Annales Univ. Sci. Budapest., Sect. Comp. 49 (2019) 325–345

INTRODUCING p-EIGENVECTORS, EXACT SOLUTIONS FOR SOME SIMPLE MATRICES

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Communicated by Sándor Fridli

(Received March 31, 2019; accepted July 3, 2019)

Abstract. A common way to define a norm of a matrix is to take the supremum of the fraction of the vector norms of the matrix-vector product and the nonzero vector, with respect to a given vector norm, i.e. the least upper bound for the norm of the vectors of the transformed unit sphere. In this paper we examine the above mentioned fraction, defining induction curves and surfaces, we show that there exist some vectors, such that this fraction is independent of the applied p-norm (and are not eigenvectors). These are to be called p-eigenvectors. Exact solutions are constructed for some simple matrices. No previous work was found in this topic so far.

1. Introduction

The starting point of the research to be presented is the investigation of the p-norm or power norm. Solutions for linear systems or approximation problems using different power norms is a traditional, widely and actively studied field in mathematics and signal processing [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17]. Approximation in 2-norm is considered a classical problem, the case of $p = 1$

Key words and phrases: Matrix norm, power norm, p-norm, induction sets, induction curves, eigenvectors, p-eigenvectors.
2010 Mathematics Subject Classification: 15A18, 47A30, 65F15, 65F35.
EFOP-3.6.3-VEKOP-16-2017-00001: Talent Management in Autonomous Vehicle Control Technologies – The Project is supported by the Hungarian Government and co-financed by the European Social Fund.
This research was also supported by the Hungarian Scientific Research Funds (OTKA) No. K115804.

and $p = \infty$ are well-studied with both classical results and recent achievements, and the cases of $1 < p < \infty$ are also often considered in various research areas.

This paper introduces a new problem related to p-norms of matrices, considering a generalization of eigenvectors, called p-eigenvectors. As a part of this novel approach, we introduce induction sets of matrices as well. No previous work was found dealing with these topics so far. Furthermore we give exact answers for the raised questions in simple cases.

Recall the power norms or p-norms for vectors of $\mathbb{R}^n$ with $2 \leq n \in \mathbb{N}$:

$$\|.\|_p : \mathbb{R}^n \to \mathbb{R}, \qquad \|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \qquad (p \in [1, \infty)),$$

and

$$\|x\|_\infty = \max_{k=1}^{n} |x_k| .$$

It is well known that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ $(x \in \mathbb{R}^n)$. Let us now consider a matrix $A \in \mathbb{R}^{n\times n}$. The p-norm of A is defined as

$$\|.\|_p : \mathbb{R}^{n\times n} \to \mathbb{R}, \qquad \|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \qquad (p \in [1, \infty]) .$$

As equivalent descriptions one may consider the supremum only for the elements of the unit sphere, i.e. $\|x\|_p = 1$, thus the fraction becomes unnecessary.

The p-norm of a matrix is also called the matrix norm *induced* by the corresponding vector p-norm. Induced norms are also called natural matrix norms. (A well-known counterexample is the Frobenius norm, which is not a natural matrix norm.)

2. Basic notions

In topics of mathematical analysis and numerical methods related to matrix norm, the detailed analysis of the fraction behind the supremum in the definition of the matrix p-norm is usually skipped. The research presented here stems from these examinations neglected so far (at least to the knowledge of the author).

2.1. Induction sets

As a general definition, let us first formulate a geometric representation of this fraction.

Definition 1. Given a matrix $A \in \mathbb{R}^{n\times n}$ with $2 \le n \in \mathbb{N}$ and $p \in [1,\infty]$, the set of points

$$\mathcal{I}_p(A) := \left\{ \frac{\|Ax\|_p}{\|x\|_p} \cdot \frac{x}{\|x\|_2} \in \mathbb{R}^n \;:\; 0 \ne x \in \mathbb{R}^n \right\} \subset \mathbb{R}^n$$

is called the *induction set* of A with parameter p. The induction set may be called induction curve for $n = 2$, induction surface for $n = 3$, induction manifold in general.

Note that if $x' = c \cdot x$ $(0 < c \in \mathbb{R}, x, x' \in \mathbb{R}^n \setminus \{0\})$, then

$$\frac{\|Ax'\|_p \cdot x'}{\|x'\|_p \cdot \|x'\|_2} = \frac{\|Acx\|_p \cdot cx}{\|cx\|_p \cdot \|cx\|_2} = \frac{c \cdot \|Ax\|_p \cdot c \cdot x}{c \cdot \|x\|_p \cdot c \cdot \|x\|_2} = \frac{\|Ax\|_p \cdot x}{\|x\|_p \cdot \|x\|_2},$$

thus for vectors of the same direction the induction set contains only one point of $\mathbb{R}^n$ in the same direction; for all possible directions. This means that $\mathcal{I}_p(A)$ is actually a curve around the origin for $n = 2$, a surface surrounding the origin for $n = 3$ etc., taking into account also that the p-norm is a continuous function of $x \in \mathbb{R}^n$. (Provided that A is invertible.) However if we repeat the above calculations for $0 > c \in \mathbb{R}$, e.g. $c = -1$, we see that the induction set is symmetric with respect to the origin, i.e. $y \in \mathcal{I}_p(A) \iff -y \in \mathcal{I}_p(A)$.

Remark 1. We will not take advantage of the triangle inequality, so in Definition 1 the restriction $p \ge 1$ is not important, we may also consider $p \in (0,1)$, quasi-norms.

Remark 2. In Definition 1 the Euclidean-norm is also presented in the denominator. Actually any p-norm could be used here, such that $x/\|x\|_p$ will define a "direction" in $\mathbb{R}^n$. The reason behind using the 2-norm is basically to have the plots in accordance with the common sense of visual perception. However it is important to have it fixed, such that induction sets for different p values may be compared. For completeness we mention, that one may define $\mathcal{I}_{p,q}(A)$ with $q \in [1,\infty]$ playing the role of the Euclidean norm, and then choose $q = 2$.

Example 1. Figure 1 presents ten examples for induction curves in $\mathbb{R}^2$. We used 5 values for p, namely 1, 4/3, 2, 4 and ∞ with light to dark gray in case of 2 matrices. On the left-hand-side, we considered the matrix A_1 below corresponding to the rotation with 45° in negative direction with a multiplication factor of $\sqrt{2}$, similar to the Walsh-matrix. On the right-hand-side the induction set of the diagonal matrix A_2 is presented. Figure 5 features an additional, rather artistic example with the symmetric matrix A_3, using $p = 4/8$, $5/8$, $6/8$, $7/8$, 1 and their reciprocals, so with quasi-norms also involved.

$$A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

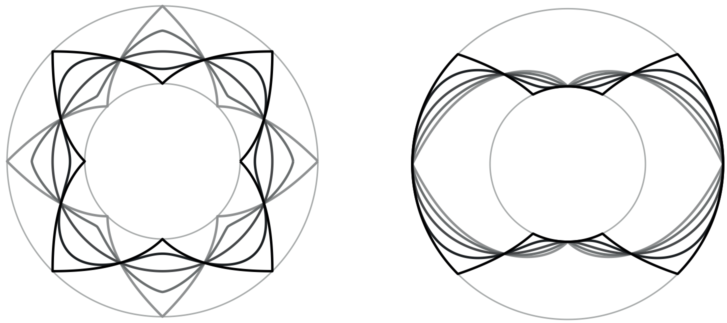

Figure 1. Some examples for induction curves in $\mathbb{R}^2$ in case of a Walsh-like and a diagonal matrix, with five p-norms. Circles denote the radial units. Shades of gray indicate curves for different p-norms.

As some explanation for the plots of Figure 1 we make the following statements. From these images also some norms of these matrices can be easily read observing the maximum absolute values of the curves (their furthest points from the origin), which are also easy to verify by the usual formulas for these norms. Namely

$$\|A_1\|_1 = 2, \ \|A_1\|_\infty = 2, \ \|A_2\|_1 = 2, \ \|A_2\|_2 = 2, \ \|A_2\|_\infty = 2.$$

Note also that $\mathcal{I}_2(A_1)$ is a circle (of radius $\sqrt{2}$), i.e. $\|A_1\|_2 = \sqrt{2}$. Furthermore one may observe, conjure that $\sqrt{2} < \|A_1\|_p < 2$, and $\|A_2\|_p = 2$ $(p \in (1, \infty))$.

Further examples related to later Sections can be seen on Figure 4.

Remark 3. For practical purposes (e.g. implementation, creating illustrations, birthday presents*) it is useful to have a parametric description of induction curves, using polar coordinates:

$$\mathcal{I}_p(A) = \{ (r \cdot \cos\varphi, r \cdot \sin\varphi) \, : \, \varphi \in [0, 2\pi) \}, \quad \text{with} \quad r = f(\varphi) = (n \circ v)(\varphi),$$

$$v \colon \mathbb{R} \to \mathbb{R}^2, \quad v(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \qquad n \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, \quad n(x) = \frac{\|Ax\|_p}{\|x\|_p}.$$

Of course a similar formalization is possible for induction surfaces too.

*The exact same graphic printout (with a concise parametric description) as on the left of Figure 1 was presented by the author as a birthday present to Prof. Ferenc Schipp, Prof. Péter Simon and late Prof. William R. Wade on the occasion of their 75th, 65th and 70th birthdays respectively in Tállya (Tokaj wine region), during the festive dinner of the Conference on Dyadic Analysis and Related Fields with Applications (DARFA) organized by the Institute of Mathematics and Computer Science, College of Nyíregyháza, Hungary, in early June, 2014.

Remark 4. The induction sets $\mathcal{I}_p(A)$ are not to be confused with the transformed unit sphere (in a p-norm):

$$\mathcal{T}_p(A) := \left\{ Ax \in \mathbb{R}^n \ : \ x \in \mathbb{R}^n, \, \|x\|_p = 1 \right\} \subset \mathbb{R}^n.$$

Although this is a similar structure, its definition is essentially different. In case of $\mathcal{T}_p(A)$ the position vectors of the unit sphere are transformed by the matrix (so their direction may change, as well as their norms), while in case of $\mathcal{I}_p(A)$ the position vectors of the unit sphere are multiplied with a factor by the norm of the transformed vector (so their direction does not change).

A detailed description and analysis of induction curves, surfaces etc. for specific (classes of) matrices and norms lies beyond the scope of the current paper. Instead we will focus on one peculiar property, some specific points of interest of induction curves. Observe that for each matrix the curves for different p values intersect in one common point. More precisely the set

$$\bigcap_{p\in[1,\infty]} \mathcal{I}_p(A)$$

is not empty, e.g. in the presented cases it contains 8 elements. This means that in these cases (with a fixed A and x), the fraction $\|Ax\|_p \,/\, \|x\|_p$ is independent of the value of p, or in other words, the function

$$f\colon [1,\infty] \to \mathbb{R}, \quad f(p) = \frac{\|Ax\|_p}{\|x\|_p}$$

is constant in p. The next section is devoted to the description of this phenomenon.

2.2. Introducing p-eigenvectors

Based on the previously stated observation, verified for several matrices, let us give the next definition. Statements of some basic properties shall follow, the one-line proofs are left as an exercise.

Definition 2. Given a matrix $A \in \mathbb{R}^{n\times n}$ $(2 \le n \in \mathbb{N})$, a vector $x \in \mathbb{R}^n, x \neq 0$ shall be called a *p-eigenvector* of A, if there exists a constant $\gamma \in \mathbb{R}_0^+$, such that for all $p \in [1,\infty]$:

$$\frac{\|Ax\|_p}{\|x\|_p} = \gamma.$$

The value γ is the *p-eigenvalue* associated with the p-eigenvector x.

Remark. It is clear from the definition that a p-eigenvalue cannot be less than zero, since it is a fraction of nonnegative numbers. Thus $\gamma \in \mathbb{R}_0^+$. Furthermore similarly to Remark 1, we note that $p \in (0,1)$ may also be included.

Using the terms "eigenvector" and "eigenvalue" is justified by the following straightforward proposition.

Proposition 1. *If $0 \neq v \in \mathbb{R}^n$ is an eigenvector of $A \in \mathbb{R}^{n\times n}$ with eigenvalue $\lambda \in \mathbb{R}$, then v is also a p-eigenvector of A with p-eigenvalue $|\lambda|$.*

Thus p-eigenvectors may be considered a generalization of (regular) eigenvectors. This motivates the following definition.

Definition 3. Such p-eigenvectors of a matrix that are not eigenvectors of the matrix (in the regular sense) are called *non-trivial p-eigenvectors*, while real valued eigenvectors of matrices may be referred to as *trivial p-eigenvectors.*

The theory and numerical methods for eigenvectors are well-developed. Therefore in cases where it is not misleading, we may shortly just write "p-eigenvectors", instead of "non-trivial p-eigenvectors".

Proposition 2. *If $0 \neq x \in \mathbb{R}^n$ is a p-eigenvector of $A \in \mathbb{R}^{n\times n}$, then the vector $x' = c \cdot x \in \mathbb{R}^n$ $(c \in \mathbb{R}, c \neq 0)$ is also a p-eigenvector of A associated with the same p-eigenvalue.*

Thus one may also speak of *p-eigendirections, p-eigensubspaces*, in this case together with 0 also allowed.

Proposition 3. *If $0 \neq x \in \mathbb{R}^n$ is a p-eigenvector of $A \in \mathbb{R}^{n\times n}$ with p-eigenvalue $\gamma \in \mathbb{R}_0^+$, then x is also a p-eigenvector of $c \cdot A$ $(c \in \mathbb{R}, c \neq 0)$ associated with the p-eigenvalue $|c| \cdot \gamma$.*

The following questions naturally arise:

- For a given matrix $A \in \mathbb{R}^{n\times n}$, are there any p-eigenvectors? (Existence.)
- How many p-eigenvectors exist for a given matrix?
- How can one construct the p-eigenvectors, and p-eigenvalues of a matrix (if they exist)?
- Do p-eigenvectors exist for every matrix in $\mathbb{R}^{n\times n}$? If not, then how can the family of matrices with non-trivial p-eigenvectors be characterized?
- What can be stated about the generalization to $\mathbb{C}$?

Many of the above stated general problems are open questions. Nevertheless in the next section we will give some answers and exact solutions in some relatively simple cases.

3. Main results

The first results related to p-eigenvectors presented in this paper concern diagonal matrices as the most simple class of matrices. Notice that already in case of arbitrary $A \in \mathbb{R}^{2\times 2}$, with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{thus} \quad Ax = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix},$$

finding the appropriate x to A where the function

$$f\colon [1,\infty) \to \mathbb{R}, \quad f(p) = \frac{\|Ax\|_p}{\|x\|_p} = \frac{\big(|ax_1 + bx_2|^p + |cx_1 + dx_2|^p \big)^{1/p}}{\big(|x_1|^p + |x_2|^p \big)^{1/p}}$$

is constant in p seems intimidating. Therefore as a first approach we aim to solve the problem for diagonal 2×2 matrices.

Remark. Unfortunately $f(\infty)$ cannot be expanded in the above manner, thus now ∞ is removed from the domain of f. Of course $f(\infty)$ may be (and should be) still considered as a limit, or as expanded with the maximum operator.

3.1. 2×2 diagonal matrices

Let us formalize the problem, and construct the p-eigenvectors. Introduce

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{thus} \quad Ax = \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix},$$

with $0 \neq a, b \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$. If both $x_1 = 0$ and $x_2 = 0$, then we would have the 0 vector, which is ruled out by definition. If exactly one of x_1 and x_2 is 0, then we would arrive at the trivial p-eigenvectors of A. Thus we may assume $x_1 \neq 0$ and $x_2 \neq 0$. With these notations

$$f\colon [1,\infty) \to \mathbb{R}, \quad f(p) = \frac{\|Ax\|_p}{\|x\|_p} = \frac{\big(|ax_1|^p + |bx_2|^p \big)^{1/p}}{\big(|x_1|^p + |x_2|^p \big)^{1/p}}.$$

In this case it is easy to see that the signs of x_1, x_2, a, b do not matter, thus we may assume that these values are positive. (This is not true for not diagonal matrices.) Note that this also means that

$$\mathcal{I}_p \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} = \mathcal{I}_p \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix},$$

and that if $x = (x_1, x_2)^T$ is a p-eigenvector, then so are

$$x' = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}, \quad x'' = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \quad x''' = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}.$$

Of course from Proposition 2 it was already clear that if x is a p-eigenvector, then so is x''', and similarly to the case of x' and x''.

In light of Propositions 2 and 3 we may further simplify the form of function f using $c = |b/a| > 0$ and $m = |x_2/x_1| > 0$ to

$$f(p) = \left(\frac{1 + (cm)^p}{1 + m^p} \right)^{1/p} =: F(p)^{1/p},$$

considering

$$A = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad \text{thus} \quad Ax = \begin{pmatrix} 1 \\ cm \end{pmatrix},$$

with our goal now rephrased as to find the appropriate m to a given c, such that f is constant in p. The obvious choice is to examine the derivative of f, and find its zeros. Note that f is differentiable (if we also consider $p \in (0, 1)$ then also at $p = 1$), and

$$\begin{aligned} f'(p) &= F(p)^{1/p} \cdot \left(\frac{F'(p)}{p \cdot F(p)} - \frac{\ln F(p)}{p^2} \right) \\ &= \underbrace{\frac{F(p)^{-1+1/p}}{p^2}}_{\neq 0} \cdot \underbrace{\Big(p \cdot F'(p) - F(p) \cdot \ln F(p) \Big)}_{H(p)}. \end{aligned}$$

The first factor is always positive, so $f'(p)$ can be 0, iff. the second factor, $H(p) := p \cdot F'(p) - F(p) \cdot \ln F(p) = 0$. We remark that

$$F'(p) = \frac{(cm)^p \ln(cm)}{1 + m^p} - \frac{(1 + (cm)^p)\, m^p \ln(m)}{(1 + m^p)^2}.$$

With this $H(p)$ can be written with parameters c and m. Proving $H(p) = 0$ for all p (such that $f(p)$ becomes constant) is still tedious, and also unnecessary if we would just like to find the relation between c and m. Thus it is sufficient to substitute $p = 1$, and investigate $H(1) = 0$.

$$H(1) = \frac{cm \ln(cm)}{1 + m} - \frac{(1 + cm)\, m \ln m}{(1 + m)^2} - \frac{1 + cm}{1 + m} \ln \left(\frac{1 + cm}{1 + m} \right).$$

The common denominator $(1 + m)^2$ is strictly positive (also greater than 1), so we shall write

$$H(1) = H_{c,m}(1) = \frac{G(c, m)}{(1 + m)^2},$$

with

$$\begin{aligned}G(c,m) &= cm\,\ln(cm) + cm^2\,\ln(cm) - m\,\ln(m) - cm^2\,\ln(m)\\ &\quad - \ln(1+cm) + \ln(1+m) - cm\,\ln(1+cm) + cm\,\ln(1+m)\\ &\quad -m\,\ln(1+cm) + m\,\ln(1+m) - cm^2\,\ln(1+cm) + cm^2\,\ln(1+m).\end{aligned}$$

The zeros of this 12 term sum with logarithmic expressions are not plain to see. Some numerical calculations were carried out to aid our search for roots.

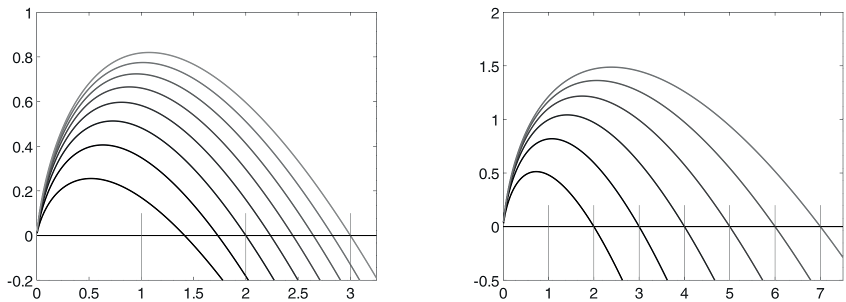

Figure 2. Some sections as functions of m with fixed c values of the function $G(c,m)$. Zeros are located at integer abscissae.

On Figure 2 we plotted some sections of $G(c,m)$ with fixed c values as a function of one positive variable, m. The left-hand-side figure shows the functions for values

$$c = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{9}.$$

Notice that for values 1/4 and 1/9 the curves intersect the horizontal axis at integer points $m = 2$ and $m = 3$. To further elaborate this conjecture, the right-hand-side figure shows the functions for values elements of the set

$$\left\{ c = \frac{1}{k^2} \;:\; k = 2, 3, \ldots, 7 \right\}.$$

The intersections clearly seem to arise again at integer abscissae, thus this underpins the conjecture, that the zero of $G(c,m)$ for a given $c > 0$ is exactly at $m = 1/\sqrt{c}$, i.e.

$$G\left(c, \frac{1}{\sqrt{c}}\right) = 0 \quad (0 < c \in \mathbb{R}), \quad \text{or } (\Leftrightarrow) \quad G\left(m^2, \frac{1}{m}\right) = 0 \quad (0 < m \in \mathbb{R}).$$

Symbolic substitution and evaluation confirms the above statements.

Based on the above calculations, we can state the following lemma.

Lemma 1. *Let $A \in \mathbb{R}^{2\times 2}$ be a diagonal matrix with non-zero diagonal elements. Apart from the trivial p-eigenvectors, the canonical unit vectors, there are exactly 2 further non-trivial p-eigenvectors $x', x'' \in \mathbb{R}^2$ of A independent of each other. Namely*

$$\text{if} \quad A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \text{then} \quad x' = \begin{pmatrix} \sqrt{|b|} \\ \sqrt{|a|} \end{pmatrix} \quad \text{and} \quad x'' = \begin{pmatrix} -\sqrt{|b|} \\ \sqrt{|a|} \end{pmatrix},$$

and the associated p-eigenvalue is $\sqrt{|ab|}$ (in both cases).

Proof. The calculations before the statement of the theorem may be considered as a constructive, "a-priori" proof. But with the appropriate values now in our hands, we shall formulate a significantly more simple "a-posteriori" proof. It is sufficient to consider only $x := x'$, and we may assume $a, b > 0$, since absolute value is taken everywhere in the norms.

$$\frac{\|Ax\|_p}{\|x\|_p} = \frac{\left\|(a\sqrt{b}, b\sqrt{a})^T\right\|_p}{\left\|(\sqrt{b}, \sqrt{a})^T\right\|_p} = \frac{\left\|\sqrt{ab}\cdot(\sqrt{a}, \sqrt{b})^T\right\|_p}{\left\|(\sqrt{b}, \sqrt{a})^T\right\|_p} = \sqrt{ab},$$

independent of p. So the effect of multiplying a p-eigenvector by the diagonal matrix is basically to switch the roles, invert the ratio of the elements of the vector.

Uniqueness (for vectors with positive elements) follows e.g. from the analysis of the function G above. ■

Corollary 1. *In case of the diagonal matrix $A \in \mathbb{R}^{n\times n}$ of Lemma 1, with $a, b \neq 0$ the set*

$$S := \bigcap_{p\in[1,\infty]} \mathcal{I}_p(A) \subset \mathbb{R}^2,$$

contains exactly 8 distinct elements, namely

$$S = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} -a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ -b \end{pmatrix}, \begin{pmatrix} b' \\ a' \end{pmatrix}, \begin{pmatrix} -b' \\ a' \end{pmatrix}, \begin{pmatrix} b' \\ -a' \end{pmatrix}, \begin{pmatrix} -b' \\ -a' \end{pmatrix} \right\},$$

where $a' = r \cdot \sqrt{|a|}$, $b' = r \cdot \sqrt{|b|}$, and $r = \sqrt{|ab|/(|a| + |b|)}$.

Proof. The first 4 elements of the set S correspond to the trivial p-eigenvectors. The second 4 elements are variants (with the signs of the elements) of the vector $x = (\sqrt{|b|}, \sqrt{|a|})^T$. According to the definition of the induction set (see Definition 1) and Lemma 1:

$$x = \begin{pmatrix} \sqrt{|b|} \\ \sqrt{|a|} \end{pmatrix}, \qquad \|x\|_2 = \sqrt{|a| + |b|}, \qquad \frac{\|Ax\|_p}{\|x\|_p} \equiv \sqrt{|ab|}.$$

Then

$$\frac{\|Ax\|_p}{\|x\|_p} \cdot \frac{x}{\|x\|_2} = \sqrt{\frac{|ab|}{|a|+|b|}} \cdot \begin{pmatrix} \sqrt{|b|} \\ \sqrt{|a|} \end{pmatrix} \in \mathcal{I}_p(A) \quad (p \in [1, \infty]),$$

thus this vector is in S. ■

Example 2. For the matrix $A_2 \in \mathbb{R}^{2\times 2}$ (with induction curves presented on Figure 1, right-hand-side), we have the independent non-trivial p-eigenvectors x' and x'' as follows:

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad x' = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad x'' = \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix},$$

and for the set of intersection of the induction curves we have

$$S = \left\{ \begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} \pm\sqrt{2/3} \\ \pm\sqrt{4/3} \end{pmatrix}, \right\}.$$

Figure 3 provides a visual feedback about the correctness of the calculation. Only a smaller portion of the plot is presented, zoomed in on the point of interest.

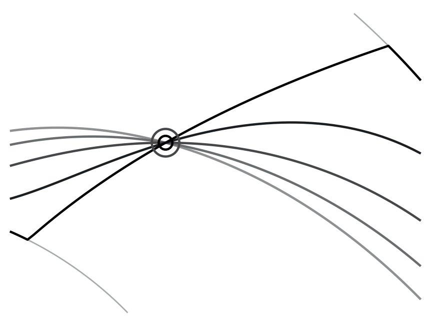

Figure 3. Visual verification of the calculation of the common intersection point (marked with circles) in case of the diagonal matrix as in Figure 1, close up.

Lemma 1 assumes that none of the two diagonal elements of the matrix is zero. The case when both of them are is trivial. However the case when exactly one of them is zero deserves some consideration.

Lemma 2. *Let $A \in \mathbb{R}^{2\times 2}$ be a diagonal matrix with one zero and one non-zero diagonal element. Then A has only trivial p-eigenvectors.*

Proof. Justify that the canonical unit vectors are trivial p-eigenvectors.

To show that no non-trivial p-eigenvectors exist, we follow the steps of the calculations in Section 3.1. Because of Propositions 2 and 3 we may consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad \text{thus} \quad Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

with $m > 0$. Note that the case when the other diagonal element is non-zero can be reduced to this case by switching elements of x too. Graphically this would mean a reflection on the diagonal line $x_1 = x_2$.

With the above notations it turns out that

$$f(p) = \left(\frac{1}{1+m^p} \right)^{1/p} =: F(p)^{1/p} \quad \text{and} \quad F'(p) = -\frac{m^p \ln(m)}{(1+m^p)^2}.$$

With $H(p) := p \cdot F'(p) - F(p) \cdot \ln F(p)$ again, we use $H(1) = 0$ to determine the solutions:

$$H(1) = -\frac{m \ln m}{(1+m)^2} - \frac{1}{1+m} \ln \left(\frac{1}{1+m} \right).$$

Since $(1+m)^2$ is still strictly positive (also greater than 1), so

$$H(1) = H_m(1) = \frac{G(m)}{(1+m)^2},$$

with

$$G(m) = -m \ln(m) + \ln(1+m) + m \ln(1+m).$$

We show that $G(m) > 0$ for all $m > 0$. Indeed

$$\lim_{m \to 0} G(m) = 0, \quad \text{and} \quad G'(m) = \ln \frac{1+m}{m} > 0 \quad (m > 0).$$

Thus there is no $m > 0$ for which the derivative of f equals 0 at $p = 1$, so there is no (non-trivial) vector with constant p-norm. ■

Example 3. The left-hand-side image in Figure 4 shows the induction curves for the matrix

$$A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

for the same p values as before. One may observe the trivial p-eigenvectors and the lack of non-trivial ones.

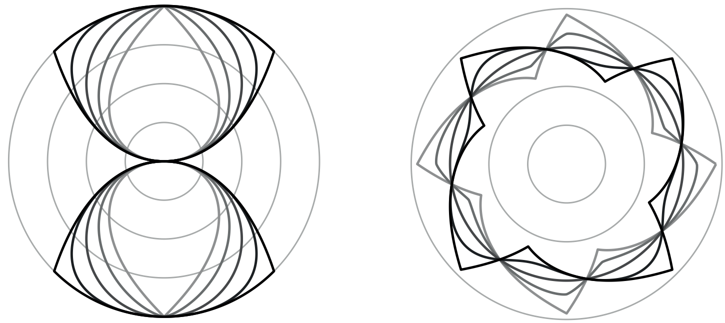

Figure 4. Some further examples for induction curves in $\mathbb{R}^2$ in case of a diagonal matrix (with only one non-zero element) and a rotation matrix (with scaling), with five p-norms. Circles denote the radial units. Shades of gray indicate curves for different p-norms.

Remark 5. The induction curve for $p = 1$ (plotted with light gray) looks similar to the Szegő curve [5, 14]. The Szegő curve arises as the limit curve where the zeros of the Taylor polynomials of the complex exponential function converge (scaled by $1/n$), originally pointing to the right. It is usually described using complex notions as the set $\left\{ z \in \mathbb{C} : \left| z\, e^{1-z} \right| = 1 \right\}$. A description using matrices, or its approximation with induction curves seem to be interesting questions.

3.2. General diagonal matrices

Lemma 1 may be generalized to diagonal matrices of $\mathbb{R}^{n\times n}$.

Theorem 1. *Let $D \in \mathbb{R}^{n\times n}$, $n \geq 2$ be a diagonal matrix with distinct non-zero elements $d_1, d_2, \ldots, d_n \in \mathbb{R} \setminus \{0\}$ in the diagonal. Apart from the n trivial p-eigenvectors, the canonical unit vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$, the following $n(n-1)$ vectors are further distinct non-trivial p-eigenvectors of D. For all $1 \leq k < l \leq n$:*

$$x'_{kl} = \sqrt{|d_l|} \cdot e_k + \sqrt{|d_k|} \cdot e_l \quad \text{and} \quad x''_{kl} = -\sqrt{|d_l|} \cdot e_k + \sqrt{|d_k|} \cdot e_l.$$

The associated p-eigenvalue for x'_{kl} and x''_{kl} is $\sqrt{|d_k d_l|}$ $(1 \leq k < l \leq n)$.

Proof. Basically we are considering the coordinate planes, and apply Lemma 1, since the effect of applying D to the vectors of the coordinate plane is the same as of a corresponding 2×2 diagonal matrix. Namely for a given pair

$1 \le k < l \le n$:

$$\begin{aligned} Dx'_{kl} &= d_k\sqrt{|d_l|}\cdot e_k + d_l\sqrt{|d_k|}\cdot e_l = \\ &= \sqrt{|d_k d_l|}\cdot\Big((\operatorname{sgn} d_k)\cdot\sqrt{|d_k|}\cdot e_k + (\operatorname{sgn} d_l)\cdot\sqrt{|d_l|}\cdot e_l\Big), \end{aligned}$$

thus

$$\|Dx'_{kl}\|_p = \sqrt{|d_k d_l|}\cdot\left(\sqrt{|d_k|}^{\,p} + \sqrt{|d_l|}^{\,p}\right)^{1/p} = \sqrt{|d_k d_l|}\cdot\|x'_{kl}\|_p$$

with the factor independent of p. The same goes for x''_{kl}.

The number of the above considered non-trivial p-eigenvectors is

$$2\cdot\binom{n}{2} = n(n-1). \qquad \blacksquare$$

Remark 6. Theorem 1 did not state that the listed p-eigenvectors are all the p-eigenvectors that exist in case of the diagonal matrix. It is posed as a question whether further non-trivial p-eigenvectors can be found that do not lie on a coordinate plane, i.e. have more than 2 non-zero components.

Remark 7. Theorem 1 assumes that the diagonal elements are all distinct, and non-zero. If some elements are equal, then the corresponding canonical unit vectors span a (regular) eigensubspace, therefore reducing the number of non-trivial p-eigenvectors of the above form. Similarly if some elements are zero, then the coordinate planes containing the corresponding canonical unit vectors will again not contain non-trivial p-eigenvectors (c.f. Lemma 2). Calculating the exact number of solutions lost these ways is left as an exercise. The combined case of having zeros and equal diagonal elements may also be considered.

Example 4. Consider the below matrix and vectors.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \text{ and } \begin{pmatrix} \pm 2 \\ \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 3 \\ 0 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 3 \\ \pm 2 \end{pmatrix}.$$

The listed vectors are p-eigenvectors of D with the associated p-eigenvalues $2, 3$ and 6 respectively. The set of all (known) p-eigenvalues is $\{1, 2, 3, 4, 6, 9\}$.

3.3. A note on linearity

While in case of (regular) eigenvectors it is true, that if v_1 and v_2 are linearly independent eigenvectors of a matrix associated with the same λ eigenvalue, then any (non-trivial) linear combination of these vectors is also an eigenvector with the same eigenvalue, thus one may speak of higher dimensional eigen-subspaces. However in case of (non-trivial) p-eigenvectors, this is not true in general, witnessed by the following counterexamples.

Example 5. Observe already in the most simple case of

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathbb{R}^{2\times 2} \quad \text{and} \quad x_1 = \begin{pmatrix} \sqrt{b} \\ \sqrt{a} \end{pmatrix}, \qquad x_2 = \begin{pmatrix} -\sqrt{b} \\ \sqrt{a} \end{pmatrix},$$

with $a, b > 0$, that x_1 and x_2 are both p-eigenvectors associated with the p-eigenvalue $\sqrt{ab}$ (c.f. Lemma 1), but $x_1 + x_2 = 2\sqrt{a} \cdot e_2$ is specifically a (trivial) p-eigenvector associated with the p-eigenvalue b, but $2 \cdot x_1 + x_2$ (or any other linear combination with coefficients of different absolute value) is not a p-eigenvector at all.

Example 6. As a numeric example of higher dimensions, examine

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in \mathbb{R}^{3\times 3} \quad \text{and} \quad x_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

The vectors x_1 and x_2 are p-eigenvectors both associated with the p-eigenvalue 2, but any linear combination of them with non-zero coefficients is not a p-eigenvector.

It is a question whether one could find such sets of non-trivial eigenvectors for some matrices, which are not composed of distinct points, directions, but rather have some positive measure, i.e. define an interval, a closed curve or area etc. These—if such exist—may be called *p-eigensets*. Or non-trivial p-eigenvectors may only be found as separate points?

3.4. Rotation

In this section real 2×2 matrices corresponding to rotations of the plane around the origin shall be considered. Let us use the following notation:

$$R(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in \mathbb{R}^{2\times 2} \qquad (\varphi \in \mathbb{R}).$$

$R(\varphi)$ corresponds to the rotation with angle φ in positive direction.

Theorem 2. *The matrix $R(\varphi) \in \mathbb{R}^{2\times 2}$ ($\varphi \in \mathbb{R}$, $\varphi \neq k \cdot \pi/2$, $k \in \mathbb{Z}$) has no trivial p-eigenvectors, but has 4 non-trivial p-eigenvectors as follows:*

$$x_k = \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix}, \quad \textit{with} \quad \theta_k = -\frac{\varphi}{2} + k \cdot \frac{\pi}{4} \qquad (k = 0, 1, 2, 3).$$

If $\varphi = k \cdot \pi/2$ for some $k \in \mathbb{Z}$ then every $0 \neq x \in \mathbb{R}^2$ is a p-eigenvector. The associated p-eigenvalue is always 1.

Remark. So in the first case the angles of the 4 p-eigenvectors are evenly distributed along the half unit circle, the 8 intersection points of the induction sets along the (full) unit circle.

Proof. Based on whether φ is a multiple of $\pi/2$ or not, we distinguish two cases.

Case 1. If $\varphi = k \cdot \pi/2$ for some $k \in \mathbb{Z}$, then there are 4 possibilities for the rotated vector $R(\varphi) \cdot x$ as follows:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{\,0\,\}\,,\; x' = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},\; x'' = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix},\; x''' = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}.$$

Thus $\|R(\varphi) \cdot x\|_p = \|x\|_p$ clearly holds in each case.

Case 2. If $\varphi \neq k \cdot \pi/2$, $k \in \mathbb{Z}$, then the vectors are rotated such that their directions always changes, no real valued eigenvectors are present. In this case the 4 p-eigenvectors may be divided into 2 groups based on their similar behavior

Case 2.1. Group 1 consists of vectors with $k = 0, 2$. In this case the multiplication with $R(\varphi)$ results in changing the sign of one component of the vector x_k. In case of $k = 0$, the calculation of $R(\varphi) \cdot x_0$ involving trigonometric identities, addition formulas goes as follows.

$$R(\varphi) \cdot \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix} = R(\varphi) \cdot \begin{pmatrix} \cos(-\varphi/2) \\ \sin(-\varphi/2) \end{pmatrix} = \begin{pmatrix} \cos(\varphi/2) \\ \sin(\varphi/2) \end{pmatrix} = \begin{pmatrix} \cos\theta_0 \\ -\sin\theta_0 \end{pmatrix}.$$

In case of $k = 2$ we have

$$R(\varphi) \cdot \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = R(\varphi) \cdot \begin{pmatrix} \cos\left(\dfrac{\pi}{2} - \dfrac{\varphi}{2}\right) \\ \sin\left(\dfrac{\pi}{2} - \dfrac{\varphi}{2}\right) \end{pmatrix} = \begin{pmatrix} -\sin(\varphi/2) \\ \cos(\varphi/2) \end{pmatrix} = \begin{pmatrix} -\cos\theta_2 \\ \sin\theta_2 \end{pmatrix}.$$

Clearly no p-norm changes by the transformation.

Case 2.2. The other group consists of vectors with $k = 1, 3$. The multiplication with $R(\varphi)$ results now in switching the components of the vector x_k (while the signs may also change). Briefly summarizing the calculation for $k = 1$, using also the identity for complementary angles in the form $\cos(\pi/4 - \varphi/2) = \sin(\pi/4 + \varphi/2)$ and vice versa we have

$$R(\varphi) \cdot \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = R(\varphi) \cdot \begin{pmatrix} \cos\left(\dfrac{\pi}{4} - \dfrac{\varphi}{2}\right) \\ \sin\left(\dfrac{\pi}{4} - \dfrac{\varphi}{2}\right) \end{pmatrix} = \begin{pmatrix} \sin\left(\dfrac{\pi}{4} - \dfrac{\varphi}{2}\right) \\ \cos\left(\dfrac{\pi}{4} - \dfrac{\varphi}{2}\right) \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \end{pmatrix}.$$

And for $k = 3$ the result is as follows:

$$R(\varphi) \cdot \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} = R(\varphi) \cdot \begin{pmatrix} \cos\left(\frac{3\pi}{4} - \frac{\varphi}{2}\right) \\ \sin\left(\frac{3\pi}{4} - \frac{\varphi}{2}\right) \end{pmatrix} = \begin{pmatrix} -\sin\theta_3 \\ \cos\theta_3 \end{pmatrix}.$$

Again, clearly no p-norm changes by the transformation. Imagining the act of rotation for the given vectors with a given angles is encouraged. ■

Scaling may also be involved aside a rotation. As a direct consequence of Theorem 2 and Proposition 3 we arrive at the following result.

Proposition 4. *The matrix $c \cdot R(\varphi) \in \mathbb{R}^{2\times 2}$ ($0 \neq c \in \mathbb{R}$, $\varphi \in \mathbb{R}$, $\varphi \neq k \cdot \pi/2$, $k \in \mathbb{Z}$) has no trivial p-eigenvectors, but has the same 4 non-trivial p-eigenvectors as $R(\varphi)$. If $\varphi = k \cdot \pi/2$ for some $k \in \mathbb{Z}$ then every $0 \neq x \in \mathbb{R}^2$ is a p-eigenvector. The associated p-eigenvalue is always $|c|$.*

Example 7. First we refer again to the left-hand-side image in Figure 1 with the matrix corresponding to rotation with 45 degrees. And as a second example the right-hand-side image in Figure 4 shows the induction curves for the matrix

$$A_5 = 3 \cdot R(-\pi/9)$$

for the same p values as before. (Both together with a scaling factor.) Note the 8 intersection points spread equally along the circle. In case of A_5 with $p = 2$ the induction curve is a circle of radius 3.

4. Conclusions and further research

In depth examination of the fraction in the definition of power norms led us to the definition of *induction sets*. Along this paper we presented many examples in case of $\mathbb{R}^2$, i.e. induction curves. Noticing common intersection points of these motivated the definition of *p-eigenvectors,* for which the above mentioned fraction is constant, independent of the parameter of the power norm (or quasi-norm).

These examinations seem to be neglected, overlooked so far in the treatise of related subjects, no previous results were found in the literature. Thus the author claims the introduction and examination of induction sets and p-eigenvectors as his own new results.

We discussed basic general properties of p-eigenvectors (Propositions of Section 2.2), and constructed exact solutions for diagonal matrices (Lemmas 1, 2

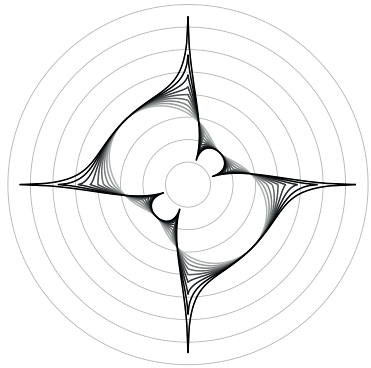

Figure 5. An artistic utilization of induction curves with quasi-norms also involved, resembling a butterfly.

and Theorem 1) and rotation matrices, possibly with a scaling factor (Theorem 2 and Proposition 4). Some corollaries and many examples were given.

The scripts and other software created during the research are available to download at:

http://numanal.inf.elte.hu/~locsi/indsets/

Throughout the article we mentioned several questions yet unsolved. To sum these up:

- Detailed description and analysis of induction curves, surfaces etc. for some (classes) of matrices may be carried out.
- How can we characterize the p-eigenvectors of further 2×2 matrices? Is a full description for $\mathbb{R}^{2\times 2}$ possible?
- Are there any further p-eigenvectors for general diagonal matrices not stated in Theorem 1?
- The existence and construction problem of non-trivial p-eigenvectors for general matrices of $\mathbb{R}^{n\times n}$ ($n \geq 2$). May an analytic form of non-trivial p-eigenvectors exist, or only numerical estimation can be given?
- We restricted the discussion to real valued matrices and vectors. What can be stated about the generalization to complex numbers?
- Is the resemblance in Figure 4 to the Szegő curve [5, 14] just by chance, or could any relations be found?
- Would *p-eigensets* be always composed of discrete directions?

Based on our observations, we make the following conjectures.

Conjecture 1. *Every* $A \in \mathbb{R}^{2\times 2}$ *invertible matrix has exactly* 4 *distinct* p*-eigendirections* (*including trivial ones*).

Conjecture 2. *Every* $A \in \mathbb{R}^{n\times n}$ $(n \geq 3)$ *invertible matrix has (non-trivial)* p*-eigendirections.*

Furthermore we mention the following possible directions of further investigations.

- Examine special classes of matrices (e.g. orthogonal, diagonalizable, symmetric, projection).
- We already have some basic graphics and observations in $\mathbb{R}^3$. Aside theoretical questions, this also leads to the problem of proper visualization of induction surfaces, possibly connected to the Thomson problem [6, 15]. This problem is concerned with the optimal constallation of a number of particles with equal charge on the unit sphere. As the usual parametrization of the sphere provides varying density of discretization points along the surface, visualization of induction surfaces utilizing (approximate) solutions of the Thomson problem may have some advantages.
- May these questions be further generalized to the case of infinite dimensional (e.g. integral) operators? More specific questions may arise in case of Fourier, Gabor or wavelet transforms.
- Further classes of norms may be investigated.
- Inverse problem: could the elements of the matrix and the used parameter p be reconstructed given the induction curve/surface/manifold, or a sampled subset of such an object.
- Application perspectives of these curves, functions e.g. in case of signal processing and sound design are to be explored.

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