AN ESTIMATE OF THE REGULARITY INDEX OF FAT POINTS IN SOME CASES

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Abstract. We estimate the regularity index of a set of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in three cases: all points $P_1, \ldots, P_s$ are on two lines; $Z$ consists at most five fat points; $Z = m_1P_1 + \cdots + P_{n+3}P_{n+3}$ is non-degenerate in $\mathbb{P}^n$.

1. Introduction

Let $\mathbb{P}^n := \mathbb{P}^n_K$ be a $n$-dimensional projective space over an algebraically closed field $K$ and $R := K[X_0, \ldots, X_n]$ be the polynomial ring in $n+1$ variables $X_0, \ldots, X_n$ with coefficients in $K$. Let $P_1, \ldots, P_s \in \mathbb{P}^n$ be distinct points and denote by $\wp_i \subset R$ the homogeneous prime ideal defining by the points $P_i$, $i = 1, \ldots, s$. Let $m_1, \ldots, m_s$ be positive integers, it is well known that the ideal $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ consists all forms $f \in R$ vanishing at $P_i$ with the multiplicity $\geq m_i$, $i = 1, \ldots, s$; we denote by $Z$ the zero-scheme defined by $I$ and call

$$Z := m_1P_1 + \cdots + m_sP_s$$

a set of fat points in $\mathbb{P}^n$. In case $m_1 = \cdots = m_s = m$ the $Z$ is called a set of equimultiple fat points.

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The homogeneous coordinate ring of $Z$ is $A := R/I$. This is a graded ring, $A = \bigoplus_{t \geq 0} A_t$. For every $t \in \mathbb{N}$, the graded part $A_t$ is a finite dimensional $K$-vector space. Then the function

$$H_Z(t) := \dim_K A_t$$

is called the Hilbert function of $Z$. This function allows us to estimate the size of all forms of degree $t$ vanishing at every point $P_i$ with multiplicity $\geq m_i$. In fact, our knowledge about $H_Z(t)$ is now very thin.

It is also well known that the number $e(A) = \sum_{i=1}^{s} (m_i + n - 1)$ is the multiplicity of the ring $A$ and the Hilbert function $H_Z(t)$ strictly increases until it reaches the multiplicity $e(A)$, at which it stabilizes. The regularity index of $Z$ is defined to be

$$\text{reg} (Z) := \min \{ t \in \mathbb{N} \mid H_A(t) = e(A) \}.$$ 

So the vector space dimension of the degree $t$ polynomials in $I$ is known if $t \geq \text{reg} (Z)$. In geometric language, the set of fat points $Z$ imposes independent conditions on forms of degree at least to be $\text{reg} (Z)$. In fact, the calculation $\text{reg} (Z)$ is very difficult. So, instead of finding $\text{reg} (Z)$, one gave upper bounds for the $\text{reg} (Z)$. We can find different upper bounds for $\text{reg} (Z)$ in [1], [2], [4], [6], [7].

For a set of fat points $Z = m_1 P_1 + \cdots + m_s P_s$ in $\mathbb{P}^n$, we put

$$T_{jZ} = \max \left\{ \left\lfloor \frac{\sum_{l=1}^{q} m_i + j - 2}{j} \right\rfloor \mid P_{i_1}, \ldots, P_{i_q} \text{ lie on a linear } j\text{-space} \right\}$$

and

$$T_Z = \max \{ T_{jZ} \mid j = 1, \ldots, n \}.$$ 

A set of points $X = \{P_1, \ldots, P_s\}$ in $\mathbb{P}^n$ is called a non-degenerate set if $X$ does not lie on a hyperplane of $\mathbb{P}^n$. A set of fat points $Z = m_1 P_1 + \cdots + m_s P_s$ is called to be non-degenerate if $X = \{P_1, \ldots, P_s\}$ is non-degenerate. In 2016, E. Ballico, O. Dumitrescu and E. Postinghel [1, Theorem 2.1] proved

$$\text{reg} (Z) \leq T_Z$$

for $Z = m_1 P_1 + \cdots + m_{n+3} P_{n+3}$ is a set of non-degenerate fat points in $\mathbb{P}^n$. Recently, U. Nagel and B. Trok [5, Theorem 5.3] proved the above upper bound to be true for any set of fat points in $\mathbb{P}^n$.

Recall that the calculation of $\text{reg} (Z)$ is very difficult. There were a few results on the calculation of $\text{reg} (Z)$ which were published by prestigious journals as follows.
In 1984, E.D. Davis and A.V. Geramita [3, Corollary 2.3] successfully calculated the regularity of fat points $Z = m_1 P_1 + \cdots + m_s P_s$ in the case all points lie on a line in $\mathbb{P}^n$:

$$\text{reg}(Z) = m_1 + \cdots + m_s - 1.$$ 

A set of points $\{P_1, \ldots, P_s\}$ in $\mathbb{P}^n$ is said in general position if no $j + 2$ points of $\{P_1, \ldots, P_s\}$ lie on a $j$-plane for every $j < n$. A set of fat points $Z = m_1 P_1 + \cdots + m_s P_s$ is called in general position in $\mathbb{P}^n$ if the points $P_1, \ldots, P_s$ are in general position. A rational normal curve in $\mathbb{P}^n$ is a curve of degree $n$ that may be given parametrically as the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^n$$

$$(s, t) \mapsto (s^n, s^{n-1} t, s^{n-2} t^2, \ldots, t^n).$$

In 1993, M.V. Catalisano, N.V. Trung and G. Valla [2] showed a formula to compute the regularity index of fat points $Z = m_1 P_1 + \cdots + m_s P_s$ in $\mathbb{P}^n$, with $m_1 \geq \cdots \geq m_s$, in two following cases:

- If $s \geq 2$ and the points $P_1, \ldots, P_s$ lie on a rational normal curve [2, Proposition 7], then

$$\text{reg}(Z) = \max \left\{ m_1 + m_2 - 1, \left[ \frac{\sum_{i=1}^{s} m_i + n - 2}{n} \right] \right\}.$$

- If $n \geq 3$, $2 \leq s \leq n + 2$, $2 \leq m_1$ and $P_1, \ldots, P_s$ are in general position in $\mathbb{P}^n$ [2, Corollary 8], then

$$\text{reg}(Z) = m_1 + m_2 - 1.$$ 

It is well known that if $P_1, \ldots, P_s$ lie on a rational normal curve in $\mathbb{P}^n$, then they are in general position in $\mathbb{P}^n$. In above cases we have $T_{1Z} = m_1 + m_2 - 1 \geq T_{jZ}$ for $j = 2, \ldots, n - 1$, and thus $T_Z = \max\{T_{1Z}, T_{nZ}\}$.

In 2012, P.V. Thien [8, Theorem 3.4] showed

$$\text{reg}(Z) = T_Z$$

in the case the points $P_1, \ldots, P_s$ are not on a linear $(s - 3)$-space in $\mathbb{P}^n$. In 2017, P.V. Thien and T.N. Sinh [10, Theorem 4.6] showed

$$\text{reg}(Z) = T_Z$$

in the case the points $P_1, \ldots, P_s$ are not on a linear $(r - 1)$-space in $\mathbb{P}^n$, $s \leq r + 3$, and $m_1 = \cdots = m_s = m \neq 2$. The conjecture $\text{reg}(Z) = T_Z$ for a set of arbitrary
fat points $Z$ in $\mathbb{P}^n$ is false because U. Nagel and B. Trok [5, Example 5.7] showed: if $Z = mP_1 + \cdots + mP_s$ is a set of fat points in $\mathbb{P}^n$, where $X = \{P_1, \ldots, P_s\}$ consisting of five arbitrary points and $\binom{d+n}{d}$ generic points for some $d \geq 5$, then
\[ \text{reg} (Z) < T_Z \]
for sufficiently large $d$ (or $n$).

In this paper we prove that
\[ T_Z - 1 \leq \text{reg} (Z) \leq T_Z \]
in the following cases:

- All $P_1, \ldots, P_s$ are on two lines.
- The scheme $Z$ consists at most five fat points.
- $Z = m_1P_1 + \cdots + m_sP_s$ is a set of non-degenerate fat points in $\mathbb{P}^n$.

2. Preliminaries

It is well known that if $\mathbb{R}/I$ is the coordinate ring of a set of fat points $Z$, then the regularity index $\text{reg} (Z)$ is equal to the Castelnuovo–Mumford regularity index $\text{reg} (\mathbb{R}/I)$.

We need use the following results for the next section.

Lemma 2.1. ([9, Proposition 6]) Let $P_1, \ldots, P_s$ be distinct points in $\mathbb{P}^n$ and $m_1, \ldots, m_s$ be positive integers. Let $n_1, \ldots, n_s$ be non-negative integers with $(n_1, \ldots, n_s) \neq (0, \ldots, 0)$ and $m_i \geq n_i$ for $i = 1, \ldots, s$. Put $I = \mathbb{P}^{m_1} \cap \cdots \cap \mathbb{P}^{m_s}$ and $J = \mathbb{P}^{n_1} \cap \cdots \cap \mathbb{P}^{n_s}$ ($\mathbb{P}^{n_i} = \mathbb{R}$ if $n_i = 0$). Then
\[ \text{reg} (\mathbb{R}/J) \leq \text{reg} (\mathbb{R}/I). \]

So, if $Y = n_1P_1 + \cdots + n_sP_s$ and $Z = m_1P_1 + \cdots + m_sP_s$, then ([5, Lemma 3.1(b)])
\[ \text{reg} (Y) \leq \text{reg} (Z). \]

In 2000, P.V. Thien proved the following result.
Lemma 2.2. ([7, Theorem 1]) Let \( Z = m_1 P_1 + \cdots + m_s P_s \) be an arbitrary set of fat points in \( \mathbb{P}^3 \). Then

\[
\text{reg}(Z) \leq \max\{T_{1Z}, T_{3Z}, T_{3Z}\}.
\]

Consider a set of fat points \( Z \) in \( \mathbb{P}^n \). In 2012, B. Benedetti, G. Fatabbi and A. Lorenzini showed the following property when the support of \( Z \) is contained in a linear subspace of \( \mathbb{P}^n \).

Lemma 2.3. ([1, Theorem 2.1]) Let \( Z = m_1 P_1 + \cdots + m_s P_s \) be a set of fat points in \( \mathbb{P}^n \) such that \( \{P_1, \ldots, P_s\} \) is contained in a linear \( r \)-space \( \alpha \). We may consider the linear \( r \)-space \( \alpha \) as a \( r \)-dimensional projective space \( \mathbb{P}^r \) containing the points \( P'_1 := P_1, \ldots, P'_s := P_s \), and \( Z_\alpha = m_1 P'_1 + \cdots + m_s P'_s \) as a set of fat points in \( \mathbb{P}^r \). If there is a non-negative integer \( t \) such that \( \text{reg}(Z_\alpha) \leq t \) in \( \mathbb{P}^r \), then

\[
\text{reg}(Z) \leq t
\]
in \( \mathbb{P}^n \).

Recall that a set of fat points \( Z = m_1 P_1 + \cdots + m_s P_s \) in \( \mathbb{P}^n \) is called non-degenerate if all the points \( P_1, \ldots, P_s \) are not on a linear \( (n-1) \)-space of \( \mathbb{P}^n \). In 2016, E. Ballico, O. Dumitrescu and E. Postinghel [1, Theorem 2.1] proved the following result.

Lemma 2.4. ([1, Theorem 2.1]) Let \( Z = m_1 P_1 + \cdots + m_{n+3} P_{n+3} \) be a set of non-degenerate fat points in \( \mathbb{P}^n \). Then

\[
\text{reg}(Z) \leq T_Z.
\]

The following result of E.D. Davis and A.V. Geramita help us to compute the regularity index of fat points with support on a line.

Lemma 2.5. ([3, Corollary 2.3]) Let \( Z = m_1 P_1 + \cdots + m_s P_s \) be a set of arbitrary fat points in \( \mathbb{P}^n \). Then

\[
\text{reg}(Z) = m_1 + \cdots + m_s - 1
\]
if and only if the points \( P_1, \ldots, P_s \) lie on a line.

The points \( P_1, \ldots, P_s \in \mathbb{P}^n \) is called to be in \( \text{Rnc}_j \) (see [9]) if there is a rational normal curve \( C \) in \( \mathbb{P}^j \) and a monomorphism \( \varphi : \mathbb{P}^j \to \mathbb{P}^n \) such that \( P_1, \ldots, P_s \) are on the image \( \varphi(C) \). In 2016, P.V. Thien proved:

Lemma 2.6. ([9, Proposition 10]) Let \( Z = m_1 P_1 + \cdots + m_s P_s \) be a set of fat points in \( \mathbb{P}^n \) such that \( P_1, \ldots, P_s \) are in \( \text{Rnc}_j \). Then

\[
\text{reg}(Z) = \max\{D_j \mid j = 1, \ldots, t\},
\]
where

\[
D_j = \max\left\{\left\lfloor \frac{\sum_{i=1}^q m_i + j - 2}{j} \right\rfloor \mid P_{i_1}, \ldots, P_{i_q} \text{ are in } \text{Rnc}_j \right\}.
\]
3. Results

Let \( X = \{P_1, \ldots, P_s\} \) be a set of distinct points in \( \mathbb{P}^n \), \( Z = m_1P_1 + \cdots + m_sP_s \) be a set of fat points in \( \mathbb{P}^n \) and \( L \) be a linear space in \( \mathbb{P}^n \). Assume that \( L \cap X = \{P_{i_1}, \ldots, P_{i_r}\} \), we put

\[
s(L \cap Z) := m_{i_1}P_{i_1} + \cdots + m_{i_r}P_{i_r},
\]

and

\[
w(s(L \cap Z)) := m_{i_1} + \cdots + m_{i_r}.
\]

From the Lemma 2.1 we get:

**Remark 3.1.** If \( Z = m_1P_1 + \cdots + m_sP_s \) is a set of fat points in \( \mathbb{P}^n \) and \( L \) is a linear space in \( \mathbb{P}^n \), then

\[
\text{reg} \ (s(L \cap Z)) \leq \text{reg} \ (Z).
\]

By using the above results we get:

**Lemma 3.2.** If \( Z = m_1P_1 + \cdots + m_sP_s \) is a set of fat points in \( \mathbb{P}^n \), then

\[
\text{reg} \ (Z) \geq T_1Z.
\]

**Proof.** By the definition of \( T_1Z \), there is a linear 1-space \( l \) in \( \mathbb{P}^n \) such that

\[
T_1Z = w_s(l \cap Z) - 1.
\]

By Remark 3.1 and Lemma 2.5, we have

\[
\text{reg} \ (Z) \geq \text{reg} \ (s(l \cap Z)) = w_s(l \cap Z) - 1.
\]

Therefore

\[
\text{reg} \ (Z) \geq T_1Z.
\]

**Lemma 3.3.** If \( Z = m_1P_1 + \cdots + m_sP_s \) is a set of fat points in \( \mathbb{P}^n \) such that \( P_1, \ldots, P_s \) are on a linear 3-space, then

\[
\text{reg} \ (Z) \leq \max\{T_1Z, T_2Z, T_3Z\} = T_Z.
\]

**Proof.** Assume that \( P_1, \ldots, P_s \) are on a linear 3-space, say \( \alpha \). Put \( P'_1 := P_1, \ldots, P'_s := P_s \) and consider \( Z_\alpha := m_1P'_1 + \cdots + m_sP'_s \) as a set of fat points in \( \mathbb{P}^3 \cong \alpha \). By the Lemma 2.2 we get

\[
\text{reg} \ (Z_\alpha) \leq \max\{T_{1Z_\alpha}, T_{2Z_\alpha}, T_{3Z_\alpha}\}.
\]
By using Lemma 2.3 we get
\[ \text{reg} (Z) \leq \max \{ T_{1Z}, T_{2Z}, T_{3Z} \}. \]

It is easy to see that
\[ T_j Z = T_j Z_\alpha \]
for \( j = 1, 2, 3 \). So
\[ \text{reg} (Z) \leq \max \{ T_{1Z}, T_{2Z}, T_{3Z} \}. \]

Since all \( P_1, \ldots, P_s \) are on a linear 3-space, we get \( T_{3Z} \geq T_j Z \) for \( j = 4, \ldots, n \).

We thus get
\[ \max \{ T_{1Z}, T_{2Z}, T_{3Z} \} = T_{\alpha}. \]

We now can estimate the regularity index of a set of fat points with support on two lines.

**Theorem 3.4.** Let \( Z = m_1 P_1 + \cdots + m_s P_s \) be a set of fat points in \( \mathbb{P}^n \) such that all \( P_1, \ldots, P_s \) are on two lines of \( \mathbb{P}^n \). Then
\[ T_{Z} - 1 \leq \text{reg} (Z) \leq T_{Z}. \]

**Proof.** Assume that the points \( P_1, \ldots, P_s \) are on two lines, say \( l_1 \) and \( l_2 \), in \( \mathbb{P}^n \). Then \( l_1 \cup l_2 \) is on a linear 3-space in \( \mathbb{P}^n \). We consider two following cases:

**Case 1:** \( l_1 \cup l_2 \) does not lie on any linear 2-space in \( \mathbb{P}^n \). We consider two following cases.

**Case 1.a:** \( w_{s(l_1 \cap Z)} \neq w_{s(l_2 \cap Z)} \). Without loss of generality we can assume that \( w_{s(l_1 \cap Z)} > m_s(l_1 \cap Z) \), then
\[ w_{s(l_1 \cap Z)} - 1 \geq \left\lfloor \frac{w_{s(l_1 \cap Z)} - w_{s(l_2 \cap Z)}}{2} \right\rfloor \geq \left\lfloor \frac{m_1 + \cdots + m_s}{2} \right\rfloor \geq \max \{ T_{2Z}, T_{3Z} \}. \]

By the definition of \( T_{1Z} \), we have \( T_{1Z} \geq w_{s(l_1 \cap Z)} - 1 \). It follows that
\[ T_{1Z} = \max \{ T_{1Z}, T_{2Z}, T_{3Z} \}. \]

Moreover, since \( P_1, \ldots, P_s \) are on a linear 3-space, from Lemma 3.2 and Lemma 3.3 we get in **Case 1.a:**
\[ \text{reg} (Z) = T_{1Z} = T_{Z}. \]

**Case 1.b:** \( w_{s(l_1 \cap Z)} = w_{s(l_2 \cap Z)} \). Then
\[ w_{s(l_1 \cap Z)} - 1 = \left\lfloor \frac{w_{s(l_1 \cap Z)} + w_{s(l_2 \cap Z)} - 1}{2} \right\rfloor. \]
Since $l_1 \cup l_2$ does not lie on a linear 2-space and lie on a linear 3-space, we have

$$\left\lfloor \frac{w_s(l_1 \cap Z) + w_s(l_2 \cap Z) - 1}{2} \right\rfloor \geq T_{2Z},$$

and

$$\left\lfloor \frac{w_s(l_1 \cap Z) + w_s(l_2 \cap Z) - 1}{2} \right\rfloor \geq \left\lfloor \frac{w_s(l_1 \cap Z) + w_s(l_2 \cap Z) + 1}{3} \right\rfloor = T_{3Z}.$$ 

Therefore,

$$w_s(l_1 \cap Z) - 1 \geq \max\{T_{2Z}, T_{3Z}\}.$$ 

But $T_{1Z} \geq w_s(l_1 \cap Z) - 1$, it follows that

$$T_{1Z} = \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.$$ 

Moreover, from Lemma 3.2 and Lemma 3.3 we get in Case 1.b:

$$\text{reg}(Z) = T_{1Z} = T_Z.$$ 

Case 2: $l_1 \cup l_2$ lie on a linear 2-space, say $\beta \subset \mathbb{P}^n$. Then $T_{2Z} \geq T_{3Z}$, so $T_{2Z} = \max\{T_{2Z}, T_{3Z}\}$. We consider two following cases:

Case 2.a: $w_s(l_1 \cap Z) \neq w_s(l_2 \cap Z)$. Without loss of generality we can assume that $w_s(l_1 \cap Z) > m_s(l_2 \cap Z)$, then

$$w_s(l_1 \cap Z) - 1 \geq \left\lfloor \frac{w_s(l_1 \cap Z) + w_s(l_2 \cap Z)}{2} \right\rfloor \geq \left\lfloor \frac{m_1 + \cdots + m_s}{2} \right\rfloor = T_{2Z} = \max\{T_{2Z}, T_{3Z}\}.$$ 

But $T_{1Z} \geq w_s(l_1 \cap Z) - 1$. Hence

$$T_{1Z} = \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.$$ 

Moreover, from Lemma 3.2 and Lemma 3.3 we get in Case 2.a:

$$\text{reg}(Z) = T_{1Z} = T_Z.$$ 

Case 2.b: $w_s(l_1 \cap Z) = w_s(l_2 \cap Z)$. Then

$$w_s(l_1 \cap Z) = \left\lfloor \frac{w_s(l_1 \cap Z) + w_s(l_2 \cap Z)}{2} \right\rfloor \geq T_{2Z}.$$ 

By defining of $T_{1Z}$, we have $w_s(l_1 \cap Z) - 1 \leq T_{1Z}$. 


If either \( w_s(l_i \cap Z) - 1 < T_{1Z} \) or \( w_s(l_i \cap Z) = T_{1Z} \) and \( l_1 \cap l_2 \cap \{P_1, \ldots, P_s\} \neq \emptyset \), then \( T_{1Z} \geq T_{2Z} = \max\{T_{2Z}, T_{3Z}\} \). So

\[
T_{1Z} = \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.
\]

Moreover, from Lemma 3.2 and Lemma 3.3 we get

\[
\text{reg}(Z) = T_{1Z} = T_{2Z}.
\]

If \( w_s(l_i \cap Z) = T_{1Z} \) and \( l_1 \cap l_2 \cap \{P_1, \ldots, P_s\} = \emptyset \), then

\[
T_{2Z} = T_{1Z} + 1 = \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.
\]

Moreover, from Lemma 3.2 and Lemma 3.3 we get

\[
T_{Z} - 1 = T_{1Z} \leq \text{reg}(Z) \leq T_{2Z} = T_{Z}.
\]

Hence in Case 2.b we get

\[
T_{Z} - 1 \leq \text{reg}(Z) \leq T_{Z}.
\]

The proof of Theorem 3.4 is completed. ■

Next we also can estimate the regularity index of a set consisting at most five fat points.

**Proposition 3.5.** Let \( Z = m_1 P_1 + \cdots + m_s P_s \) be a set of fat points in \( \mathbb{P}^n \), \( s \leq 5 \). Then

\[
T_{Z} - 1 \leq \text{reg}(Z) \leq T_{Z}.
\]

**Proof.** If \( P_1, \ldots, P_s \) lie on two lines, then by the above theorem we get

\[
T_{Z} - 1 \leq \text{reg}(Z) \leq T_{Z}.
\]

If \( P_1, \ldots, P_s \) do not lie on two lines, then \( s = 5 \) and there are two following cases for \( P_1, \ldots, P_5 \):

**Case 1:** All \( P_1, \ldots, P_5 \) lie on a linear 2-space in \( \mathbb{P}^n \). Then \( P_1, \ldots, P_5 \) are in \( Rn=2 \) because \( P_1, \ldots, P_5 \) are not on two lines. By Lemma 2.6 we have

\[
\text{reg}(Z) = \max\{D_1, D_2\}.
\]

Since \( D_1 = T_{1Z} \) and \( D_2 = T_{2Z} \geq T_{jZ} \) for \( j = 3, \ldots, n \), we get

\[
\text{reg}(Z) = T_{Z}.
\]
Case 2: $P_1, \ldots, P_5$ do not lie on a linear 2-space in $\mathbb{P}^n$. Then by [8, Theorem 3.4] we get

$$\text{reg}(Z) = T_Z.$$ \hfill ■

For $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ is a set of non-degenerate fat points in $\mathbb{P}^n$, E. Ballico, O. Dumitrescu and E. Postinghel [1] proved $\text{reg}(Z) \leq T_Z$. We now prove that $\text{reg}(Z)$ is bounded lowerly by $T_Z - 1$.

**Theorem 3.6.** Let $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ be a set of non-degenerate fat points in $\mathbb{P}^n$. Then

$$T_Z - 1 \leq \text{reg}(Z) \leq T_Z.$$

**Proof.** Without loss of generality, we can assume that $m_1 \geq m_2 \geq \cdots \geq m_{n+3}$. By Lemma 2.4 we have

$$\text{reg}(Z) \leq T_Z$$

with

$$T_Z = \max\{T_{jZ} \mid j = 1, \ldots, n\}$$

and

$$T_{jZ} = \max\left\{\left\lfloor \frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \ldots, P_{i_q} \text{ lie on a linear } j\text{-space}\right\}.$$  

So, in the remainder we only need prove that $\text{reg}(Z) \geq T_Z - 1$.

Since $P_1, \ldots, P_{n+3}$ are in non-degenerate in $\mathbb{P}^n$, there are at most $j + 3$ points of them are on a linear $j$-space for $j = 1, \ldots, n - 1$. This implies

$$m_1 + m_2 \geq T_{jZ}$$

for $j = 3, \ldots, n$. So

$$T_Z = \max\{T_{1Z}, T_{2Z}\}.$$  

We consider two following cases:

**Case 1:** $T_{2Z} \leq T_{1Z}$. Then $T_Z = T_{1Z}$, by Lemma 3.2 we get

$$\text{reg}(Z) \geq T_{1Z} = T_Z.$$

**Case 2:** $T_{2Z} > T_{1Z}$. Since $P_1$ and $P_2$ are on a line, we have $T_{1Z} \geq m_1 + m_2 - 1$ by defining of $T_{1Z}$. So, $T_{2Z} \geq m_1 + m_2$. On the other hand, by defining of $T_{2Z}$ there is a linear 2-space, say $\alpha$, such that

$$T_{2Z} = \left\lfloor \frac{w_{\alpha}(\alpha \cap Z)}{2} \right\rfloor.$$
Suppose that $\alpha \cap Z = m_1P_{i_1} + \cdots + m_qP_{i_q}$, then

$$\left\lfloor \sum_{i=1}^{q} \frac{m_i}{2} \right\rfloor = \left\lfloor \frac{w_s(\alpha \cap Z)}{2} \right\rfloor \geq m_1 + m_2.$$  

Since $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_{n+3}$, we have $q \geq 4$. We consider two following cases for $q$.

**Case $q = 4$:** Then $m_1 = m_2 = m_3 = m_4 = m$ and $T_{2Z} = 2m = T_Z = T_{1Z} + 1$. By Lemma 3.2 we get

$$\text{reg} (Z) \geq T_{1Z} = T_Z - 1.$$  

**Case $q \geq 5$:** Since $P_1, \ldots, P_{n+3}$ are in non-degenerate in $\mathbb{P}^n$, there are at most five points on the linear 2-space. Thus $q = 5$ because $\alpha$ is a linear 2-space. By using Proposition 3.5 we get

$$\text{reg} (Z) \geq T_Z - 1.$$  

References


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