

UNSOLVED PROBLEMS

THREE NEW CONJECTURES RELATED TO THE VALUES OF ARITHMETIC FUNCTIONS AT CONSECUTIVE INTEGERS

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1. Introduction

Let \mathcal{M}_1^* stand for the set of completely multiplicative functions f such that $|f(n)| = 1$ for all integers $n \geq 1$ and let \mathcal{A}^* be the set of completely additive functions. Given $f \in \mathcal{M}_1^*$, we set $\delta_f(n) := f(n+1)\overline{f(n)}$ for each integer $n \geq 1$, whereas given $f \in \mathcal{A}^*$, we set $\Delta_f(n) := f(n+1) - f(n)$ for each integer $n \geq 1$.

Given $f \in \mathcal{M}_1^*$, we say that $w \in \mathbb{C}$ is a *strong limit point* of the sequence $(\delta_f(n))_{n \geq 1}$ if there exists an infinite sequence of positive integers $n_1 < n_2 < \dots$ such that $\lim_{j \rightarrow \infty} \delta_f(n_j) = w$ and $\liminf_{x \rightarrow \infty} \frac{1}{x} \#\{n_j < x\} = c$ for some constant positive c . Similarly, given $f \in \mathcal{A}^*$, we say that $w \in \mathbb{C}$ is a *strong logarithmic limit point* of the sequence $(\delta_f(n))_{n \geq 1}$ if there exists an infinite sequence of positive integers $n_1 < n_2 < \dots$ such that $\lim_{j \rightarrow \infty} \delta_f(n_j) = w$ and such that $\liminf_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n_j < x} \frac{1}{n_j} = c$ for some constant positive c .

Here, letting \mathcal{H} (resp. \mathcal{H}_{\log}) stand for the set of those $f \in \mathcal{M}_1^*$ which have at least one strong limit point (resp. at least one strong logarithmic limit point), we conjecture that all functions in \mathcal{H} are necessarily of a certain particular form and we also conjecture that the set \mathcal{H}_{\log} is the same as the set \mathcal{H} .

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Similarly, we let \mathcal{K} be the set of those $f \in \mathcal{A}^*$ for which there exists some real number $\lambda \in [0, 1)$ and an infinite sequence of positive integers $n_1 < n_2 < \dots$ such that $\lim_{j \rightarrow \infty} \|\Delta_f(n_j) - \lambda\| = 0$ and $\liminf_{x \rightarrow \infty} \frac{1}{x} \#\{n_j < x\} = c$ for some positive constant c . Here, we conjecture that all functions in $f \in \mathcal{K}$ can be written as $f(n) = d \log n + u(n) + v(n)$ for some constant $d > 0$ and where $u(n)$ and $v(n)$ are some basic functions belonging to the set \mathcal{K} .

2. Characterisation of those functions belonging to \mathcal{H}

Consider the following three categories of functions.

- (A) Those functions f of the form $f(n) = n^{it}$ for some real number t . Clearly, all these functions belong to \mathcal{H} .
- (B) Those functions $f \in \mathcal{M}_1^*$ such that for some $k \in \mathbb{N}$, we have $f^k(n) = 1$ for all integers $n \geq 1$. Clearly, all these functions also belong to \mathcal{H} .
- (C) Let \mathcal{B} a set of primes such that $\sum_{p \in \mathcal{B}} 1/p < \infty$. Moreover, let $\mathcal{N}(\mathcal{B})$ be the multiplicative semigroup generated by \mathcal{B} . We construct a particular function $f \in \mathcal{M}_1^*$ as follows. Let ξ be an arbitrary point on the unit circle. We then define f on the primes p by

$$f(p) = \begin{cases} \xi & \text{if } p \in \mathcal{B}, \\ 1 & \text{if } p \notin \mathcal{B}. \end{cases}$$

One can prove that such functions f belong to \mathcal{H} . Indeed, let f be such a function and let $b_1, b_2 \in \mathcal{N}(\mathcal{B})$ be such that $(b_1, b_2) = 1$. Since the set of those positive integers n of the form $n = b_1 \nu$ and for which $b_2 \mu - b_1 \nu = 1$ with $(\mu, \nu) = 1$ is of positive density, we may therefore conclude that $f \in \mathcal{H}$.

Given three arithmetic functions f_A, f_B, f_C belonging to the categories A, B, C, respectively, consider the arithmetic function $f(n) := f_A(n) \cdot f_B(n) \cdot f_C(n)$. One can easily prove that $f \in \mathcal{H}$.

Conjecture 1. *If $f \in \mathcal{H}$, then $f(n) = f_A(n) \cdot f_B(n) \cdot f_C(n)$ for some functions f_A, f_B, f_C belonging to the categories A, B, C, respectively.*

Conjecture 2. *The set \mathcal{H}_{\log} is the same as the set \mathcal{H} .*

3. Characterisation of those functions belonging to \mathcal{K}

Consider the following three categories of functions.

- (A1) Those functions f of the form $f(n) = d \log n$ for some real number d . Clearly, all these functions belong to \mathcal{K} .
- (B1) Let $u \in \mathcal{A}^*$ be such that $ku(n) \equiv 0 \pmod{1}$ for some $k \in \mathbb{N}$. One can easily see that all such functions u belong to \mathcal{K} .
- (C1) Let \mathcal{B} a set of primes such that $\sum_{p \in \mathcal{B}} 1/p < \infty$. We construct a particular function $v \in \mathcal{A}^*$ as follows. Let ξ be an arbitrary point on the unit circle. We then define v on the primes p by

$$v(p) = \begin{cases} \xi & \text{if } p \in \mathcal{B}, \\ 0 \pmod{1} & \text{if } p \notin \mathcal{B}. \end{cases}$$

One can prove that such functions v belong to \mathcal{K} .

Given any real number d and two arithmetic functions u and v belonging respectively to the categories B1 and C1, consider the arithmetic function $f(n) := d \log n + u(n) + v(n)$. One can easily prove that $f \in \mathcal{K}$.

Conjecture 3. *If $f \in \mathcal{K}$, then $f(n) = d \log n + u(n) + v(n)$ for some real number d and some functions u and v belonging respectively to the categories B1 and C1.*

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