EXTENSIONS OF VIEIRA'S THEOREMS ON THE ZEROS OF SELF-INVERSIVE POLYNOMIALS

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Abstract. Recently R.S.Vieira [18] found sufficient conditions for self-inversive polynomials to have some of their zeros on the unit circle. We extend his results by giving the location of those zeros. In case of fourth degree real reciprocal polynomials we compare Vieira's sufficient conditions with the necessary and sufficient conditions obtained by help of Chebyshev transformation.

1. Introduction

Let $P_m(z) = \sum\limits_{k=0}^{m} A_k z^k = A_m \prod\limits_{k=0}^{m} (z - z_k) \in \mathbb{C}[z]$ be a polynomial of degree m with zeros $z_1, \ldots, z_m$. Further let P_m^* be the polynomial defined by $P_m^*(z) :=$ $:= z^m \bar{P}(1/z) = \sum\limits_{k=0}^{m} \bar{A}_k z^{n-k} = \bar{A}_0 \prod\limits_{k=0}^{m} (z - z_k^*)$ whose zeros are $z_k^* = 1/\bar{z}_k$, $k = 0, \ldots, m$ (the inverses of z_k with respect to the unit circle $\{z \in \mathbb{C} : |z| = 1\}$).

Definition 1.1. A polynomial $P_m(z)$ of degree m is said to be *self-inversive* if there exists an $\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$ such that $P_m^*(z) = \varepsilon P_m(z)$.

There are several equivalent definitions of self-inversive polynomials. It is well-known (see e.g. [14]) that for a polynomial $P_m(z) = \sum\limits_{k=0}^{m} A_k z^k$ of degree m the following statements are equivalent

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1. P_m is self-inversive,
2. $\bar{A}_k = \varepsilon A_{m-k}, k = 0, \ldots, m,$ where $|\varepsilon| = 1$,
3. for the zeros z_k of P_m we have $\{z_1, z_2, \ldots, z_m\} = \{1/\bar{z}_1, 1/\bar{z}_2, \ldots, 1/\bar{z}_m\}$.

It follows that the zeros of a self-inversive polynomial of degree m can be divided into three groups: $0 \leq l \leq m/2$ zeros inside the complex unit circle, the same number of zeros outside the unit circle, and $m - 2l$ zeros on the unit circle.

If a polynomial with real coefficients is self-inversive then ε is necessarily real hence either $\varepsilon = 1$ our polynomial is called reciprocal, or $\varepsilon = -1$ and our polynomial is called antireciprocal.

There is an extensive literature dealing with polynomials all of whose zeros are on the unit circle. There are *necessary and sufficient conditions* by Cohn [2] (see also [14] p.14, Theorem 2.1.6) and several *sufficient conditions* (Chen [1], DiPippo and Howe [3], Lakatos [7], Schinzel [16], Lakatos and Losonczi [8], [9], [10], Losonczi and Schinzel [13], Kim and Park [4], Petersen and Sinclair [15], Sinclair and Vaaler [17], Kwon [5], [6]) for the coefficients of reciprocal and self-inversive polynomials to have all their zeros on the unit circle.

Our starting point is the paper of R. S. Vieira [18] who proved the following nice result (the notations and formulations are slightly changed).

Theorem 1.1. (Vieira) *Let* $P_m(z) = \sum\limits_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$ *be a* m*-degree self-inversive polynomial. If the inequality*

$$|A_l| > \frac{1}{2}\left(\frac{m}{m-2l}\right) \sum_{\substack{k=0 \\ k \neq \{l, m-l\}}}^{m} |A_k|, \qquad l < m/2 \tag{1.1}$$

holds then P_m *has exactly* $m - 2l$ *simple zeros on the unit circle. Moreover if* m *is even and* $l = m/2$ *then* P_m *has no zeros on the unit circle if the inequality*

$$|A_{m/2}| > 2 \sum_{k=0, k \neq m/2}^{m} |A_k| \tag{1.2}$$

is satisfied.

For $l = 0$ (1.1) goes over into $|A_m| > 1/2 \sum_{k=0}^{m} |A_k|$ which (with $>$ replaced by $\geq$) is exactly the sufficient condition of Lakatos and Losonczi [9] for P_m to have all of its zeros on the unit circle. Moreover in [9] the location of the zeros and conditions for multiple zeros were given. Vieira proved another sufficient condition which we formulate as

Theorem 1.2. (Vieira) *Let $P_m(z)=\sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$ be a m-degree self-inversive polynomial. If the inequality*

$$|A_l| > \frac{1}{2} \sum_{\substack{k=0 \\ k\neq\{l,m-l\}}}^{m} |A_k|, \qquad l < m/2 \tag{1.3}$$

holds then P_m has at least $m-2l$ zeros on the unit circle.

In the proof also the positions of those $m-2l$ zeros were described.

2. Results

Here we study the case of equality in (1.3), show that in this case P_m also has at least $m-2l$ zeros on the unit circle, give the location of these zeros and the conditions for multiple zeros. As we mentioned, the (ii)-1 part of the next theorem was essentially proved by Vieira but for the sake of completeness we give the complete proof here.

Theorem 2.1. (i) *Let $P_m(z)=\sum_{k=0}^{m} A_k z^k$ be a m-degree self-inversive polynomial. If*

$$|A_l| \geq \frac{1}{2} \sum_{\substack{k=0 \\ k\neq\{l,m-l\}}}^{m} |A_k|, \qquad l < m/2 \tag{2.1}$$

then P_m has at least $m-2l$ zeros on the unit circle.

Let

$$\begin{aligned} \beta_k : &= \arg\left(\left(\frac{A_m}{A_0}\right)^{-\frac{1}{2}} A_k\right) \ (k=0,\dots,m), \\ \varphi_s : &= \frac{2(s\pi-\beta_l)}{m-2l} \ (s=0,\dots,m-2l). \end{aligned} \tag{2.2}$$

Let us call the points $e^{i\varphi_s}$ $(s=0,\dots,m-2l)$ division points on the unit circle, clearly $e^{i\varphi_{m-2l}}=e^{i(2\pi+\varphi_0)}=e^{i\varphi_0}$. Thus there are $m-2l$ division points on the unit circle forming a regular $m-2l$-gon.

(ii)-1 *If (2.1) holds with strict inequality then P_m has at least one zero on each open arc between two neighboring division points $e^{i\varphi_{s-1}}$ and $e^{i\varphi_s}$ $(s=1,\dots,m-2l)$*

i.e. P_m *has at least* $m-2l$ *(not necessarily simple) zeros* e^{iu_s} $(s=1,\dots,m-2l)$ *on the unit circle such that*

$$\varphi_{s-1} < u_s < \varphi_s \qquad (s=1,\dots,m-2l). \tag{2.3}$$

(ii)-2 *If* (2.1) *holds with equality then some division points may become* (*necessarily multiple*) *zeros of* P_m. *In particular* $e^{i\varphi_s}$ $(s=1,\dots,m-2l)$ *is* (*necessarily multiple*) *zero of* P_m *if and only if the coefficients of* P_m *satisfy the conditions*

$$\cos\left(\beta_k + \left(\frac{m}{2}-k\right)\varphi_s\right) + (-1)^s = 0 \tag{2.4}$$

for all $k=0,1,\dots,[m/2], k\neq l$ *for which* $A_k \neq 0$.

(ii)-3 *If* (2.1) *holds with equality and two neighboring division points* $e^{i\varphi_{s-1}}$ *and* $e^{i\varphi_s}$ $(s=1,\dots,m-2l)$ *are not zeros of* P_m *then in the open arc between them there is a zero of* P_m.

(ii)-4 *If* (2.1) *holds with equality then two neighboring division points* $e^{i\varphi_{s-1}}$ *and* $e^{i\varphi_s}$ $(s=1,\dots,m-2l)$ *cannot both be (necessarily multiple) zeros of* P_m.

Proof. In the proof we follow the arguments of [9] with suitable changes. Let $\varepsilon = \overline{\frac{A_m}{A_0}}$, $B_k = \varepsilon^{-\frac{1}{2}}A_k$, then $B_k = \overline{B_{m-k}}$ $(k=0,\dots,m)$. If $m=2n+1$ is odd, $z=e^{i\varphi}$ then

$$\begin{aligned}\varepsilon^{-\frac{1}{2}} z^{-\frac{m}{2}} P_m(z) &= \sum_{k=0}^{n}\left(\overline{B_{m-k}z^{\frac{m}{2}-k}} + B_{m-k}z^{\frac{m}{2}-k}\right) = \\ &= \sum_{k=0}^{n} 2|B_{m-k}|\cos\left(\beta_{m-k} + \left(\tfrac{m}{2}-k\right)\varphi\right) = \\ &= \sum_{k=0}^{n} 2|B_k|\cos\left(\beta_k + \left(\tfrac{m}{2}-k\right)\varphi\right),\end{aligned}$$

where $\beta_k = \arg\, B_k = \arg\, \varepsilon^{-\frac{1}{2}}A_k$ $(k=0,\dots,m)$.

For even $m=2n$ we have similarly

$$\varepsilon^{-\frac{1}{2}} z^{-\frac{m}{2}} P_m(z) = \sum_{k=0}^{n-1} 2|B_k|\cos\left(\beta_k + \left(\frac{m}{2}-k\right)\varphi\right) + |B_n|\cos\beta_n$$

where $\beta_n = 0$ or π as B_n is real.

Let $F_l(\varphi) := \dfrac{\varepsilon^{-\frac{1}{2}}}{2|B_l|} z^{-\frac{m}{2}} P_m(z)|_{z=e^{i\varphi}}$ then we have

$$F_l(\varphi) = \cos\left(\beta_l + \left(\frac{m}{2}-l\right)\varphi\right) + f_l(\varphi),$$

where
(2.5)

$$f_l(\varphi)=\begin{cases} \sum\limits_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n} \left|\frac{B_k}{B_l}\right| \cos\left(\beta_k+\left(\frac{m}{2}-k\right)\varphi\right) & \text{if} m=2n+1 \\ \sum\limits_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n-1} \left|\frac{B_k}{B_l}\right| \cos\left(\beta_k+\left(\frac{m}{2}-k\right)\varphi\right) + \left|\frac{B_n}{2B_l}\right| \cos\beta_n & \text{if} m=2n. \end{cases}$$

It is easy to check that

$$F_l(\varphi+2\pi) = \begin{cases} -F_l(\varphi) & \text{if } m=2n+1 \\ F_l(\varphi) & \text{if } m=2n. \end{cases} \tag{2.6}$$

Next we rewrite (2.1) as

$$1 \geq \begin{cases} \sum\limits_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n} \left|\frac{B_k}{B_l}\right| & \text{if } m=2n+1 \\ \sum\limits_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n-1} \left|\frac{B_k}{B_l}\right| + \left|\frac{B_n}{2B_l}\right| & \text{if } m=2n. \end{cases} \tag{2.7}$$

(ii)-1 Suppose now that there is strict inequality in (2.1) then the same true for (2.7). From (2.5) and (2.7) we have $|f_l(\varphi)| < 1$. Therefore we have for $s = 0, \ldots, m-2l$
(2.8)

$$F_l(\varphi_s) = \cos\left(\beta_l + \left(\frac{m}{2} - l\right)\varphi_s\right) + f_l(\varphi_s) = (-1)^s + f_l(\varphi_s) \begin{array}{ll} > 0 & \text{if } s \text{ is even} \\ < 0 & \text{if } s \text{ is odd.} \end{array}$$

By (2.8) F_l assumes values of different signs at the endpoints of each of the intervals $]\varphi_{s-1}, \varphi_s[$ $(s = 1, \ldots, m-2l)$. This is also true for the last interval since using (2.6)

$$F_l(\varphi_{m-2l-1}) \begin{array}{ll} < 0 & \text{if } m \text{ is even,} \\ > 0 & \text{if } m \text{ is odd,} \end{array}$$

$$F_l(\varphi_{m-2l}) \begin{array}{lll} = F_l(2\pi+\varphi_0) = & F_l(\varphi_0) > 0 & \text{if } m \text{ is even,} \\ = F_l(2\pi+\varphi_0) = & -F_l(\varphi_0) < 0 & \text{if } m \text{ is odd.} \end{array}$$

By the intermediate value theorem it follows that each interval contains at least one zero u_s of F_l which means that e^{iu_s} $(s = 1, \ldots, m-2l)$ are zeros of P_l proving (ii)-1.

(ii)-2 Assume now that equality holds in (2.1) then there is also equality in (2.7) and we have

$$F_l(\varphi_s) = \cos\left(\beta_l + \left(\frac{m}{2} - l\right)\varphi_s\right) + f_l(\varphi_s) = (-1)^s + f_l(\varphi_s) = 0, \tag{2.9}$$

or using the form (2.5) of f_l and replacing $(-1)^s$ by $(-1)^s$ multiplied by the right hand side of (2.7) we get

(2.10)
$$F_l(\varphi_s)=\begin{cases}\displaystyle\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n}\left|\frac{B_k}{B_l}\right|\left[\cos\left(\beta_k+\left(\frac{m}{2}-k\right)\varphi_s\right)+(-1)^s\right]=0 & \text{if } m=2n+1\\ \displaystyle\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n-1}\left|\frac{B_k}{B_l}\right|\left[\cos\left(\beta_k+\left(\frac{m}{2}-k\right)\varphi_s\right)+(-1)^s\right] & \\ \displaystyle\qquad+\left|\frac{B_n}{2B_l}\right|\left[\cos\beta_n+(-1)^s\right]=0 & \text{if } m=2n.\end{cases}$$

Since all expressions of (2.10) in brackets are non-negative for even s (and non-positive for odd s) and the same is true for $F_l(\varphi_s)$, we conclude that (2.10) i.e. $F_l(\varphi_s)=0$ can hold if and only if the all expressions in brackets are zero provided that in their coefficients $B_k\neq 0$ proving (2.4).

If (2.4) holds then $e^{i\varphi_s}$ is a multiple zero as in this case

$$\begin{aligned}F_l'(\varphi_s)= & -\left(\frac{m}{2}-l\right)\sin\left(\beta_l+\left(\frac{m}{2}-l\right)\varphi_s\right)-\\ & -\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{[\frac{m-1}{2}]}\left|\frac{B_k}{B_l}\right|\left(\frac{m}{2}-k\right)\sin\left(\beta_k+\left(\frac{m}{2}-k\right)\varphi_s\right)=0\end{aligned}$$

since all sin factors (or B_k's) are zero. For the second derivative we have

$$\begin{aligned}&F_l''(\varphi_s)=-\left(\frac{m}{2}-l\right)^2\cos(\beta_l+(\frac{m}{2}-l)\varphi_s)-\\&-\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{[\frac{m-1}{2}]}\left|\frac{B_k}{B_l}\right|\left(\frac{m}{2}-k\right)^2\cos\left(\beta_k+\left(\frac{m}{2}-k\right)\varphi_s\right)=\\&=-\left(\frac{m}{2}-l\right)^2(-1)^s-\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{[\frac{m-1}{2}]}\left|\frac{B_k}{B_l}\right|\left(\frac{m}{2}-k\right)^2(-1)^{s+1}=\\&=(-1)^{s+1}\left[\left(\frac{m}{2}-l\right)^2-\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{[\frac{m-1}{2}]}\left|\frac{B_k}{B_l}\right|\left(\frac{m}{2}-k\right)^2\right]=\\&=\begin{cases}\displaystyle(-1)^{s+1}\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n}\left|\frac{B_k}{B_l}\right|\left[\left(\frac{m}{2}-l\right)^2-\left(\frac{m}{2}-k\right)^2\right] & \text{if} m=2n+1\\ \displaystyle(-1)^{s+1}\left[\left|\frac{B_n}{2B_l}\right|\left(\frac{m}{2}-l\right)^2+\sum_{\substack{k=0\\ k\neq\{l,m-l\}}}^{n-1}\left|\frac{B_k}{B_l}\right|\left[\left(\frac{m}{2}-l\right)^2-\left(\frac{m}{2}-k\right)^2\right]\right] & \text{if} m=2n.\end{cases}\end{aligned}$$

Since $\operatorname{sgn}\left(\left(\frac{m}{2}-l\right)^2-\left(\frac{m}{2}-k\right)^2\right)=\operatorname{sgn}(k-l)(m-(k+l)=\operatorname{sgn}(k-l)$ the second derivative $F_l''(\varphi_s)$ may become zero and then $e^{i\varphi_s}$ is at least a triple zero.

(ii)-3 Suppose that (2.1) holds with equality, $e^{i\varphi_{s-1}}$ and $e^{i\varphi_s}$ are not zeros of P_m for some $s \in \{1, \ldots, m-2l\}$. Then $F_l(\varphi_{s-1}) \neq 0$, $F_l(\varphi_s) \neq 0$ thus

$$F_l(\varphi_{s-1}) \neq 0 \text{ and by (2.10) } (-1)^{s-1} F_l(\varphi_{s-1}) = 1 + (-1)^{s-1} f_l(\varphi_{s-1}) \geq 0$$

since $|f_l(\varphi_{s-1})| \leq 1$ implying that $(-1)^{s-1} F_l(\varphi_{s-1}) > 0$. Arguing similarly we get that $(-1)^s F_l(\varphi_s) > 0$ therefore

$$\operatorname{sgn} F_l(\varphi_{s-1}) \neq \operatorname{sgn} F_l(\varphi_s).$$

By intermediate value theorem there is at least one zero u_s strictly between φ_{s-1} and φ_s and thus e^{iu_s} is a zero of P_m.

(ii)-4 For $l = 0$ the statement (ii)-4 has been proved in [9], Lemma 1, therefore in the following we may suppose that $0 < l < m/2$.

Assume that contrary to (ii)-4 there is equality in (2.1), $e^{i\varphi_{s-1}}$ and $e^{i\varphi_s}$ are both zeros of P_m for some $s \in \{1, \ldots, m-2l\}$. Then $|B_0| = |B_m| \neq 0$ and $0 \neq l$ therefore (2.4) holds for $s-1, s$ and for $k = 0$:

$$\begin{aligned} \cos\left(\beta_0 + \tfrac{m}{2}\varphi_{s-1}\right) + (-1)^{s-1} &= 0 \\ \cos\left(\beta_0 + \tfrac{m}{2}\varphi_s\right) + (-1)^s &= 0. \end{aligned} \tag{2.11}$$

Case 1: If $m = 2n+1$ is odd or $0 < l < n-1$ then using the equality $\varphi_{s-1} = = \varphi_s + \frac{2\pi}{m-2l}$ it follows from (2.11) that

$$\begin{aligned} (-1)^{s+1} &= \cos\left(\beta_0 + \tfrac{m}{2}\varphi_s\right) \\ &= \cos\left(\beta_0 + \tfrac{m}{2}\varphi_{s-1}\right)\cos\tfrac{2\pi}{m-2l} = (-1)^s \cos\tfrac{2\pi}{m-2l} \end{aligned}$$

thus $\cos\frac{2\pi}{m-2l} = -1$ which is a contradiction since the previous equality holds if and only if $m - 2l = 2$, $m = 2(l+1) = 2n$ i.e. if m is even and $l = n-1$.

Case 2: If $m = 2n$ is even and $l = n-1$ then the equations (2.11) are equivalent to

$$\beta_0 + n\varphi_{s-1} = \begin{cases} 2p\pi & \text{if } s \text{ is even} \\ (2p+1)\pi & \text{if } s \text{ is odd} \end{cases}$$

$$\beta_0 + n\left(\varphi_{s-1} + \tfrac{\pi}{n-l}\right) = \begin{cases} (2q+1)\pi & \text{if } s \text{ is even} \\ 2q\pi & \text{if } s \text{ is odd} \end{cases}$$

with some $p, q \in \mathbb{Z}$. Subtracting the first equation from the second and dividing by π we get

$$\frac{n}{n-l} = 2r + 1 \text{ or } (2r+1)l = 2rn,$$

where $r = p - q - 1$ if s is even and $r = p - q$ if s is odd. In both cases we get a contradiction since in the last equality one side is odd the other is even, Summarizing: in the case of equality in (2.1) some pairs of zeros $e^{iu_s}, e^{iu_{s+1}}$ $(s = 1, \dots, m-2l)$ (with $u_{s+1} = u_1$) satisfying (2.3) pull together to the (at least double) zero $e^{i\varphi_s}$ while other zeros remain in the same position. ■

Since $\frac{m}{m-2l} > 1$ (1.1) clearly implies (1.3) and by Theorem 2.1 we get

Theorem 2.2. (Extension of Vieira's Theorem 1.1) *Let* $P_m(z) = \sum_{k=0}^{m} A_k z^k \in \mathbb{C}[z]$ *be a* m*-degree self-inversive polynomial. If the inequality* (1.1) *holds then* P_m *has exactly* $m-2l$ *simple zeros* e^{iu_s} $(s = 1, \dots, m-2l)$ *on the unit circle such that*

$$\varphi_{s-1} < u_s < \varphi_s \qquad (s = 1, \dots, m-2l).$$

where φ_s *is given by* (2.2).

3. Comparison of Vieira's conditions to necessary and sufficient conditions by degree four reciprocal polynomials

Here we consider the reciprocal polynomial $P_4(z) := z^4 + A_1 z^3 + A_2 z^2 + A_1 z + 1$ $(A_1, A_2 \in \mathbb{R})$ of degree four with real coefficients. Using the method of Chebyshev transforms (see Lakatos [7]) we find criteria for the coefficients of P_4 to have all zeros, two zeros or no zeros on the unit circle, and we compare these with the sufficient conditions given by Vieira's theorem, and also the conditions given by Theorem 3.

The Chebyshev transform of P_4 is (see [7] p.659)

$$\mathcal{T}P_4(x) = x^2 + A_1 x + A_2 - 2$$

whose zeros are

$$x_1 = \left(-A_1 + \sqrt{A_1^2 - 4(A_2 - 2)}\right)/2, \quad x_2 = \left(-A_1 - \sqrt{A_1^2 - 4(A_2 - 2)}\right)/2$$

By Corollary 1 of [12]

(i) all zeros of P_4 are on the unit circle if and only if x_1, x_2 are both real and lie in the interval $[-2, 2]$,

(ii) exactly two zeros of P_4 are on the unit circle if and only if x_1, x_2 are both real and one of them is in the interval $[-2, 2]$ and the other is not,

(iii) P_4 has no zeros on the unit circle if and only if x_1, x_2 are both real and none of them lie in the interval $[-2, 2]$ or if x_1, x_2 are both complex.

It is easy to see that the zeros are real if and only if $|A_1| \geq \Delta := \sqrt{\max\{A_2 - 2, 0\}}$.

The case (i) has been settled in [7] pp. 659-660, proving that both zeros are in $[-2, 2]$ if and only if

$$2\sqrt{\max\{A_2 - 2, 0\}} \leq |A_1| \leq \min\{4, (A_2 + 2)/2\}.$$

(ii) holds exactly if $|A_1| \geq \Delta$ and either $-2 \leq x_2 \leq 2$ and $x_1 > 2$ or $-2 \leq$ $\leq x_1 \leq 2$ and $x_2 < -2$. An elementary but a bit long calculation shows that these hold exactly if

$$|A_1| \geq \frac{1}{2}|A_2 - 2| \text{ except the line segments } |A_1| = \frac{1}{2}|A_2 - 2|,\, A_2 \in]-2, 6[.$$

(iii) holds exactly if either x_1, x_2 are real (i.e. $|A_1| \geq \Delta$) and $x_2 > 2$ or $x_2 < -2$ and $x_1 > 2$ or $x_1 < -2$ or x_1, x_2 are complex (i.e. $|A_1| < \Delta$). An elementary calculation gives that these conditions (in the above order) are satisfied exactly if

$$\begin{aligned} & |A_1| < \tfrac{1}{2}|A_2 - 2| \text{ if } A_2 < -2 \\ \text{or} \quad & |A_1| < \sqrt{A_2 - 2} \text{ if } 2 \leq A_2 \leq 6 \text{ and } |A_1| < \tfrac{1}{2}|A_2 - 2| \text{ if } 6 < A_2. \end{aligned}$$

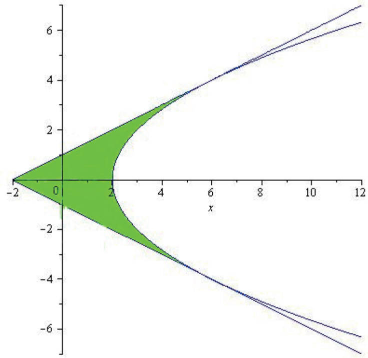
By Vieira's inequalities (1.1) and (1.2) we get the following sufficient conditions for P_4 to have 4, 2 and no zeros on the unit circle

$$\begin{aligned} &|A_1| < 1 - \tfrac{1}{2}|A_2|, \\ &|A_1| > 2 + |A_2| \text{ and} \\ &|A_1| > \tfrac{1}{4}|A_2| - 1, \end{aligned}$$

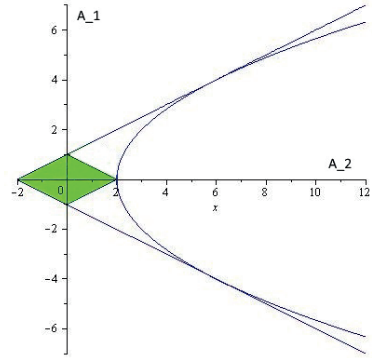
respectively. Finally the condition which ensures that P_4 has at least two zeros on the unit circle is by (2.1)

$$|A_1| \geq 1 + \frac{1}{2}|A_2|.$$

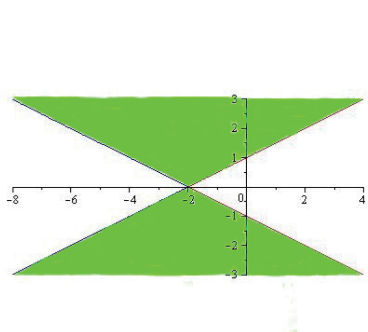
In figures C_4, C_2, C_0 and S_4, S_2, S_0 we draw the domains in the (A_2, A_1) plane which give criteria and Vieira's sufficient conditions for P_4 to have four, two, or none zeros on the unit circle respectively. The last figure L_2 gives the domain where P_4 has at least two zeros on the unit circle.



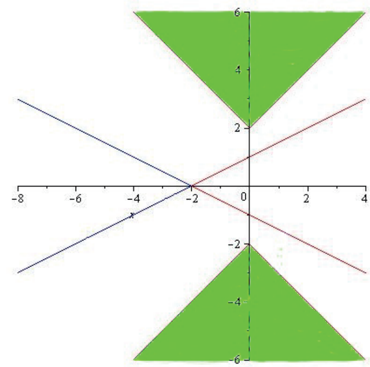
C_4



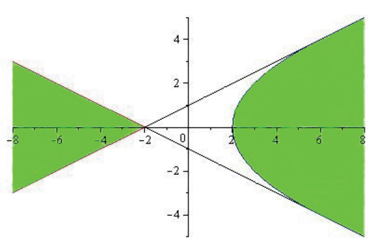
S_4



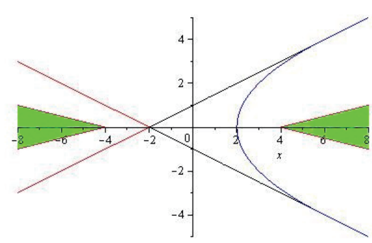
C_2



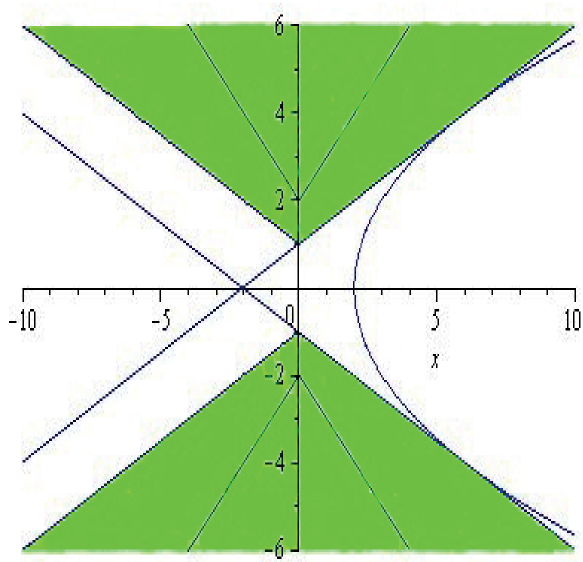
S_2



C_0



S_0



L_2

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