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A NOTE ON A RESULT OF B. BOJAN

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Abstract. We determine all solutions of those functions $f : \mathbb{N} \to \mathbb{C}$ for which $f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K$ is satisfied for all positive integers n, m, where a non-negative integer k and $K \in \mathbb{C}$ are given. This improves the result of B. Bojan.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the set positive integers, integers and complex numbers, respectively.

In 2014 B. Bojan [1] determined all solutions of those $f : \mathbb{N} \to \mathbb{C}$ for which

$$f(n^2 + m^2) = f^2(n) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N},$$

namely

$$f(n) \in \{0, \pm\frac{1}{2}, \pm n\}.$$

Our purpose in this note is to improve the result of B. Bojan [1].

Theorem. *Assume that a non-negative integer k, a complex number K and an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation*

$$f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K \quad \textit{for every} \quad n, m \in \mathbb{N}.$$

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Then one of the following assertions holds:

a) $f(n) = \pm\chi_2(n), \quad f(n^2+m^2+k) = \chi_2(n) + \chi_2(m) - 1 \quad$ *and* $\quad 2 \nmid k, \quad K = -1,$

b) $f(n) = \pm\chi_2(n+1), \quad f(n^2+m^2+k) = \chi_2(n+1) + \chi_2(m+1) - 1 \quad$ *and* $\quad 2 \mid k, \quad K = -1,$

c) $f(n) = \dfrac{\varepsilon(n)}{4}(1 - \sqrt{-8K+1}),$

d) $f(n) = \dfrac{\varepsilon(n)}{4}(1 + \sqrt{-8K+1}),$

e) $f(n) = \dfrac{\varepsilon(n)}{4} a(n)\sqrt{-8\chi_3(n)+9} \quad$ *and* $\quad K = -\dfrac{7}{8}$ for $\mathrm{k} \equiv 1 \pmod 3,$

f) $f(n) = \varepsilon(n) b(n) \chi_3(n) \quad$ *and* $\quad K = -1$ for $\mathrm{k} \equiv 2 \pmod 3,$

g) $f(n) = \dfrac{\varepsilon(n)}{4} b(n)\sqrt{-8\chi_3(n)+9} \quad$ *and* $\quad K = -\dfrac{3}{8}$ for $\mathrm{k} \equiv 0 \pmod 3.$

h) $f(n) = \varepsilon(n) n,$

where $\chi_2(n) \pmod 2$, $\chi_3(n) \pmod 3$ *are principal Dirichlet characters,* $\varepsilon(n) \in \in \{-1, 1\}$ *and* $\varepsilon(n^2+m^2+k) = 1$ *for every* $n, m \in \mathbb{N}$, $a(n)$, $b(n)$ *are periodic sequences* $\pmod 3$ *with* $a(1) = 1$, $a(2) = a(3) = -1$, $b(1) = b(3) = 1$, $b(2) = -1$.

2. Lemmas

Assume that a non-negative integer k, a complex number K and an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation

$$f(n^2+m^2+k) = f^2(n) + f^2(m) + K \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{2.1}$$

Lemma 1. *Let* $F(n) := f^2(n)$. *Then*

$$\begin{aligned} F(\ell+12) = F(\ell+9) + F(\ell+8) + F(\ell+7) - \\ - F(\ell+5) - F(\ell+4) - F(\ell+3) + F(\ell) \end{aligned} \tag{2.2}$$

holds for every $\ell \in \mathbb{N}$ *and*

$$\begin{cases} F(7) = 2F(5) - F(1), \\ F(8) = 2F(5) + F(4) - 2F(1), \\ F(9) = F(6) + 2F(5) - F(2) - F(1), \end{cases} \tag{2.3}$$

$$\text{(2.3)} \qquad \begin{cases} F(10) = F(6) + 3F(5) - F(3) - 2F(1), \\ F(11) = F(6) + 4F(5) - F(3) - F(2) - 2F(1), \\ F(12) = F(6) + 4F(5) + F(4) - F(2) - 4F(1). \end{cases}$$

Proof. We infer from (2.1) and the next relations

$$(2n+1)^2+(n-2)^2 = (2n-1)^2+(n+2)^2, \ (2n+3)^2+(n-6)^2 = (2n-3)^2+(n+6)^2$$

that

$$f\Big[(2n+1)^2 + (n-2)^2 + k\Big] = f\Big[(2n-1)^2 + (n+2)^2 + k\Big]$$

and

$$f\Big[(2n+3)^2 + (n-6)^2 + k\Big] = f\Big[(2n-3)^2 + (n+6)^2 + k\Big].$$

These imply that

$$F(2n+1) + F(n-2) = F(2n-1) + F(n+2)$$

and

$$F(2n+3) + F(n-6) = F(2n-3) + F(n+6).$$

We infer from the last relations that

$$\begin{aligned} F(n+6) - F(n-6) &= F(2n+3) - F(2n-3) = \\ &= \big[F(2n+3) - F(2n+1)\big] + \big[F(2n+1) - F(2n-1)\big] + \\ &\quad + \big[F(2n-1) - F(2n-3)\big] = \\ &= \big[F(n+3) - F(n-1)\big] + \big[F(n+2) - F(n-2)\big] + \\ &\quad + \big[F(n+1) - F(n-3)\big], \end{aligned}$$

consequently

$$\begin{aligned} F(n+6) = F(n+3) + F(n+2) + F(n+1) - F(n-1) - \\ - F(n-2) - F(n-3) + F(n-6). \end{aligned}$$

This with $m = \ell + 6$ proves (2.2).

In the order to prove (2.3), we note from (2.1) that

$$\text{(2.4)} \qquad \text{If } \ x^2 + y^2 = u^2 + v^2, \ \text{ then } \ \mathrm{F(x) + F(y) = F(u) + F(v)}.$$

Applications of (2.4) in the cases

$$\begin{aligned} (x, y, u, v) \in \big\{&(5, 5, 7, 1), \ (7, 4, 8, 1), \ (7, 6, 9, 2), \\ &(9, 7, 11, 3), \ (9, 8, 12, 1), \ (10, 5, 11, 2)\big\} \end{aligned}$$

prove that (2.3) holds for $F(7), F(8), F(9), F(11), F(12)$ and $F(10)$.

Thus, (2.3) and Lemma 1 is proved. ■

Lemma 2. *Assume that the function $F:\mathbb{N}\to\mathbb{C}$ satisfy* (2.2) *and* (2.3). *Let*

$$\begin{aligned}
A :=& \frac{1}{120}\Big(F(6)+4F(5)-F(3)-F(2)-3F(1)\Big),\\
\Gamma_2 :=& \frac{-1}{8}\Big(F(6)-4F(5)+4F(4)-F(3)+3F(2)-3F(1)\Big),\\
\Gamma_3 :=& \frac{-1}{3}\Big(F(6)-2F(5)+2F(3)-F(2)\Big),\\
\Gamma_4 :=& \frac{1}{4}\Big(F(6)-2F(4)-F(3)+F(2)+F(1)\Big),\\
\Gamma_5 :=& \frac{1}{5}\Big(F(6)-F(5)-F(3)-F(2)+2F(1)\Big),\\
\Gamma :=& \frac{1}{4}\Big(F(6)-4F(5)+2F(4)+3F(3)+F(2)+F(1)\Big),\\
B_k :=& \Gamma_2\chi_2(k)+\Gamma_3\chi_3(k)+\Gamma_4\chi_4(k-1)+\Gamma_5\chi_5(k)+\Gamma,
\end{aligned}$$

where $\chi_2(k) \pmod 2$, $\chi_3(k) \pmod 3$ *are the principal Dirichlet characters and* $\chi_4(k) \pmod 4$, $\chi_5(k) \pmod 5$ *are the real, non-principal Dirichlet characters, i.e.*

$$\begin{gathered}
\chi_2(0)=0, \chi_2(1)=1, \chi_3(0)=0, \chi_3(1)=\chi_3(2)=1,\\
\chi_4(0)=\chi_4(2)=0, \chi_4(1)=1, \chi_4(3)=-1,\\
\chi_5(2)=\chi_5(3)=-1, \chi_5(1)=\chi_5(4)=1.
\end{gathered}$$

Then we have

$$F(\ell)=A\ell^2+B_\ell \quad \textit{for every} \quad \ell\in\mathbb{N}. \tag{2.5}$$

Proof. The proof is very similar to that of Lemma 2 in [4]. Here we omit the proof. ∎

3. Proof of the Theorem

Assume that a non-negative integer k, a complex number K and an arithmetical function $f:\mathbb{N}\to\mathbb{C}$ satisfy the equation (2.1). Let $F(n):=f^2(n)$ for every $n\in\mathbb{N}$. From (2.1) and Lemma 2, we have

$$F(n^2+m^2+k)=(F(n)+F(m)+K)^2 \tag{3.1}$$

and

$$F(\ell)=f^2(\ell)=A\ell^2+B_\ell \quad \text{for every} \quad \ell\in\mathbb{N}, \tag{3.2}$$

where

$$B_\ell := \Gamma_2\chi_2(\ell) + \Gamma_3\chi_3(\ell) + \Gamma_4\chi_4(\ell-1) + \Gamma_5\chi_5(\ell) + \Gamma.$$

Lemma 3. *We have*

$$A \in \{0, 1\}.$$

Proof. We obtain from (3.1) and (3.2) that

$$A(n^2+m^2+k)^2 + B_{n^2+m^2+k} = \Big(A(n^2+m^2) + B_n + B_m + K\Big)^2 \tag{3.3}$$

for every $n, m \in \mathbb{N}$. Since

$$|B_\ell| \le |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma| \quad \text{for every} \quad \ell \in \mathbb{N}$$

and

$$n^2 + m^2 + k \to \infty \quad \text{as} \quad n, m \to \infty,$$

we infer from (3.3) that

$$\begin{aligned} A &= \lim_{n,m\to\infty} \left[A\Big(1 + \frac{k}{n^2+m^2+k}\Big)^2 + \frac{B_{n^2+m^2+k}}{(n^2+m^2+k)^2} \right] = \\ &= \lim_{n,m\to\infty} \left[\Big(A + \frac{-k + B_n + B_m + K}{n^2+m^2+k}\Big)^2 \right] = A^2. \end{aligned}$$

Therefore, we have $A \in \{0, 1\}$. Lemma 3 is proved. ∎

Lemma 4. *Assume that $A = 1$, i.e.*

$$F(\ell) = f^2(\ell) = \ell^2 + B_\ell \quad \textit{for every} \quad \ell \in \mathbb{N},$$

then

$$B_m = 0, \quad f(m) = \pm m \quad \textit{and} \quad f(n^2+m^2+k) = n^2+m^2+k$$

for every $n, m \in \mathbb{N}$.

Proof. From (3.3) we obtain that

$$\begin{aligned} &(n^2+m^2+k)^2 + B_{n^2+m^2+k} - \\ &- \Big((n^2+m^2) + B_n + B_m + K\Big)^2 = \\ &= 2(k - B_n - B_m - K)n^2 + W(n,m) = 0, \end{aligned} \tag{3.4}$$

holds for every $n, m \in \mathbb{N}$, where

$$W(n,m) := B_{n^2+m^2+k} + (m^2+k)^2 - (m^2 + B_n + B_m + K)^2. \tag{3.5}$$

Now let $m \in \mathbb{N}$ be fixed, $n \in \mathbb{N}$, $n \equiv a \pmod{60}$ with some $a \in \mathbb{N}$, $0 \leq a < 60$. Since the sequence $\{B_\ell\}_0^\infty$ is periodic $\pmod{60}$, we have

$$|W(n,m)| = |W(a,m)| < \infty$$

and so we obtain from (3.4) that

$$k - B_a - K - B_m = \lim_{\substack{n\to\infty \\ n\equiv a \pmod{60}}} \frac{-W(n,m)}{2n^2} = 0$$

and

$$W(a,m) = 0$$

hold for each $a, m \in \mathbb{N}$. It follows from these that

$$B_m = k - B_a - K = c \quad \text{for every} \quad m \in \mathbb{N}$$

and

$$W(a,m) := c + (m^2+k)^2 - (m^2+2c+K)^2 = 0 \quad \text{for every} \quad m \in \mathbb{N},$$

where $c \in \mathbb{C}$ is some fixed constant. These show that

$$k - K = 2c \quad \text{and} \quad c + k^2 - (2c+K)^2 = 0. \tag{3.6}$$

Finally, we obtain from the last relations that

$$k = 2c + K \quad \text{and} \quad 0 = c + k^2 - (2c+K)^2 = c + k^2 - k^2 = c.$$

Thus we have

$$B_m = c = 0 \quad \text{and} \quad F(m) = m^2 \quad \text{for every} \quad m \in \mathbb{N}$$

and from (3.6) we have $k = K$. Thus, in the case $A = 1$, we proved that

$$f^2(m) = m^2, f(n) = \pm m$$

and

$$f(n^2+m^2+k) = f^2(n) + f^2(m) + K = n^2 + m^2 + k$$

for every $n, m \in \mathbb{N}$. Lemma 4 is proved. ■

Lemma 5. *Assume that $A = 0$. Then*

$$\Gamma_4 = 0, \tag{3.7}$$

where Γ_4 is defined in Lemma 2.

Proof. We infer from the fact $A=0$ and from (3.1), (3.2) that

$$B_{n^2+m^2+k}=(B_n+B_m+K)^2 \quad \text{for every} \quad n,m\in\mathbb{N}. \tag{3.8}$$

Since $\{B_\ell\}_0^\infty$ is periodic $\pmod{60}$, therefore

$$B_{2^2+2^2+k}=B_{8^2+2^2+k} \quad \text{and} \quad B_{2^2+2^2+k}=B_{8^2+8^2+k},$$

which with (3.8) imply that

$$(B_2+B_2+K)^2=(B_8+B_2+K)^2 \quad \text{and} \quad (B_2+B_2+K)^2=(B_8+B_8+K)^2.$$

Consequently

$$(B_2-B_8)\Big(3B_2+B_8+2K\Big)=0 \tag{3.9}$$

and

$$4(B_2-B_8)\Big(B_2+B_8+K\Big)=0. \tag{3.10}$$

Since

$$B_2=\Gamma_3+\Gamma_4-\Gamma_5+\Gamma \quad \text{and} \quad B_8=\Gamma_3-\Gamma_4-\Gamma_5+\Gamma,$$

we have $\Gamma_4=0$ if $B_2=B_8$. Assume now that $B_2-B_8\neq 0$. Then we infer from (3.9) and (3.10) that

$$3B_2+B_8+2K=0 \quad \text{and} \quad B_2+B_8+K=0,$$

consequently

$$B_2=B_8.$$

This contradicts to the assumption $B_2-B_8\neq 0$. Thus, $\Gamma_4=0$ follows and the proof of Lemma 5 is completed. ■

Lemma 6. *Assume that $A=0$, $\Gamma_4=0$ and $\Gamma_2\neq 0$. Then $K=-1$, furthermore either*

$$2\nmid k,\quad f(n)=\pm\chi_2(n) \quad \textit{and} \quad f(n^2+m^2+k)=\chi_2(n)+\chi_2(m)-1$$

or

$$2\mid k,\quad f(n)=\pm\chi_2(n+1) \quad \textit{and}$$
$$f(n^2+m^2+k)=\chi_2(n+1)+\chi_2(m+1)-1$$

for every $n,m\in\mathbb{N}$.

Proof. Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 \neq 0$. Then

$$B_k := \Gamma_2\chi_2(k) + \Gamma_3\chi_3(k) + \Gamma_5\chi_5(k) + \Gamma \quad \text{for every} \quad k \in \mathbb{N}, \tag{3.11}$$

and

$$B_n = B_m \quad \text{if} \quad n \equiv m \pmod{30}. \tag{3.12}$$

It is clear from (3.12) that if $n^2 + m^2 \equiv u^2 + v^2 \pmod{30}$, then

$$n^2 + m^2 + k \equiv u^2 + v^2 + k \pmod{30} \quad \text{and} \quad B_{n^2+m^2+k} = B_{u^2+v^2+k}.$$

Let

$$\begin{aligned} E(n,m,u,v) &:= (B_n + B_m + K)^2 - (B_u + B_v + K)^2 = \\ &= (B_n + B_m - B_u - B_v)(B_n + B_m + B_u + B_v + 2K). \end{aligned} \tag{3.13}$$

We obtain from the above relations and from (3.8) that

$$\text{If} \quad \mathrm{n}^2 + \mathrm{m}^2 \equiv \mathrm{u}^2 + \mathrm{v}^2 \pmod{30}, \quad \text{then} \quad \mathrm{E}(\mathrm{n},\mathrm{m},\mathrm{u},\mathrm{v}) = 0. \tag{3.14}$$

With the help of a computer and Maple program, we find those positive integers n, m, u, v such that $n^2 + m^2 \equiv u^2 + v^2 \pmod{30}$, therefore we obtain the equations $E(n, m, u, v) = 0$.

It is clear from (3.11) that

$$\begin{cases} B_1 = \Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma, \\ B_2 = \Gamma_3 - \Gamma_5 + \Gamma, \\ B_3 = \Gamma_2 - \Gamma_5 + \Gamma, \\ B_4 = \Gamma_3 + \Gamma_5 + \Gamma, \\ B_5 = \Gamma_2 + \Gamma_3 + \Gamma, \\ B_6 = \Gamma_5 + \Gamma. \end{cases}$$

By using (3.13), we infer from these and from the following relations

$$\begin{cases} E(1,1,4,4) \;\;= 4\Gamma_2(2\Gamma + 2\Gamma_5 + 2\Gamma_3 + \Gamma_2 + K) = 0, \\ E(1,5,4,10) = 4\Gamma_2(2\Gamma + \Gamma_5 + 2\Gamma_3 + \Gamma_2 + K) = 0, \\ E(1,3,2,6) \;\;= 4\Gamma_2(2\Gamma + \Gamma_3 + \Gamma_2 + K) = 0 \end{cases}$$

that

$$\begin{cases} 2\Gamma + 2\Gamma_5 + 2\Gamma_3 + \Gamma_2 + K = 0, \\ 2\Gamma + \Gamma_5 + 2\Gamma_3 + \Gamma_2 + K = 0, \\ 2\Gamma + \Gamma_3 + \Gamma_2 + K = 0. \end{cases}$$

Solve these equations to get

$$\Gamma_3 = 0, \Gamma_5 = 0 \quad \text{and} \quad \Gamma_2 = -(2\Gamma + K).$$

Therefore the sequence $B_n = \Gamma_2 \chi_2(n) + \Gamma$ is periodic $\pmod 2$ and

$$B_1 = \Gamma_2 + \Gamma = -(\Gamma + K) \quad \text{and} \quad B_2 = \Gamma.$$

On the other hand, we obtain from (3.8) that

$$B_k = (2\Gamma + K)^2 \quad \text{and} \quad B_{k+1} = 0.$$

Thus, two possibilities exist: either

$$\text{(a)} \quad \mathrm{B_1 = B_k} \quad \text{and} \quad \mathrm{B_2 = B_{k+1} = 0}$$

or

$$\text{(b)} \quad \mathrm{B_1 = B_{k+1} = 0} \quad \text{and} \quad \mathrm{B_2 = B_k}.$$

Case (a). Assume that $B_k = B_1$ and $B_{k+1} = B_2$. Then $k \equiv 1 \pmod 2$ and

$$\begin{cases} B_k - B_1 & = (2\Gamma + K)^2 + (\Gamma + K) = 0, \\ B_{k+1} - B_2 & = 0 - \Gamma = 0. \end{cases}$$

Since $\Gamma_2 = -(2\Gamma + K) \neq 0$, the above system implies that $\Gamma = 0$ and $K = -1$. Consequently

$$B_1 = 1, \; B_2 = 0 \quad \text{and} \quad F(n) = B_n = \chi_2(n)$$

and

$$f(n^2 + m^2 + k) = \chi_2(n) + \chi_2(m) - 1, \quad k \equiv 1 \pmod 2.$$

Case (b). Assume that $B_k = B_2$ and $B_{k+1} = B_1$. Then

$$\begin{cases} B_k - B_2 & = (2\Gamma + K)^2 - \Gamma = 0, \\ B_{k+1} - B_1 & = 0 + (\Gamma + K) = 0. \end{cases}$$

These imply that $\Gamma = -K$ and $0 = (2\Gamma + K)^2 - \Gamma = K^2 + K$. Since

$$\Gamma_2 = -(2\Gamma + K) = -(-2K + K) = K \neq 0,$$

the relation $K^2 + K = 0$ gives $K = -1$, and so $\Gamma = -K = 1$, $\Gamma_2 = -(2\Gamma + K) = = -(2-1) = -1$. Thus, we have proved that

$$B_1 = -(\Gamma + K) = 0, \; B_2 = \Gamma = 1 \quad \text{and} \quad F(n) = B_n = \chi_2(n+1).$$

Thus all assertions of Lemma 6 are proved. ∎

Lemma 7. *Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 = 0$. Then one of the following assertions holds:*

(A) $f(n) = \varepsilon(n)\dfrac{1 \pm \sqrt{-8K+1}}{4}$,

(B) $f(n) = \dfrac{\varepsilon(n)}{4}a(n)\sqrt{-8\chi_3(n)+9}$ *and* $K = -\dfrac{7}{8}$ for $\mathrm{k} \equiv 1 \pmod 3$,

(C) $f(n) = \varepsilon(n)b(n)\chi_3(n)$ *and* $K = -1$ for $\mathrm{k} \equiv 2 \pmod 3$,

(D) $f(n) = \dfrac{\varepsilon(n)}{4}b(n)\sqrt{-8\chi_3(n)+9}$ *and* $K = -\dfrac{3}{8}$ for $\mathrm{k} \equiv 0 \pmod 3$,

where $\chi_3(n) \pmod 3$ is a principal Dirichlet character, $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(n^2 + m^2 + k) = 1$ for every $n, m \in \mathbb{N}$, $a(n)$, $b(n)$ are periodic sequences $\pmod 3$ with $a(1) = 1$, $a(2) = a(3) = -1$, $b(1) = b(3) = 1$, $b(2) = -1$.

Proof. Assume that $\Gamma_2 = 0$ and $\Gamma_4 = 0$. Then the sequence

$$B_k = \Gamma_3\chi_3(k) + \Gamma_5\chi_5(k) + \Gamma \quad \text{for every} \quad k \in \mathbb{N}$$

is periodic $\pmod{15}$. The elements of sequence are:

$$\begin{cases} B_1 = B_4 = B_{11} = B_{14} & = \Gamma_3 + \Gamma_5 + \Gamma \\ B_2 = B_7 = B_8 = B_{13} & = \Gamma_3 - \Gamma_5 + \Gamma \\ B_3 = B_{12} & = -\Gamma_5 + \Gamma \\ B_5 = B_{10} & = \Gamma_3 + \Gamma \\ B_6 = B_9 & = \Gamma_5 + \Gamma \\ B_{15} & = \Gamma. \end{cases} \tag{3.15}$$

By using (3.8) and (3.15) we have

$$\begin{cases} B_{k+1} & = & B_{1^2+15^2+k} & = & (\Gamma_3 + \Gamma_5 + 2\Gamma + K)^2 \\ B_{k+2} & = & B_{1^2+1^2+k} & = & (2\Gamma_3 + 2\Gamma_5 + 2\Gamma + K)^2 \\ B_{k+3} & = & B_{3^2+3^2+k} & = & (-2\Gamma_5 + 2\Gamma + K)^2 \\ B_{k+4} & = & B_{2^2+15^2+k} & = & (\Gamma_3 - \Gamma_5 + 2\Gamma + K)^2 \\ B_{k+5} & = & B_{1^2+2^2+k} & = & (2\Gamma_3 + 2\Gamma + K)^2 \\ B_{k+6} & = & B_{15^2+6^2+k} & = & (\Gamma_5 + 2\Gamma + K)^2 \end{cases} \tag{3.16}$$

Since the sequence $\{B_\ell\}_0^\infty$ is periodic $\pmod{15}$, we have

$$B_{16} = B_1,\ B_{17} = B_2,\ B_{18} = B_3,\ B_{19} = B_4,\ B_{20} = B_5.$$

Let I be the set of 15 different 6 consecutive elements of sequence $\{B_\ell\}_0^\infty$, i.e

$$I := \big\{(B_1, B_2, B_3, B_4, B_5, B_6),\ (B_2, B_3, B_4, B_5, B_6, B_7), \ldots, \\ (B_{15}, B_1, B_2, B_3, B_4, B_5).\big\}$$

It is obvious that

$$\big(B_{k+1},\, B_{k+2},\, B_{k+3},\, B_{k+4},\, B_{k+5},\, B_{k+6}\big) \in I,$$

which with (3.15) and (3.16) give 15 system of equations. By using the computations with Maple, we solve these systems and we have 4 solutions:

$$\begin{aligned}
&\text{(A)}\;\; \Gamma_3 = 0,\; \Gamma_5 = 0 \quad \text{and} \quad \Gamma = \frac{-4K+1 \pm \sqrt{-8K+1}}{8} \quad \text{for every} \quad n \in \mathbb{N},\\
&\text{(B)}\;\; \Gamma_3 = -\frac{1}{2},\; \Gamma_5 = 0,\; \Gamma = \frac{9}{16} \quad \text{and} \quad K = -\frac{7}{8} \;\; \text{for} \;\; \mathrm{k} \equiv 1 \pmod 3,\\
&\text{(C)}\;\; \Gamma_3 = 1,\; \Gamma_5 = 0,\; \Gamma = 0 \quad \text{and} \quad K = -1 \;\; \text{for} \;\; \mathrm{k} \equiv 2 \pmod 3,\\
&\text{(D)}\;\; \Gamma_3 = -\frac{1}{2},\; \Gamma_5 = 0,\; \Gamma = \frac{9}{16} \quad \text{and} \quad K = -\frac{3}{8} \;\; \text{for} \;\; \mathrm{k} \equiv 0 \pmod 3.
\end{aligned}$$

In the case (A) we infer from

$$\Gamma = \frac{-4K+1 \pm \sqrt{-8K+1}}{8} = \left(\frac{1 \pm \sqrt{-8K+1}}{4}\right)^2,$$

and

$$f^2(n) = B_n = \Gamma = \left(\frac{1 \pm \sqrt{-8K+1}}{4}\right)^2$$

that

$$f(n) = \pm\left(\frac{1 \pm \sqrt{-8K+1}}{4}\right),$$

and so

$$f(n^2+m^2+k) = f^2(n) + f^2(m) + K = 2\Gamma + K = \frac{1 \pm \sqrt{-8K+1}}{4}.$$

Thus in the cases (A)–(D) we have

$$\begin{aligned}
&\text{(A)}\;\; f(n) = \varepsilon(n)\frac{1 \pm \sqrt{-8K+1}}{4},\\
&\text{(B)}\;\; f(n) = \frac{\varepsilon(n)}{4} a(n)\sqrt{-8\chi_3(n)+9} \quad \text{and} \quad K = -\frac{7}{8} \;\; \text{for} \;\; \mathrm{k} \equiv 1 \pmod 3,\\
&\text{(C)}\;\; f(n) = \varepsilon(n) b(n) \chi_3(n) \quad \text{and} \quad K = -1 \;\; \text{for} \;\; \mathrm{k} \equiv 2 \pmod 3,\\
&\text{(D)}\;\; f(n) = \frac{\varepsilon(n)}{4} b(n)\sqrt{-8\chi_3(n)+9} \quad \text{and} \quad K = -\frac{3}{8} \;\; \text{for} \;\; \mathrm{k} \equiv 0 \pmod 3.
\end{aligned}$$

Thus, Lemma 7 is proved. ∎

Proof of the Theorem. Our theorem follows from Lemma 4, Lemma 6 and Lemma 7. ■

Remarks. I. Kátai and B. M. Phong posed the following conjecture:

Conjecture. (I. Kátai and B. M. Phong [2]) *Assume that the number $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation*

$$f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N}.$$

Then one of the following assertions holds:

$$\begin{aligned} &a) \quad f(n) = 0 \quad \textit{for every} \quad n \in \mathbb{N}, \\ &b) \quad f(n) = \frac{\epsilon(n)}{D+1} \quad \textit{for every} \quad n \in \mathbb{N}, \\ &c) \quad f(n) = \epsilon(n)n \quad \textit{for every} \quad n \in \mathbb{N}, \end{aligned}$$

where $E := \{n^2 + Dm^2 | n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$.

This conjecture is proved in our paper [4]. Some special cases of this result have been proven in [3], [5], [6] and [7].

In a next article we determine all functions $f, g : \mathbb{N} \to \mathbb{C}$ for which for which $f(n^2 + Dm^2 + k) = g^2(n) + Dg^2(m) + K$ is satisfied for all $n, m \in \mathbb{N}$, where a non-negative integer k, $D \in \mathbb{N}$ and $K \in \mathbb{C}$ are given.

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