Annales Univ. Sci. Budapest., Sect. Comp. 49 (2019) 399–410

AN ESTIMATE OF THE REGULARITY INDEX OF FAT POINTS IN SOME CASES

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Communicated by Bui Minh Phong

(Received January 28, 2019; accepted March 3, 2019)

Abstract. We estimate the regularity index of a set of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in three cases: all points $P_1, \ldots, P_s$ are on two lines; Z consists at most five fat points; $Z = m_1P_1 + \cdots + P_{n+3}P_{n+3}$ is non-degenerate in $\mathbb{P}^n$.

1. Introduction

Let $\mathbb{P}^n := \mathbb{P}^n_K$ be a n-dimensional projective space over an algebraically closed field K and $R := K[X_0, \ldots, X_n]$ be the polynomial ring in $n+1$ variables $X_0, \ldots, X_n$ with coefficients in K. Let $P_1, \ldots, P_s \in \mathbb{P}^n$ be distinct points and denote by $\wp_i \subset R$ the homogeneous prime ideal defining by the points P_i, $i = 1, \ldots, s$. Let $m_1, \ldots, m_s$ be positive integers, it is well known that the ideal $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ consists all forms $f \in R$ vanishing at P_i with the multiplicity $\geq m_i$, $i = 1, \ldots, s$; we denote by Z the zero-scheme defined by I and call

$$Z := m_1P_1 + \cdots + m_sP_s$$

a set of fat points in $\mathbb{P}^n$. In case $m_1 = \cdots = m_s = m$ the Z is called a set of equimultiple fat points.

Key words and phrases: Fat points, regularity index, Zero-scheme.
2010 Mathematics Subject Classification: Primary 14C20, Secondary 13D40.

The homogeneous coordinate ring of Z is $A := R/I$. This is a graded ring, $A = \underset{t\geq 0}{\oplus} A_t$. For every $t \in \mathbb{N}$, the graded part A_t is a finite dimensional $K-$vector space. Then the function

$$H_Z(t) := \dim_K A_t$$

is called the Hilbert function of Z. This function allows us to estimate the size of all forms of degree t vanishing at every point P_i with multiplicity $\geq m_i$. In fact, our knowledge about $H_Z(t)$ is now very thin.

It is also well known that the number $e(A) = \sum_{i=1}^{s} \binom{m_i + n - 1}{n}$ is the multiplicity of the ring A and the Hilbert function $H_Z(t)$ strictly increases until it reaches the multiplicity $e(A)$, at which it stabilizes. The regularity index of Z is defined to be

$$\operatorname{reg}(Z) := \min\{t \in \mathbb{N} \mid H_A(t) = e(A)\}.$$

So the vector space dimension of the degree t polynomials in I is known if $t \geq \operatorname{reg}(Z)$. In geometric language, the set of fat points Z imposes independent conditions on forms of degree at least to be $\operatorname{reg}(Z)$. In fact, the calculation $\operatorname{reg}(Z)$ is very difficult. So, instead of finding $\operatorname{reg}(Z)$, one gave upper bounds for the $\operatorname{reg}(Z)$. We can find different upper bounds for $\operatorname{reg}(Z)$ in [1], [2], [4], [6], [7].

For a set of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$, we put

$$T_{jZ} = \max\left\{ \left\lfloor \frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j\text{-space} \right\}$$

and

$$T_Z = \max\{T_{jZ} \mid j = 1, \dots, n\}.$$

A set of points $X = \{P_1, \dots, P_s\}$ in $\mathbb{P}^n$ is called a non-degenerate set if X does not lie on a hyperplane of $\mathbb{P}^n$. A set of fat points $Z = m_1P_1 + \cdots + m_sP_s$ is called to be non-degenerate if $X = \{P_1, \dots, P_s\}$ is non-degenerate. In 2016, E. Ballico, O. Dumitrescu and E. Postinghel [1, Theorem 2.1] proved

$$\operatorname{reg}(Z) \leq T_Z$$

for $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ is a set of non-degenerate fat points in $\mathbb{P}^n$. Recently, U. Nagel and B. Trok [5, Theorem 5.3] proved the above upper bound to be true for any set of fat points in $\mathbb{P}^n$.

Recall that the calculation of $\operatorname{reg}(Z)$ is very difficult. There were a few results on the calculation of $\operatorname{reg}(Z)$ which were published by prestigious journals as follows.

In 1984, E.D. Davis and A.V. Geramita [3, Corollary 2.3] successfully calculated the regularity of fat points $Z = m_1P_1+\cdots+m_sP_s$ in the case all points lie on a line in $\mathbb{P}^n$:

$$\operatorname{reg}(Z) = m_1 + \cdots + m_s - 1.$$

A set of points $\{P_1, \ldots, P_s\}$ in $\mathbb{P}^n$ is said in general position if no $j + 2$ points of $\{P_1, \ldots, P_s\}$ lie on a j-plane for every $j < n$. A set of fat points $Z = m_1P_1+\cdots+m_sP_s$ is called in general position in $\mathbb{P}^n$ if the points $P_1, \ldots, P_s$ are in general position. A rational normal curve in $\mathbb{P}^n$ is a curve of degree n that may be given parametrically as the image of the map

$$\begin{aligned} &\mathbb{P}^1 \to \mathbb{P}^n \\ &(s, t) \mapsto (s^n, s^{n-1}t, s^{n-2}t^2, \ldots, t^n). \end{aligned}$$

In 1993, M.V. Catalisano, N.V. Trung and G. Valla [2] showed a formular to compute the regularity index of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$, with $m_1 \geq \cdots \geq m_s$, in two following cases:

◦ If $s \geq 2$ and the points $P_1, \ldots, P_s$ lie on a rational normal curve [2, Proposition 7], then

$$\operatorname{reg}(Z) = \max\left\{m_1 + m_2 - 1, \left\lfloor \Big(\sum_{i=1}^{s} m_i + n - 2\Big)/n \right\rfloor\right\}.$$

◦ If $n \geq 3$, $2 \leq s \leq n + 2$, $2 \leq m_1$ and $P_1, \ldots, P_s$ are in general position in $\mathbb{P}^n$ [2, Corllary 8], then

$$\operatorname{reg}(Z) = m_1 + m_2 - 1.$$

It is well known that if $P_1, \ldots, P_s$ lie on a rational normal curve in $\mathbb{P}^n$, then they are in general position in $\mathbb{P}^n$. In above cases we have $T_{1Z} = m_1 + m_2 - 1 \geq T_{jZ}$ for $j = 2, \ldots, n - 1$, and thus $T_Z = \max\{T_{1Z}, T_{nZ}\}$.

In 2012, P.V. Thien [8, Theorem 3.4] showed

$$\operatorname{reg}(Z) = T_Z$$

in the case the points $P_1, \ldots, P_s$ are not on a linear $(s - 3)$-space in $\mathbb{P}^n$. In 2017, P.V. Thien and T.N. Sinh [10, Theorem 4.6] showed

$$\operatorname{reg}(Z) = T_Z$$

in the case the points $P_1, \ldots, P_s$ are not on a linear $(r-1)$-space in $\mathbb{P}^n$, $s \leq r+3$, and $m_1 = \cdots = m_s = m \neq 2$. The conjecture $\operatorname{reg}(Z) = T_Z$ for a set of arbitrary fat points Z in $\mathbb{P}^n$ is false because U. Nagel and B. Trok [5, Example 5.7] showed: if $Z = mP_1 + \cdots + mP_s$ is a set of fat points in $\mathbb{P}^n$, where $X = \{P_1, \ldots, P_s\}$ consisting of five arbitrary points and $\binom{d+n}{d}$ generic points for some $d \geq 5$, then

$$\operatorname{reg}(Z) < T_Z$$

for sufficiently large d (or n).

In this paper we prove that

$$T_Z - 1 \leq \operatorname{reg}(Z) \leq T_Z$$

in the following cases:

- All $P_1, \ldots, P_s$ are on two lines.
- The scheme Z consists at most five fat points.
- $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ is a set of non-degenerate fat points in $\mathbb{P}^n$.

2. Preliminaries

It is well known that if R/I is the coordinate ring of a set of fat points Z, then the regularity index $\operatorname{reg}(Z)$ is equal to the Castelnuovo–Mumford regularity index $\operatorname{reg}(R/I)$.

We need use the following results for the next section.

Lemma 2.1. ([9, Proposition 6]) *Let $P_1, \ldots, P_s$ be distinct points in $\mathbb{P}^n$ and $m_1, \ldots, m_s$ be positive integers. Let $n_1, \ldots, n_s$ be non-negative integers with $(n_1, \ldots, n_s) \neq (0, \ldots, 0)$ and $m_i \geq n_i$ for $i = 1, \ldots, s$. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ and $J = \wp_1^{n_1} \cap \cdots \cap \wp_s^{n_s}$ ($\wp_i^{n_i} = R$ if $n_i = 0$). Then*

$$\operatorname{reg}(R/J) \leq \operatorname{reg}(R/I).$$

So, if $Y = n_1P_1 + \cdots + n_sP_s$ and $Z = m_1P_1 + \cdots + m_sP_s$, then ([5, Lemma 3.1(b)])

$$\operatorname{reg}(Y) \leq \operatorname{reg}(Z).$$

In 2000, P.V. Thien proved the following result.

Lemma 2.2. ([7, Theorem 1]) *Let $Z = m_1P_1 + \cdots + m_sP_s$ be an arbitrary set of fat points in $\mathbb{P}^3$. Then*

$$\operatorname{reg}(Z) \le \max\{T_{1Z}, T_{3Z}, T_{3Z}\}.$$

Consider a set of fat points Z in $\mathbb{P}^n$. In 2012, B. Benedetti, G. Fatabbi and A. Lorenzini showed the following property when the support of Z is contained in a linear subspace of $\mathbb{P}^n$.

Lemma 2.3. ([1, Theorem 2.1]) *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$ such that $\{P_1, \ldots, P_s\}$ is contained in a linear r-space α. We may consider the linear r-space α as a r-dimensional projective space $\mathbb{P}^r$ containing the points $P'_1 := P_1, \ldots, P'_s := P_s$, and $Z_\alpha = m_1P'_1 + \cdots + m_sP'_s$ as a set of fat points in $\mathbb{P}^r$. If there is a non-negative integer t such that $\operatorname{reg}(Z_\alpha) \le t$ in $\mathbb{P}^r$, then*

$$\operatorname{reg}(Z) \le t$$

in $\mathbb{P}^n$.

Recall that a set of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$ is called non-degenerate if all the points $P_1, \ldots, P_s$ are not on a linear $(n-1)$-space of $\mathbb{P}^n$. In 2016, E. Ballico, O. Dumitrescu and E. Postinghel [1, Theorem 2.1] proved the following result.

Lemma 2.4. ([1, Theorem 2.1]) *Let $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ be a set of non-degenerate fat points in $\mathbb{P}^n$. Then*

$$\operatorname{reg}(Z) \le T_Z.$$

The following result of E.D. Davis and A.V. Geramita help us to compute the regularity index of fat points with support on a line.

Lemma 2.5. ([3, Corollary 2.3]) *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of arbitrary fat points in $\mathbb{P}^n$. Then*

$$\operatorname{reg}(Z) = m_1 + \cdots + m_s - 1$$

if and only if the points $P_1, \ldots, P_s$ lie on a line.

The points $P_1, \ldots, P_s \in \mathbb{P}^n$ is called to be in *Rnc*-j (see [9]) if there is a rational normal curve $\mathcal{C}$ in $\mathbb{P}^j$ and a monomorphism $\varphi : \mathbb{P}^j \to \mathbb{P}^n$ such that $P_1, \ldots, P_s$ are on the image $\varphi(\mathcal{C})$. In 2016, P.V. Thien proved:

Lemma 2.6. ([9, Proposition 10]) *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$ such that $P_1, \ldots, P_s$ are in Rnc-j. Then*

$$\operatorname{reg}(Z) = \max\{D_j \mid j = 1, \ldots, t\},$$

where

$$D_j = \max\left\{ \left\lfloor \frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \ldots, P_{i_q} \text{ are in } Rnc - j \right\}.$$

3. Results

Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$, $Z = m_1P_1 + \cdots + +m_sP_s$ be a set of fat points in $\mathbb{P}^n$ and L be a linear space in $\mathbb{P}^n$. Assume that $L \cap X = \{P_{i_1}, \dots, P_{i_r}\}$, we put

$$s(L \cap Z) := m_{i_1}P_{i_1} + \cdots + m_{i_r}P_{i_r},$$

and

$$w_{s(L\cap Z)} := m_{i_1} + \cdots + m_{i_r}.$$

From the Lemma 2.1 we get:

Remark 3.1. If $Z = m_1P_1 + \cdots + m_sP_s$ is a set of fat points in $\mathbb{P}^n$ and L is a linear space in $\mathbb{P}^n$, then

$$\operatorname{reg}(s(L \cap Z)) \leq \operatorname{reg}(Z).$$

By using the above results we get:

Lemma 3.2. *If $Z = m_1P_1 + \cdots + m_sP_s$ is a set of fat points in $\mathbb{P}^n$, then*

$$\operatorname{reg}(Z) \geq T_{1Z}.$$

Proof. By the definition of T_{1Z}, there is a linear 1-space l in $\mathbb{P}^n$ such that

$$T_{1Z} = w_{s(l\cap Z)} - 1.$$

By Remark 3.1 and Lemma 2.5, we have

$$\operatorname{reg}(Z) \geq \operatorname{reg}(s(l \cap Z)) = w_{s(l\cap Z)} - 1.$$

Therefore

$$\operatorname{reg}(Z) \geq T_{1Z}. \qquad \blacksquare$$

Lemma 3.3. *If $Z = m_1P_1 + \cdots + m_sP_s$ is a set of fat points in $\mathbb{P}^n$ such that $P_1, \dots, P_s$ are on a linear 3-space, then*

$$\operatorname{reg}(Z) \leq \max\{T_{1Z}, T_{2Z}, T_{3Z}\} = T_Z.$$

Proof. Assume that $P_1, \dots, P_s$ are on a linear 3-space, say α. Put $P_1' := := P_1, \dots, P_s' := P_s$ and consider $Z_\alpha := m_1P_1' + \cdots + m_sP_s'$ as a set of fat points in $\mathbb{P}^3 \cong \alpha$. By the Lemma 2.2 we get

$$\operatorname{reg}(Z_\alpha) \leq \max\{T_{1Z_\alpha}, T_{2Z_\alpha}, T_{3Z_\alpha}\}.$$

By using Lemma 2.3 we get

$$\operatorname{reg}(Z) \leq \max\{T_{1Z_\alpha}, T_{2Z_\alpha}, T_{3Z_\alpha}\}.$$

It is easy to see that

$$T_{jZ} = T_{jZ_\alpha}$$

for $j = 1, 2, 3$. So

$$\operatorname{reg}(Z) \leq \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.$$

Since all $P_1, \dots, P_s$ are on a linear 3-space, we get $T_{3Z} \geq T_{jZ}$ for $j = 4, \dots, n$. We thus get

$$\max\{T_{1Z}, T_{2Z}, T_{3Z}\} = T_Z. \qquad \blacksquare$$

We now can estimate the regularity index of a set of fat points with support on two lines.

Theorem 3.4. *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$ such that all $P_1, \dots, P_s$ are on two lines of $\mathbb{P}^n$. Then*

$$T_Z - 1 \leq \operatorname{reg}(Z) \leq T_Z.$$

Proof. Assume that the points $P_1, \dots, P_s$ are on two lines, say l_1 and l_2, in $\mathbb{P}^n$. Then $l_1 \cup l_2$ is on a linear 3-space in $\mathbb{P}^n$. We consider two following cases:

Case 1: $l_1 \cup l_2$ does not lie on any linear 2-space in $\mathbb{P}^n$. We consider two following cases.

Case 1.a: $w_{s(l_1 \cap Z} \neq w_{s(l_2 \cap Z)}$. Without loss of generality we can assume that $w_{s(l_1 \cap Z)} > m_{s(l_2 \cap Z)}$, then

$$w_{s(l_1\cap Z)} - 1 \geq \left\lfloor \frac{w_{s(l_1\cap Z)} + w_{s(l_2\cap Z)}}{2} \right\rfloor \geq \left\lfloor \frac{m_1 + \cdots + m_s}{2} \right\rfloor \geq \max\{T_{2Z}, T_{3Z}\}.$$

By the definition of T_{1Z}, we have $T_{1Z} \geq w_{s(l_1\cap Z)} - 1$. It follows that

$$T_{1Z} = \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.$$

Moreover, since $P_1, \dots, P_s$ are on a linear 3-space, from Lemma 3.2 and Lemma 3.3 we get in *Case 1.a:*

$$\operatorname{reg}(Z) = T_{1Z} = T_Z.$$

Case 1.b: $w_{s(l_1\cap Z)} = w_{s(l_2\cap Z)}$. Then

$$w_{s(l_1\cap Z)} - 1 = \left\lfloor \frac{w_{s(l_1\cap Z)} + w_{s(l_2\cap Z)} - 1}{2} \right\rfloor.$$

Since $l_1 \cup l_2$ does not lie on a linear 2-space and lie on a linear 3-space, we have

$$\left\lfloor \frac{w_{s(l_1\cap Z)} + w_{s(l_2\cap Z)} - 1}{2} \right\rfloor \geq T_{2Z},$$

and

$$\left\lfloor \frac{w_{s(l_1\cap Z)} + w_{s(l_2\cap Z)} - 1}{2} \right\rfloor \geq \left\lfloor \frac{w_{s(l_1\cap Z)} + w_{s(l_2\cap Z)} + 1}{3} \right\rfloor = T_{3Z}.$$

Thererore,

$$w_{s(l_1\cap Z)} - 1 \geq \max\{T_{2Z}, T_{3Z}\}.$$

But $T_{1Z} \geq w_{s(l_1\cap Z)} - 1$, it follows that

$$T_{1Z} = \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.$$

Moreover, from Lemma 3.2 and Lemma 3.3 we get in *Case 1.b:*

$$\operatorname{reg}(Z) = T_{1Z} = T_Z.$$

Case 2: $l_1 \cup l_2$ lie on a linear 2-space, say $\beta \subset \mathbb{P}^n$. Then $T_{2Z} \geq T_{3Z}$, so $T_{2Z} = \max\{T_{2Z}, T_{3Z}\}$. We consider two following cases:

Case 2.a: $w_{s(l_1\cap Z} \neq w_{s(l_2\cap Z)}$. Without loss of generality we can assume that $w_{s(l_1\cap Z)} > m_{s(l_2\cap Z)}$, then

$$w_{s(l_1\cap Z)} - 1 \geq \left\lfloor \frac{w_{s(l_1\cap Z)} + w_{s(l_2\cap Z)}}{2} \right\rfloor \geq \left\lfloor \frac{m_1 + \cdots + m_s}{2} \right\rfloor = T_{2Z} = \max\{T_{2Z}, T_{3Z}\}.$$

But $T_{1Z} \geq w_{s(l_1\cap Z)} - 1$. Hence

$$T_{1Z} = \max\{T_{1Z}, T_{2Z}, T_{3Z}\}.$$

Moreover, from Lemma 3.2 and Lemma 3.3 we get in *Case 2.a:*

$$\operatorname{reg}(Z) = T_{1Z} = T_Z.$$

Case 2.b: $w_{s(l_1\cap Z)} = w_{s(l_2\cap Z)}$. Then

$$w_{s(l_1\cap Z)} = \left\lfloor \frac{w_{s(l_1\cap Z)} + w_{s(l_2\cap Z)}}{2} \right\rfloor \geq T_{2Z}.$$

By defining of T_{1Z}, we have $w_{s(l_1\cap Z)} - 1 \leq T_{1Z}$.

If either $w_{s(l_1\cap Z)}-1 < T_{1Z}$ or $w_{s(l_1\cap Z)}-1 = T_{1Z}$ and $l_1\cap l_2\cap\{P_1,\dots,P_s\}\neq\emptyset$, then $T_{1Z}\geq T_{2Z}=\max\{T_{2Z},T_{3Z}\}$. So

$$T_{1Z}=\max\{T_{1Z},T_{2Z},T_{3Z}\}.$$

Moreover, from Lemma 3.2 and Lemma 3.3 we get

$$\operatorname{reg}(Z)=T_{1Z}=T_Z.$$

If $w_{s(l_1\cap Z)}-1=T_{1Z}$ and $l_1\cap l_2\cap\{P_1,\dots,P_s\}=\emptyset$, then

$$T_{2Z}=T_{1Z}+1=\max\{T_{1Z},T_{2Z},T_{3Z}\}.$$

Moreover, from Lemma 3.2 and Lemma 3.3 we get

$$T_Z-1=T_{1Z}\leq\operatorname{reg}(Z)\leq T_{2Z}=T_Z.$$

Hence in *Case 2.b* we get

$$T_Z-1\leq\operatorname{reg}(Z)\leq T_Z.$$

The proof of Theorem 3.4 is completed. ■

Next we also can estimate the regularity index of a set consisting at most five fat points.

Proposition 3.5. *Let $Z=m_1P_1+\cdots+m_sP_s$ be a set of fat points in $\mathbb{P}^n$, $s\leq 5$. Then*

$$T_Z-1\leq\operatorname{reg}(Z)\leq T_Z.$$

Proof. If $P_1,\dots,P_s$ lie on two lines, then by the above theorem we get

$$T_Z-1\leq\operatorname{reg}(Z)\leq T_Z.$$

If $P_1,\dots,P_s$ do not lie on two lines, then $s=5$ and there are two following cases for $P_1,\dots,P_5$:

Case 1: All $P_1,\dots,P_5$ lie on a linear 2-space in $\mathbb{P}^n$. Then $P_1,\dots,P_5$ are in *Rnc*-2 because $P_1,\dots,P_5$ are not on two lines. By Lemma 2.6 we have

$$\operatorname{reg}(Z)=\max\{D_1,D_2\}.$$

Since $D_1=T_{1Z}$ and $D_2=T_{2Z}\geq T_{jZ}$ for $j=3,\dots,n$, we get

$$\operatorname{reg}(Z)=T_Z.$$

Case 2: $P_1, \ldots, P_5$ do not lie on a linear 2-space in $\mathbb{P}^n$. Then by [8, Theorem 3.4] we get

$$\operatorname{reg}(Z) = T_Z. \qquad \blacksquare$$

For $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ is a set of non-degenerate fat points in $\mathbb{P}^n$, E. Ballico, O. Dumitrescu and E. Postinghel [1] proved $\operatorname{reg}(Z) \leq T_Z$. We now prove that $\operatorname{reg}(Z)$ is bounded lowerly by $T_Z - 1$.

Theorem 3.6. *Let $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ be a set of non-degenerate fat points in $\mathbb{P}^n$. Then*

$$T_Z - 1 \leq \operatorname{reg}(Z) \leq T_Z.$$

Proof. Without loss of generality, we can assume that $m_1 \geq m_2 \geq \cdots \geq m_{n+3}$. By Lemma 2.4 we have

$$\operatorname{reg}(Z) \leq T_Z$$

with

$$T_Z = \max\{T_{jZ} \mid j = 1, \ldots, n\}$$

and

$$T_{jZ} = \max\left\{\left\lfloor \frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \ldots, P_{i_q} \text{ lie on a linear } j\text{-space}\right\}.$$

So, in the remainder we only need prove that $\operatorname{reg}(Z) \geq T_Z - 1$.

Since $P_1, \ldots, P_{n+3}$ are in non-degenerate in $\mathbb{P}^n$, there are at most $j+3$ points of them are on a linear j-space for $j = 1, \ldots, n-1$. This implies

$$m_1 + m_2 \geq T_{jZ}$$

for $j = 3, \ldots, n$. So

$$T_Z = \max\{T_{1Z}, T_{2Z}\}.$$

We consider two following cases:

Case 1: $T_{2Z} \leq T_{1Z}$. Then $T_Z = T_{1Z}$, by Lemma 3.2 we get

$$\operatorname{reg}(Z) \geq T_{1Z} = T_Z.$$

Case 2: $T_{2Z} > T_{1Z}$. Since P_1 and P_2 are on a line, we have $T_{1Z} \geq m_1 + {} +m_2 - 1$ by defining of T_{1Z}. So, $T_{2Z} \geq m_1 + m_2$. On the other hand, by defining of T_{2Z} there is a linear 2-space, say α, such that

$$T_{2Z} = \left\lfloor \frac{w_{s(\alpha \cap Z)}}{2} \right\rfloor.$$

Suppose that $\alpha \cap Z = m_{i_1}P_{i_1} + \cdots + m_{i_q}P_{i_q}$, then

$$\left\lfloor \frac{\sum_{l=1}^{q} m_{i_l}}{2} \right\rfloor = \left\lfloor \frac{w_{s(\alpha\cap Z)}}{2} \right\rfloor \geq m_1 + m_2.$$

Since $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_{n+3}$, we have $q \geq 4$. We consider two following cases for q.

Case $q = 4$: Then $m_1 = m_2 = m_{i_1} = m_{i_2} = m_{i_3} = m_{i_4} = m$ and $T_{2Z} = 2m = T_Z = T_{1Z} + 1$. By Lemma 3.2 we get

$$\operatorname{reg}(Z) \geq T_{1Z} = T_Z - 1.$$

Case $q \geq 5$: Since $P_1, \ldots, P_{n+3}$ are in non-degenerate in $\mathbb{P}^n$, there are at most five points on the linear 2-space. Thus $q = 5$ because α is a linear 2-space. By using Proposition 3.5 we get

$$\operatorname{reg}(Z) \geq T_Z - 1. \qquad \blacksquare$$

References

[1] **Ballico, E., O. Dumitrescu and E. Postinghel,** On Segre's bound for fat points in $\mathbb{P}^n$, *J. Pure Appl. Algebra,* **220**, (2016), 2307–2323.

[2] **Cataliano, M.V., N.V. Trung and G. Valla,** A sharp bound for the regularity index of fat points in general position, *Proc. Amer. Math. Soc.,* **118** (1993), 717–724.

[3] **Davis, E.D. and A.V. Geramita,** The Hilbert function of a special class of 1-dimensional Cohen–Macaulay graded algebras, The Curves Seminar at Queen's, *Queen's Papers in Pure and Appl. Math.,* **67** (1984), 1–29.

[4] **Fatabbi, G. and A. Lorenzini,** On a sharp bound for the regularity index of any set of fat points, *J. Pure Appl. Algebra,* **161** (2001), 91–111.

[5] **Nagel, U. and B. Trok,** Segre's regularity bound for fat point schemes, https://arxiv.org/abs/1611.06279v1 (2016).

[6] **Segre, B.,** Alcune questioni su insiemi finiti di punti in geometria algebrica, *Atti. Convergno. Intern. di Torino*, **20** (1961), 67–85.

[7] **Thien, P.V.,** Segre bound for the regularity index of fat points in $\mathbb{P}^3$, *J. Pure Appl. Algebra,* **151** (2000), 197–214.

[8] **Thien, P.V.,** Regularity index of $s+2$ fat points not on a linear $(s-1)$-space, *Comm. Algebra,* **40** (2012), 3704–3715.

[9] **Thien, P.V.,** Lower bound for the regularity index of fat points, *IJPAM*, **105** (2016), 745–755.

[10] **Thien, P.V. and T.N. Sinh,** On the regularity index of s fat points not on a linear $(r-1)$-space, $s \leq r+3$, *Comm. Algebra*, **45** (2017), 4123–4138.

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