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AN INEQUALITY FOR THE AREAS OF PONCELET TRIANGLES

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Abstract. We prove an inequality for areas of Poncelet triangles, which holds between the integral mean and the arithmetical mean of the largest and smallest areas.

1. Introduction

Euler's theorem [1] states that given a triangle, the circumradius R, the inradius r and the distance d between their centers satisfy

$$d^2 = R(R-2r). \tag{1.1}$$

Conversely, the celebrated theorem of Poncelet [2] guarantees that if discs $D_1 \subset D_2$ are given with radii r, R and distance d between their centers, with Euler's equality (1.1) holding, then there are infinitely many such triangles, with an *arbitrarily chosen* vertex on $C_2 \equiv \partial(D_2)$, the boundary of D_2.

Since the areas of these triangles may be different, it is worth examining their average measure. To this we parametrize the triangles in question. First we chose the points of the outer circle (as vertices of the 'sandwiched' triangles), however, then we hit on the paper [3] of Mirko Radić entitled *Extreme Areas of Triangles in Poncelets Closure Theorem*. He took the length of the tangent

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from a point of C_2 to C_1 as parameter, which enabled him to formulate the results in a simple way, this is why we use here those notations, and also some of his results.

Note finally that an inequality of the form

$$\frac{1}{b-a}\int_a^b f \ \leq \ \frac{1}{2}\Big(f(a)+f(b)\Big)$$

is obviously true for convex functions $f:[a,b]\to\mathbb{R}$, see e.g. [4], however our function $f=J$ will not be convex - in fact, it is equioscillating (cf. Figure 2).

2. Preliminaries

We illustrate the Poncelet situation by an example to start with.

Example 2.1. Let our circles be $C_1 = C([1,1],1)$ and $C_2 = C([2,\frac{3}{2}],\frac{5}{2})$. Then the distance of the centers is $d=\frac{\sqrt{5}}{2}$, and Euler's condition is fulfilled, thus given any point on C_2, drawing the tangent to C_1, extending the segment to C_2 etc. will take us back to the initial point in three steps. The triangle obtained in this way has C_1 and C_2 as incircle and circumcircle, resp.

Figure 1. below shows the shortest and longest tangents t_m and t_M. Note that for this we joined the centers of the circles by a straight line and took the intersection points of this line with the outer circle C_2 as starting points. At the same time we obtain as in [3]

$$t_m=\sqrt{(R-d)^2-r^2}, \qquad t_M=\sqrt{(R+d)^2-r^2}.$$

As a surplus, the vertices $[0,0],[4,0],[0,3]$ of the Pythagorean triangle – one of the sandwiched triangles – are marked with '+' sign. For this one the area is easy to calculate.

Adopting the notations in [3], let $t_1+t_2,\ t_2+t_3,\ t_3+t_1$ be the sides of a Poncelet triangle, i.e. of a triangle with C_1 and C_2 as incircle and circumcircle. We choose then t_1 as parameter, and assign to it the area of the associated Poncelet triangle, i.e. we define

$$J(t_1)=r\,(t_1+t_2+t_3).$$

Remark 2.1. Knowing the incenter, the quantities $\{t_i\}$ are easily determined. Then we apply the known formula "area = inradius · semiperimeter". To get

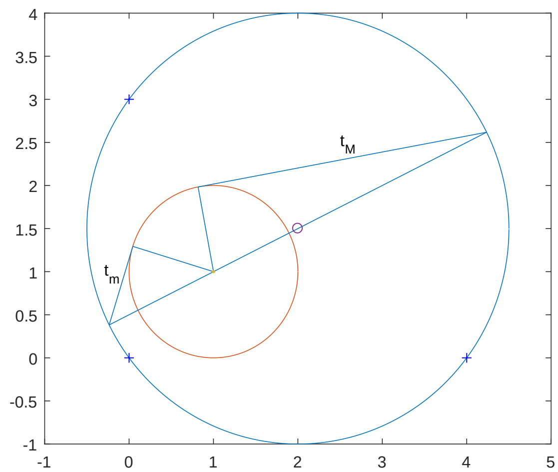

Figure 1. The shortest and longest tangents from C_2 to C_1.

the right-angled triangle mentioned, we choose the origin as a starting point laying on C_2, then we have $t_1 = 1$, with further parameters 2 and 3, resp.

The author of [3] rewrote the area function J, by calculating t_2 and t_3, as

$$J(t_1) = r\,t_1\left(1 + \frac{4Rr}{r^2 + t_1^2}\right),$$

and proved the following.

Theorem 2.1. ([3]) $J(t_m) \le J(t_1) \le J(t_M)$ *for* $t_m \le t_1 \le t_M$.

Observe that the function $J(\cdot)$ has a special graph: given an intermediate value f^* with $J(t_m) \le f^* \le J(t_M)$, there are (including multiplicity) exactly three arguments t_i, with $J(t_i) = f^*$ holding. The reason is that the vertices of a Poncelet triangle yield the same triangle, hence the same area; in other words, the function values at t_1, t_2, t_3 are identical. For $f^* = J(t_m)$ and $f^* = J(t_M)$ two of three values necessarily coincide due to the fact that the actual triangle is then obviously isosceles.

Figure 2 shows the function J for the quantities given in the Example above. (To be more informative, we drew the line related to the rectangular triangle. It is seen that the value 6 is obtained for $t = 1, 2, 3$. Also, we visualized the inflection pont with abscissa $t = r\sqrt{3} = \sqrt{3}$ and ordinate $\frac{7}{2}\sqrt{3} \approx 6.0621778$. Note, however, that a rectangular Poncelet triangle does not always exist.)

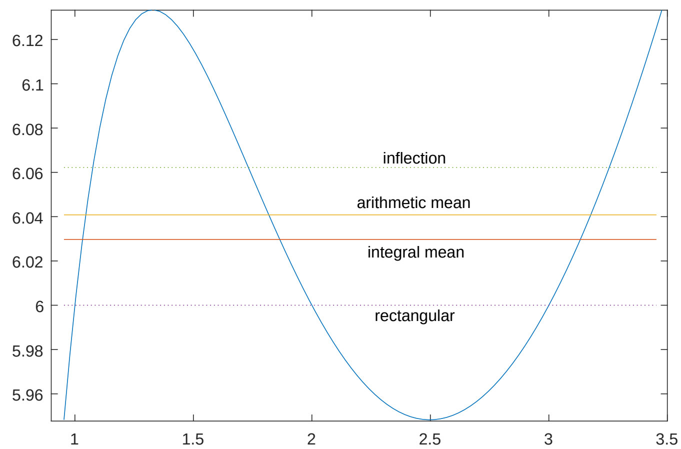

Figure 2. The area function for the above data.

3. The main theorem

Hereby we are ready to formulate our main statement.

Theorem 3.1. *With the previous notations we have*

$$\frac{1}{t_M - t_m} \int_{t_m}^{t_M} J \;\leq\; \tfrac{1}{2}\Big(J(t_m) + J(t_M)\Big),$$

i.e. the integral mean of J does not exceed the arithmetic mean of the extrema.

Remark 3.1. Before proving the theorem, observe that we do not integrate with respect to the angle of rotation, as usual, but with respect to the length t of the tangent drawn from the actual point of C_2 to the circle C_1, while this actual point runs along the larger circle C_2.

Proof. We will prove the theorem in the multiplied form

$$\int_{t_m}^{t_M} J \;\leq \tfrac{1}{2}\,(t_M - t_m)\Big(J(t_m) + J(t_M)\Big), \tag{3.1}$$

while, for simplicity, we write t for t_1. The primitive function (in other words, the antiderivative) of J is determined to be

$$\tfrac{1}{2}rt^2 + 2Rr^2\ln(r^2 + t^2).$$

Thus, integrating between the limits t_m and t_M, we obtain

$$\int_{t_m}^{t_M} J = \tfrac{r}{2}(t_M^2 - t_m^2) + 2Rr^2\Big(\ln(r^2 + t_M^2) - \ln(r^2 + t_m^2)\Big)$$

with first term

$$\tfrac{r}{2}(t_M^2 - t_m^2) = \tfrac{r}{2}\Big(\big[(R+d)^2 - r^2\big] - \big[(R-d)^2 - r^2\big]\Big) = 2Rrd.$$

The other term also can be simplified, owing to an elementary property of the logarithm function, yielding

$$\int_{t_m}^{t_M} J = 2Rrd + 4Rr^2 \ln \frac{R+d}{R-d}.$$

Next we examine the right hand side of (3.1). Note that multiplying by the length $(t_M - t_m)$ of the integration interval is practical because then quadratic terms in t_m and t_M will occur:

$$t_M^2 = \frac{(R+d)^3(3R-d)}{4R^2}, \quad t_m^2 = \frac{(R-d)^3(3R+d)}{4R^2}.$$

At the same time we calculate

$$(t_M t_m)^2 = t_M^2 t_m^2 = \frac{(R^2-d^2)^3(9R^2-d^2)}{16R^4}.$$

By definition of J it holds that

$$J(t_M) + J(t_m) = r(t_M + t_m) + 4Rr^2\left(\frac{t_M}{(R+d)^2} + \frac{t_m}{(R-d)^2}\right),$$

therefore a multiplication through $\frac{1}{2}(t_M - t_m)$ gives

$$\begin{aligned}\tfrac{1}{2}\,(t_M - t_m)\Big(J(t_m) + J(t_M)\Big) &= \tfrac{r}{2}(t_M^2 - t_m^2) + \\ +2Rr^2(t_M - t_m)&\left(\frac{t_M}{(R+d)^2} + \frac{t_m}{(R-d)^2}\right).\end{aligned}$$

The first term $\frac{r}{2}(t_M^2 - t_m^2)$ is identical with the first term of the integral in (3.1), hence our statement simplifies to proving

$$2\ln\frac{R+d}{R-d} \le (t_M - t_m)\left(\frac{t_M}{(R+d)^2} + \frac{t_m}{(R-d)^2}\right).$$

The right hand side here can be rewritten as

$$\frac{t_M^2}{(R+d)^2} - \frac{t_m^2}{(R-d)^2} + t_M t_m \left(\frac{1}{(R-d)^2} - \frac{1}{(R+d)^2} \right) =$$
$$= \frac{(R+d)(3R-d) - (R-d)(3R+d)}{4R^2} + \frac{4Rd\sqrt{9R^2-d^2}}{4R^2\sqrt{R^2-d^2}} =$$
$$= \frac{d}{R}\left(1 + \sqrt{\frac{9R^2-d^2}{R^2-d^2}}\right).$$

Thus our task reduces to showing the inequality

$$2\ln\frac{R+d}{R-d} \le \frac{d}{R}\left(1 + \sqrt{\frac{9R^2-d^2}{R^2-d^2}}\right).$$

Using its homogeneity, we can substitute

$$x = \frac{d}{R}$$

to arrive at the problem of proving the one-variable inequality

$$\varphi(x) = x\left(1 + \sqrt{\frac{9-x^2}{1-x^2}}\right) - 2\ln\frac{1+x}{1-x} \ge 0, \quad 0 < x < 1.$$

Since $\varphi(0) = 0$, it suffices to show that φ is monotone increasing. We have

$$\varphi'(x) = \frac{x^4 - 2x^2 + 9 + \sqrt{\frac{9-x^2}{1-x^2}}\left(x^4 + 2x^2 - 3\right)}{(1-x^2)^2\sqrt{\frac{9-x^2}{1-x^2}}}.$$

The denominator is positive, and the positivity of the numerator is equivalent with the validity of

$$\frac{x^4 - 2x^2 + 9}{3 - 2x^2 - x^4} > \sqrt{\frac{9-x^2}{1-x^2}}.$$

Squaring this – and rearranging the terms – gives

$$(1-x^2)(x^4 - 2x^2 + 9)^2 > (9-x^2)(3 - 2x^2 - x^4)^2,$$

where the difference of the left and right hand sides is just

$$64\,x^4(1-x^2),$$

which is positive for $0 < x < 1$. The theorem is proved. ■

Question. We wonder if a similar inequality theorem holds for quadrilaterals, pentagons, etc., i.e. for (convex) Poncelet polygons in general.

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