \to \infty} \frac{1}{x} \#\{n_j < x\} = c$ for some positive constant $c$. Here, we conjecture that all functions in $f \in K$ can be written as $f(n) = d \log n + u(n) + v(n)$ for some constant $d > 0$ and where $u(n)$ and $v(n)$ are some basic functions belonging to the set $K$.

2. Characterisation of those functions belonging to $H$

Consider the following three categories of functions.

(A) Those functions $f$ of the form $f(n) = n^t$ for some real number $t$. Clearly, all these functions belong to $H$.

(B) Those functions $f \in M_1^*$ such that for some $k \in \mathbb{N}$, we have $f^k(n) = 1$ for all integers $n \geq 1$. Clearly, all these functions also belong to $H$.

(C) Let $B$ a set of primes such that $\sum_{p \in B} 1/p < \infty$. Moreover, let $\mathcal{N}(B)$ be the multiplicative semigroup generated by $B$. We construct a particular function $f \in M_1^*$ as follows. Let $\xi$ be an arbitrary point on the unit circle. We then define $f$ on the primes $p$ by

$$f(p) = \begin{cases} \xi & \text{if } p \in B, \\ 1 & \text{if } p \notin B. \end{cases}$$

One can prove that such functions $f$ belong to $H$. Indeed, let $f$ be such a function and let $b_1, b_2 \in \mathcal{N}(B)$ be such that $(b_1, b_2) = 1$. Since the set of those positive integers $n$ of the form $n = b_1 \nu$ and for which $b_2 \mu - b_1 \nu = 1$ with $(\mu \nu, B) = 1$ is of positive density, we may therefore conclude that $f \in H$.

Given three arithmetic functions $f_A, f_B, f_C$ belonging to the categories A, B, C, respectively, consider the arithmetic function $f(n) := f_A(n) \cdot f_B(n) \cdot f_C(n)$. One can easily prove that $f \in H$.

**Conjecture 1.** If $f \in H$, then $f(n) = f_A(n) \cdot f_B(n) \cdot f_C(n)$ for some functions $f_A, f_B, f_C$ belonging to the categories A, B, C, respectively.

**Conjecture 2.** The set $H_{\log}$ is the same as the set $H$.
3. Characterisation of those functions belonging to $K$

Consider the following three categories of functions.

(A1) Those functions $f$ of the form $f(n) = d \log n$ for some real number $d$. Clearly, all these functions belong to $K$.

(B1) Let $u \in A^*$ be such that $ku(n) \equiv 0 \pmod{1}$ for some $k \in \mathbb{N}$. One can easily see that all such functions $u$ belong to $K$.

(C1) Let $B$ a set of primes such that $\sum_{p \in B} 1/p < \infty$. We construct a particular function $v \in A^*$ as follows. Let $\xi$ be an arbitrary point on the unit circle. We then define $v$ on the primes $p$ by

$$v(p) = \begin{cases} \xi & \text{if } p \in B, \\ 0 \pmod{1} & \text{if } p \notin B. \end{cases}$$

One can prove that such functions $v$ belong to $K$.

Given any real number $d$ and two arithmetic functions $u$ and $v$ belonging respectively to the categories B1 and C1, consider the arithmetic function $f(n) := d \log n + u(n) + v(n)$. One can easily prove that $f \in K$.

**Conjecture 3.** If $f \in K$, then $f(n) = d \log n + u(n) + v(n)$ for some real number $d$ and some functions $u$ and $v$ belonging respectively to the categories B1 and C1.

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J.-M. De Koninck  
Département de mathématiques  
Université Laval  
Québec  
Québec G1V 0A6  
Canada  
jmdk@mat.ulaval.ca

I. Kátai and B.M. Phong  
Department of Computer Algebra  
Faculty of Informatics  
Eötvös Loránd University  
Pázmány Péter sétány 1/C  
H-1117 Budapest, Hungary  
katai@inf.elte.hu  
bui@inf.inf.elte.hu