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WEIGHTED HARDY SPACES AND THE FEJÉR MEANS OF WALSH–FOURIER SERIES

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Abstract. In this paper we consider the A_p weights and weighted martingale Hardy spaces. We present Doob's maximal inequality and the atomic decomposition of the Hardy spaces. As application, we investigate the convergence of partial sums and Fejér summability of Walsh–Fourier series. We prove that the maximal operator of the Fejér means is bounded from the weighted dyadic Hardy space $H_p(w)$ to $L_p(w)$. This implies almost everywhere convergence of the Fejér means.

1. Introduction

In this paper we investigate the convergence of partial sums and the Fejér means of Walsh–Fourier series. It was proved by Fine [7] that the Fejér means of the Walsh–Fourier series,

$$\sigma_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1} s_k f(x) \to f$$

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almost everywhere if $f \in L_1$, where $s_k f$ denotes the kth partial sum. Schipp [19] obtained the same result by proving the weak type inequality of the maximal operator σ_* of the Fejér means. Next Fujii [8] showed that σ_* is bounded from the dyadic Hardy space H_1 to L_1 (see also Schipp and Simon [20]). Later the author proved that σ_* is bounded from H_p to L_p for $1/2 < p < \infty$.

In this paper, we generalize these results to weighted spaces. We introduce the concept of A_p weights. Next we prove the Doob's maximal inequality, i.e., the boundedness of the dyadic maximal function on the weighted $L_p(w)$ space if $w \in A_p$ for some $1 < p < \infty$. For $p = 1$ a weak type inequality holds. With the help of the weighted $L_p(w)$-norm of the dyadic maximal function, we introduce the weighted dyadic Hardy spaces and give their atomic decomposition. Using the atomic decomposition, we verify a sufficient condition for an operator to be bounded from the weighted dyadic Hardy space $H_p(w)$ to $L_p(w)$.

In the next section, we prove that the partial sums s_n are uniformly bounded on the weighted $L_p(w)$ space if $w \in A_p$ for some $1 < p < \infty$. This implies the norm convergence of the partial sums in $L_p(w)$. Finally, we show that σ_* is bounded from $H_p(w)$ to $L_p(w)$ if $w \in A_{2p}$ for some $1/2 < p < \infty$. From this we obtain the almost everywhere and norm convergence of the Fejér means to the function. Some of the results of this paper are proved in a more general form for Musielak–Orlicz type spaces in Xie et al. [32].

2. Walsh system

In this paper let w be a strictly positive integrable weight function. The weighted space $L_p(w)$ is equipped with the norm (or quasinorm)

$$\|f\|_{L_p(w)} := \left(\int_0^1 |f|^p w \, d\lambda \right)^{1/p} \qquad (0 < p \le \infty),$$

where λ is the Lebesgue measure. We use the notation $|I| = \lambda(I)$ for the Lebesgue measure of the set I and

$$w(I) := \int_I w \, d\lambda.$$

The weak $L_p(w)$ space $L_{p,\infty}(w)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}(w)} := \sup_{\rho > 0} \rho \, w(|f| > \rho)^{1/p} < \infty.$$

For $w = 1$, we write simply L_p and $L_{p,\infty}$. Note that $L_{p,\infty}(w)$ is a quasi-normed space. It is easy to see that

$$L_p(w) \subset L_{p,\infty}(w) \qquad \text{and} \qquad \|\cdot\|_{L_{p,\infty}(w)} \le \|\cdot\|_{L_p(w)}$$

for each $0 < p < \infty$.

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the *Walsh system*:

$$w_n := \prod_{k=0}^{\infty} r_k{}^{n_k} \qquad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad (0 \le n_k < 2).$$

We investigate dyadic martingales. By a *dyadic interval*, we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$ and $0 \le k < 2^n$. For any given $n \in \mathbb{N}$ and $x \in [0,1)$, denote by $I_n(x)$ the dyadic interval of length 2^{-n} which contains x. For any $n \in \mathbb{N}$, the σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in [0,1)\}$ is denoted by $\mathcal{F}_n$. It is clear that, for any $n \in \mathbb{N}$, r_n is $\mathcal{F}_{n+1}$ measurable.

Let us denote by E_n the conditional expectation operator with respect to $\mathcal{F}_n$ and the Lebesgue measure. A sequence $(f_n, n \in \mathbb{N})$ is called a martingale if f_n is $\mathcal{F}_n$ measurable and $E_{n-1} f_n = f_{n-1}$ $(n \in \mathbb{N})$. If $f \in L^1[0,1)$, for any $n \in \mathbb{N}$, the number

$$\widehat{f}(n) := \int_0^1 f w_n \, d\lambda$$

is called the nth *Walsh–Fourier coefficient* of f. We can extend this definition to martingales as follows. If $f := (f_k)_{k \in \mathbb{N}}$ is a martingale, then

$$\widehat{f}(n) := \lim_{k \to \infty} \int_0^1 f_k w_n \, d\lambda \qquad (n \in \mathbb{N}).$$

In this paper the constants C_p depend only on p and may denote different constants in different contexts.

3. Weighted dyadic Hardy spaces

To prove almost everywhere and norm convergence of the Fejér means introduced in the next section, we will need the concept of weighted dyadic Hardy spaces and their atomic decomposition.

The definition of A_p weights for martingales was introduced by Izumisawa and Kazamaki in [13].

Definition 1. Let $q \in [1, \infty)$. A weight function w is said to satisfy the A_q condition if there exists a positive constant K such that, when $q \in (1, \infty)$,

$$E_n(w) \left[E_n \left(w^{-\frac{1}{q-1}} \right) \right]^{q-1} \leq K \qquad (n \in \mathbb{N})$$

and, when $q = 1$,

$$E_n(w) \leq Kw \qquad (n \in \mathbb{N}).$$

w is said to satisfy the A_∞ condition if $w \in A_q$ for some $q \in [1, \infty)$.

It is known that for any $p, q \in (1, \infty)$ with $p \leq q$,

$$A_1 \subset A_p \subset A_q \subset A_\infty.$$

Let

$$q(w) := \inf \{ q \in [1, \infty) : w \in A_q \}.$$

The following lemma comes from Doléans-Dade and Meyer [5] (see also Long [15, Corollary 6.3.3] or Xie et al. [32]).

Lemma 1. *Let $p \in (1, \infty)$ and w be a weight. If $w \in A_p$, then there exists a positive constant ε such that $w \in A_{p-\varepsilon}$.*

We define the maximal operator of the dyadic martingale $f = (f_n, n \in \mathbb{N})$ by

$$f^* := \sup_{n \in \mathbb{N}} |f_n|.$$

The *weighted dyadic Hardy space* $H_p(w)$ $(0 < p < \infty)$ consists of all dyadic martingales $f = (f_n, n \in \mathbb{N})$ for which

$$\|f\|_{H_p(w)} := \|f^*\|_{L_p(w)} < \infty.$$

For $w = 1$, we write again H_p. It is known (see e.g. or Weisz [29]) that

$$H_p \sim L_p \qquad (1 < p \leq \infty),$$

where $\sim$ denotes the equivalence of spaces and norms. For weighted Hardy spaces we can formulate this theorem as follows.

Theorem 1. *If $w \in A_p$ for some $1 < p < \infty$, then*

$$\|f^*\|_{L_p(w)} \leq C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

Taking into account Lemma 1, we obtain

Corollary 1. *If $w \in A_\infty$ and $q(w) < p < \infty$, then*

$$\|f^*\|_{L_p(w)} \leq C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

Note that the inequality

$$\|f\|_{L_p(w)} \leq \|f^*\|_{L_p(w)}$$

is trivial. This implies

Corollary 2. *If $w \in A_\infty$ and $q(w) < p < \infty$, then*

$$H_p(w) \sim L_p(w).$$

The weak type inequality concerning the maximal operator can also be proved.

Theorem 2. *If $w \in A_1$ and $f \in L_1(w)$, then*

$$\sup_{\rho > 0} \rho\, w(f^* > \rho) \leq K \|f\|_{L_1(w)}.$$

Proof. Considering the stopping time

$$\nu := \inf\{n : |f_n| > \rho\},$$

we conclude

$$\begin{aligned}\rho\, w(f^* > \rho) &= \rho\, w(\nu < \infty) = \rho \sum_{k=0}^{\infty} w(\nu = k) \leq \\ &\leq \sum_{k=0}^{\infty} \int_{\{\nu=k\}} |f_k| w\, d\lambda \leq \sum_{k=0}^{\infty} \int_{\{\nu=k\}} E_k |f| w\, d\lambda = \\ &= \sum_{k=0}^{\infty} \int_{\{\nu=k\}} |f| E_k w\, d\lambda \leq K \sum_{k=0}^{\infty} \int_{\{\nu=k\}} |f| w\, d\lambda = \\ &= K \int_{\{\nu<\infty\}} |f| w\, d\lambda \leq K \|f\|_{L_1(w)},\end{aligned}$$

which finishes the proof. ■

The atomic decomposition provides a useful characterization of Hardy spaces. A bounded function a is a *weighted dyadic p-atom* if there exists a dyadic interval $I \subset [0,1)$ such that

(i) $\operatorname{supp} a \subset I$,

(ii) $\|a\|_\infty \leq w(I)^{-1/p}$,

(iii) $\int_I a(x)\, dx = 0$.

The atomic decomposition of the Hardy space $H_p(w)$ means that every function (more exactly, martingale) from the Hardy space can be decomposed into the sum of atoms. A first version of the atomic decomposition was introduced by Coifman and Weiss [4] in the classical case and by Herz [12] in the martingale case. The following theorem can be shown as in the unweighted case in Weisz [27, 29].

Theorem 3. *Let $0 < p \leq 1$ and $w \in A_\infty$. A martingale f is in $H_p(w)$ if and only if there exist a sequence $(\mu_{k,j,i})$ of real numbers and a sequence $(a^{k,j,i})$ $(k \in \mathbb{Z}, j, i \in \mathbb{N})$ of weighted dyadic p-atoms such that*

$$f_n = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{n-1} \sum_i \mu_{k,j,i} a_n^{k,j,i} \quad a.e., \quad n \in \mathbb{N}.$$

Moreover,

$$\|f\|_{H_p(w)} \sim \inf \left(\sum_{k\in\mathbb{Z}} \sum_{j=0}^{n-1} \sum_i |\mu_{k,j,i}|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f as above.

In the present form the theorem does not hold for $1 < p < \infty$. We can generalize the result as follows. If the supports of $a^{k,j,i}$ are the atoms $I_{k,j,i} \in \mathcal{F}_j$, which are disjoint for fixed k, then

$$\left(\sum_{k\in\mathbb{Z}}\sum_{j=0}^{n-1}\sum_{i}|\mu_{k,j,i}|^p\right)^{1/p} = \left\|\left(\sum_{k\in\mathbb{Z}}\left(\sum_{j=0}^{\infty}\sum_{i}\frac{\mu_{k,j,i}\chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{L_p(w)}}\right)^p\right)^{\frac{1}{p}}\right\|_{L_p(w)}$$

Then the next result can be proved similarly to Xie et al. [32].

Theorem 4. *Let $0 < r \le 1$, $0 < p < \infty$ and $w \in A_\infty$. A martingale f is in $H_p(w)$ if and only if there exist a sequence $(\mu_{k,j,i})$ of real numbers and a sequence $(a^{k,j,i})$ $(k \in \mathbb{Z}, j,i \in \mathbb{N})$ of weighted dyadic p-atoms with supports $I_{k,j,i} \in \mathcal{F}_j$, which are disjoint for fixed k, such that*

$$f_n = \sum_{k\in\mathbb{Z}}\sum_{j=0}^{n-1}\sum_{i}\mu_{k,j,i}a_n^{k,j,i} \quad a.e., \quad n \in \mathbb{N}.$$

Moreover,

$$\|f\|_{H_p(w)} \sim \inf\left\|\left(\sum_{k\in\mathbb{Z}}\left(\sum_{j=0}^{\infty}\sum_{i}\frac{\mu_{k,j,i}\chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{L_p(w)}}\right)^r\right)^{\frac{1}{r}}\right\|_{L_p(w)},$$

where the infimum is taken over all decompositions of f as above.

The following result gives a sufficient condition for an operator to be bounded from $H_p(w)$ to $L_p(w)$ (for the unweighted case see Weisz [29] and, for $p_0 = 1$, Schipp, Wade, Simon and Pl [21] and Móricz, Schipp and Wade [17]). For $I \subset \mathbb{T}$ let I^r be the dyadic interval, for which $I \subset I^r$ and $|I^r| = 2^r|I|$ $(r \in \mathbb{N})$.

Theorem 5. *Let $w \in A_\infty$. Suppose that V is a σ-sublinear operator and*

$$\int_{[0,1)\setminus I^r} |Va|^{p_0} w\, d\lambda \le C_{p_0} \tag{3.1}$$

for all weighted dyadic p_0-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$, where the interval I is the support of the atom. If V is bounded from $L_{p_1}(w)$ to $L_{p_1}(w)$ for some $1 < p_1 \le \infty$, then

$$\|Vf\|_{L_p(w)} \le C_p\|f\|_{H_p(w)} \qquad (f \in H_p(w)) \tag{3.2}$$

for all $p_0 \le p \le p_1$. *Moreover, if* $w \in A_1$ *and* $p_0 < 1$, *then the operator* V *is of weak type* $(1,1)$, *i.e., if* $f \in L_1(w)$ *then*

$$\sup_{\rho>0} \rho\, w(|Vf| > \rho) \le C\|f\|_{L_1(w)}. \tag{3.3}$$

Proof. Using condition (3.1) and Hlder's inequality, we can easily show that $\|Va\|_{p_0} \le C$ for all weighted dyadic p_0-atoms a. We take an atomic decomposition of f:

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k,$$

where each a^k is a weighted dyadic p_0-atom and

$$\left(\sum_{k\in\mathbb{Z}} |\mu_k|^{p_0}\right)^{1/p_0} \le C_{p_0}\|f\|_{H_{p_0}(w)}.$$

Since V is a σ-sublinear operator,

$$|Vf| \le \sum_{k=0}^{\infty} |\mu_k| \left|Va^k\right|.$$

Then the inequality

$$\|Vf\|_{L_{p_0}(w)}^{p_0} \le \sum_{k\in\mathbb{Z}} |\mu_k|^{p_0} \left\|Va^k\right\|_{L_{p_0}(w)}^{p_0} \le C_{p_0} \|f\|_{H_{p_0}(w)}^{p_0}$$

finishes the proof. ■

Note that (3.3) can be obtained from (3.2) by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [2] and Bennett and Sharpley [1] or Weisz [27, 29]. The interpolation of martingale Hardy spaces was worked out in [27]. Theorem 5 can be regarded also as an alternative tool to the Calderon–Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderon–Zygmund decomposition lemma.

4. Partial sums of Walsh–Fourier series

Denote by $s_n f$ the *nth partial sum of the Walsh–Fourier series* of a martingale f, namely,

$$s_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k \qquad (n \in \mathbb{N}).$$

If $f \in L_1$, then

$$s_n f(x) = \int_0^1 f(t) D_n(x \dot{+} t)\, dt \qquad (n \in \mathbb{N}),$$

where $\dot{+}$ denotes the dyadic addition and

$$D_n(u) := \sum_{k=0}^{n-1} w_k(u)$$

is the nth *Walsh-Dirichlet kernel* (see, for example, Schipp, Wade, Simon and Pál [21]).

It is a basic question as to whether the function f can be reconstructed from the partial sums of its Fourier series. It is easy to see that, for any martingale $f = (f_n)$,

$$s_{2^n} f = f_n \qquad (n \in \mathbb{N}).$$

Hence, by the martingale convergence theorem, we know that, for $1 \le p < \infty$ and $f \in L_p$,

$$\lim_{n\to\infty} s_{2^n} f = f \qquad \text{in the } L_p\text{-norm.}$$

This result was generalized by Schipp et al. [21, Theorem 4.1]. More precisely, they proved that, for any $1 < p < \infty$ and $f \in L_p$,

$$\lim_{n\to\infty} s_n f = f \qquad \text{in the } L_p\text{-norm.}$$

Using the boundedness of the martingale transforms, we generalize this result to $L_p(w)$.

Theorem 6. *If $w \in A_p$ for some $1 < p < \infty$, then*

$$\sup_{n\in\mathbb{N}} \|s_n f\|_{L_p(w)} \le C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

Corollary 3. *If $w \in A_\infty$ and $q(w) < p < \infty$ then,*

$$\sup_{n\in\mathbb{N}} \|s_n f\|_{L_p(w)} \le C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

The following corollary, which is due to Paley [18] in the unweighted case, follows by a usual density argument.

Corollary 4. *If $w \in A_\infty$, $q(w) < p < \infty$ and $f \in L_p(w)$, then*

$$\lim_{n\to\infty} s_n f = f \qquad \textit{in the } L_p(w)\textit{-norm.}$$

The inequality of Theorem 6 and the convergence in Corollary 4 do not hold for $p = 1$ or $p = \infty$.

5. Fejér means of Walsh–Fourier series

Though Theorem 6 and Corollary 4 are not true for $p = 1$ and $p = \infty$, with the help of some summability methods they can be generalized for these endpoint cases. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature. We refer at this time only to the books Stein and Weiss [25], Butzer and Nessel [3], Trigub and Belinsky [26], Grafakos [11] and Weisz [29, 30, 31] and the references therein.

The best known summability method is the Fejér method. In 1904 Fejér [6] investigated the arithmetic means of the partial sums of the trigonometric Fourier series, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $(f(x-0)+f(x+0))/2$. One year later Lebesgue [14] extended this theorem and obtained that every integrable function is Fejér summable at each Lebesgue point, thus almost everywhere.

In this paper we consider the *Fejér means* defined by

$$\sigma_n f(x) := \frac{1}{n}\sum_{k=0}^{n-1} s_k f(x) = \sum_{j=0}^{n}\left(1-\frac{j}{n}\right)\widehat{f}(j)w_j(x) = \int_0^1 f(x-u)K_n(u)\,du,$$

where the *Fejér kernels* are given by

$$K_n(u) := \sum_{j=0}^{n}\left(1-\frac{j}{n}\right)w_j(u) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(u).$$

The next results extend Theorem 6 and Corollaries 3 and 4 to the Fejér means.

Theorem 7. *If $w \in A_p$ for some $1 < p < \infty$, then*

$$\sup_{n\in\mathbb{N}} \|\sigma_n f\|_{L_p(w)} \le C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

Corollary 5. *If $w \in A_\infty$ and $q(w) < p < \infty$ then,*

$$\sup_{n\in\mathbb{N}} \|\sigma_n f\|_{L_p(w)} \le C_p \|f\|_{L_p(w)} \qquad (f \in L_p(w)).$$

Corollary 6. *If $w \in A_\infty$, $q(w) < p < \infty$ and $f \in L_p(w)$, then*

$$\lim_{n\to\infty} \sigma_n f = f \qquad \text{in the } L_p(w)\text{-norm.}$$

The *maximal operator* of the Fejér means are defined by

$$\sigma_* f := \sup_{n\in\mathbb{N}} |\sigma_n f| .$$

Applying Theorem 5, we can generalize the previous results to the $L_p(w)$ spaces and to the maximal operator (see e.g. [28, 29, 32]. The first inequality of Theorem 8 was proved by Fujii [8] for $w = p = 1$ (see also Schipp and Simon [20]).

Theorem 8. *If $w \in A_{2p}$ for some $1/2 < p < \infty$, then*

$$\|\sigma_* f\|_{L_p(w)} \le C_p \|f\|_{H_p(w)} \qquad (f \in H_p(w)).$$

Corollary 7. *If $w \in A_\infty$ and $q(w)/2 < p < \infty$ then,*

$$\|\sigma_* f\|_{L_p(w)} \le C_p \|f\|_{H_p(w)} \qquad (f \in H_p(w)).$$

Corollary 7 implies the following convergence results (see Xie et al. [32]).

Corollary 8. *If $w \in A_\infty$, $q(w)/2 < p < \infty$ and $f \in H_p(w)$, then $\sigma_n f$ converges almost everywhere on $[0, 1)$ as well as in the $L_p(w)$-norm as $n \to \infty$.*

For any integrable function f, the limit of $\sigma_n f$ is exactly the function. For any $k \in \mathbb{N}$, let $I \in \mathcal{F}_k$ be an atom of $\mathcal{F}_k$. The *restriction of a martingale f to the atom I* is defined by

$$f\chi_I := (E_n f \chi_I, n \ge k).$$

Corollary 9. *Let $w \in A_\infty$ and $q(w)/2 < p < \infty$. If $f \in H_p(w)$ and there exists a dyadic interval I such that the restriction $f\chi_I \in L_1(I)$, then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \qquad \text{for almost every } x \in I \text{ as well as in the } L_p(I, w)\text{-norm.}$$

If $q(w) < p < \infty$ in Corollary 9, then, by Corollary 2, $f \in L_p(w)$ and the norm convergence gives back Corollary 6.

Note that $q(w) \ge 1$. If $w = 1$ and $p \le 1/2$, then σ_* is not bounded anymore (see Simon and Weisz [24], Simon [23] and Gát and Goginava [9, 10]).

Theorem 9. *The operator σ_* is not bounded from $H_p(w)$ to $L_p(w)$ if $w = 1$ and $p \leq 1/2$.*

We get the next weak type $(1,1)$ inequality from Theorems 8 and 5 as in the unweighted case (see Weisz [28, 29], for $w = p = 1$ Schipp [19] and Simon [22]).

Corollary 10. *If $w \in A_1$ and $f \in L_1(w)$, then*

$$\sup_{\rho>0} \rho\, w(\sigma_* f > \rho) \leq C\|f\|_{L_1(w)}.$$

This weak type $(1,1)$ inequality and the density argument of Marcinkiewicz and Zygmund [16] imply the well known theorem of Fejér [6] and Lebesgue [14]. For the unweighted case it was proved in Fine [7] and Schipp [19].

Corollary 11. *If $w \in A_1$ and $f \in L_1(w)$, then*

$$\lim_{n\to\infty} \sigma_n f = f \qquad a.e.$$

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