A NOTE ON A RESULT OF B. BOJAN

Bui Minh Mai Khanh (Budapest, Hungary)

Communicated by Imre Kátai

(Received March 27, 2019; accepted June 21, 2019)

Abstract. We determine all solutions of those functions \( f: \mathbb{N} \to \mathbb{C} \) for which \( f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K \) is satisfied for all positive integers \( n, m \), where a non-negative integer \( k \) and \( K \in \mathbb{C} \) are given. This improves the result of B. Bojan.

1. Introduction

Let, as usual, \( \mathbb{N}, \mathbb{Z}, \mathbb{C} \) be the set positive integers, integers and complex numbers, respectively.

In 2014 B. Bojan [1] determined all solutions of those \( f: \mathbb{N} \to \mathbb{C} \) for which

\[ f(n^2 + m^2) = f^2(n) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}, \]

namely

\[ f(n) \in \{0, \pm \frac{1}{2}, \pm n\}. \]

Our purpose in this note is to improve the result of B. Bojan [1].

**Theorem.** Assume that a non-negative integer \( k \), a complex number \( K \) and an arithmetical function \( f: \mathbb{N} \to \mathbb{C} \) satisfy the equation

\[ f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K \quad \text{for every} \quad n, m \in \mathbb{N}. \]

**Key words and phrases:** Arithmetical functions, functional equation.

**2010 Mathematics Subject Classification:** 11A07, 11A25, 11N25, 11N64.
Then one of the following assertions holds:

a) \( f(n) = \pm \chi_2(n) \), \( f(n^2 + m^2 + k) = \chi_2(n) + \chi_2(m) - 1 \) and \( 2 \nmid k \), \( K = -1 \),
b) \( f(n) = \pm \chi_2(n + 1) \), \( f(n^2 + m^2 + k) = \chi_2(n + 1) + \chi_2(m + 1) - 1 \) and \( 2 \mid k \), \( K = -1 \),
c) \( f(n) = \frac{\varepsilon(n)}{4} (1 - \sqrt{-8K + 1}) \),
d) \( f(n) = \frac{\varepsilon(n)}{4} (1 + \sqrt{-8K + 1}) \),
e) \( f(n) = \frac{\varepsilon(n)}{4} a(n) \sqrt{-8\chi_3(n) + 9} \) and \( K = -\frac{7}{8} \) for \( k \equiv 1 \) (mod 3),
f) \( f(n) = \varepsilon(n) b(n) \chi_3(n) \) and \( K = -1 \) for \( k \equiv 2 \) (mod 3),
g) \( f(n) = \frac{\varepsilon(n)}{4} b(n) \sqrt{-8\chi_3(n) + 9} \) and \( K = -\frac{3}{8} \) for \( k \equiv 0 \) (mod 3),
h) \( f(n) = \varepsilon(n)n \),

where \( \chi_2(n) \) (mod 2), \( \chi_3(n) \) (mod 3) are principal Dirichlet characters, \( \varepsilon(n) \in \{-1, 1\} \) and \( \varepsilon(n^2 + m^2 + k) = 1 \) for every \( n, m \in \mathbb{N} \), \( a(n), b(n) \) are periodic sequences (mod 3) with \( a(1) = 1, a(2) = a(3) = -1, b(1) = b(3) = 1, b(2) = -1 \).

2. Lemmas

Assume that a non-negative integer \( k \), a complex number \( K \) and an arithmetical function \( f : \mathbb{N} \to \mathbb{C} \) satisfy the equation

\[
(2.1) \quad f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K \quad \text{for every} \quad n, m \in \mathbb{N}.
\]

**Lemma 1.** Let \( F(n) := f^2(n) \). Then

\[
F(\ell + 12) = F(\ell + 9) + F(\ell + 8) + F(\ell + 7) - F(\ell + 5) - F(\ell + 4) - F(\ell + 3) + F(\ell)
\]

holds for every \( \ell \in \mathbb{N} \) and

\[
\begin{align*}
F(7) &= 2F(5) - F(1), \\
F(8) &= 2F(5) + F(4) - 2F(1), \\
F(9) &= F(6) + 2F(5) - F(2) - F(1),
\end{align*}
\]

(2.3)
A note on a result of B. Bojan

\[ F(10) = F(6) + 3F(5) - F(3) - 2F(1), \]
\[ F(11) = F(6) + 4F(5) - F(3) - F(2) - 2F(1), \]
\[ F(12) = F(6) + 4F(5) + F(4) - F(2) - 4F(1). \]

(2.3)

Proof. We infer from (2.1) and the next relations
\[
(2n+1)^2 + (n-2)^2 = (2n-1)^2 + (n+2)^2, \quad (2n+3)^2 + (n-6)^2 = (2n-3)^2 + (n+6)^2
\]
that
\[
f[(2n + 1)^2 + (n-2)^2 + k] = f[(2n-1)^2 + (n+2)^2 + k]
\]
and
\[
f[(2n + 3)^2 + (n-6)^2 + k] = f[(2n-3)^2 + (n+6)^2 + k].
\]
These imply that
\[
F(2n + 1) + F(n-2) = F(2n-1) + F(n+2)
\]
and
\[
F(2n + 3) + F(n-6) = F(2n-3) + F(n+6).
\]
We infer from the last relations that
\[
F(n + 6) - F(n - 6) = F(2n + 3) - F(2n - 3) =
\]
\[
= [F(2n + 3) - F(2n + 1)] + [F(2n + 1) - F(2n - 1)] +
\]
\[
+ [F(2n - 1) - F(2n - 3)] =
\]
\[
= [F(n + 3) - F(n - 1)] + [F(n + 2) - F(n - 2)] +
\]
\[
+ [F(n + 1) - F(n - 3)],
\]
consequently
\[
F(n + 6) = F(n + 3) + F(n + 2) + F(n + 1) - F(n - 1) -
\]
\[
- F(n - 2) - F(n - 3) + F(n - 6).
\]
This with \( m = \ell + 6 \) proves (2.2).

In the order to prove (2.3), we note from (2.1) that
\[
(2.4) \quad \text{If } x^2 + y^2 = u^2 + v^2, \quad \text{then } F(x) + F(y) = F(u) + F(v).
\]
Applications of (2.4) in the cases
\[
(x, y, u, v) \in \{(5, 5, 7, 1), \ (7, 4, 8, 1), \ (7, 6, 9, 2), \ (9, 7, 11, 3), \ (9, 8, 12, 1), \ (10, 5, 11, 2)\}
\]
prove that (2.3) holds for \( F(7), F(8), F(9), F(11), F(12) \) and \( F(10) \).

Thus, (2.3) and Lemma 1 is proved.
Lemma 2. Assume that the function $F : \mathbb{N} \to \mathbb{C}$ satisfy (2.2) and (2.3). Let

$$A := \frac{1}{120} \left( F(6) + 4F(5) - F(3) - F(2) - 3F(1) \right),$$

$$\Gamma_2 := -\frac{1}{8} \left( F(6) - 4F(5) + 4F(4) - F(3) + 3F(2) - 3F(1) \right),$$

$$\Gamma_3 := -\frac{1}{3} \left( F(6) - 2F(5) + 2F(3) - F(2) \right),$$

$$\Gamma_4 := \frac{1}{4} \left( F(6) - 2F(4) - F(3) + F(2) + F(1) \right),$$

$$\Gamma_5 := \frac{1}{5} \left( F(6) - F(5) - F(3) - F(2) + 2F(1) \right),$$

$$B_k := \Gamma_2 \chi_2(k) + \Gamma_3 \chi_3(k) + \Gamma_4 \chi_4(k-1) + \Gamma_5 \chi_5(k) + \Gamma,$$

where $\chi_2(k) \pmod{2}$, $\chi_3(k) \pmod{3}$ are the principal Dirichlet characters and $\chi_4(k) \pmod{4}$, $\chi_5(k) \pmod{5}$ are the real, non-principal Dirichlet characters, i.e.

$$\chi_2(0) = 0, \chi_2(1) = 1, \chi_3(0) = 0, \chi_3(1) = \chi_3(2) = 1,$$

$$\chi_4(0) = \chi_4(2) = 0, \chi_4(1) = 1, \chi_4(3) = -1,$$

$$\chi_5(2) = \chi_5(3) = -1, \chi_5(1) = \chi_5(4) = 1.$$

Then we have

$$F(\ell) = A\ell^2 + B_\ell \quad \text{for every} \quad \ell \in \mathbb{N}.$$  

Proof. The proof is very similar to that of Lemma 2 in [4]. Here we omit the proof. □

3. Proof of the Theorem

Assume that a non-negative integer $k$, a complex number $K$ and an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation (2.1). Let $F(n) := f^2(n)$ for every $n \in \mathbb{N}$. From (2.1) and Lemma 2, we have

$$F(n^2 + m^2 + k) = (F(n) + F(m) + K)^2$$

and

$$F(\ell) = f^2(\ell) = A\ell^2 + B_\ell \quad \text{for every} \quad \ell \in \mathbb{N},$$
where
\[ B_\ell := \Gamma_2 \chi_2(\ell) + \Gamma_3 \chi_3(\ell) + \Gamma_4 \chi_4(\ell - 1) + \Gamma_5 \chi_5(\ell) + \Gamma. \]

**Lemma 3.** We have
\[ A \in \{0, 1\}. \]

**Proof.** We obtain from (3.1) and (3.2) that
\[
(3.3) \quad A(n^2 + m^2 + k)^2 + B_{n^2 + m^2 + k} = \left( A(n^2 + m^2) + B_n + B_m + K \right)^2
\]
for every \( n, m \in \mathbb{N} \). Since
\[
|B_\ell| \leq |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma| \quad \text{for every } \ell \in \mathbb{N}
\]
and
\[
n^2 + m^2 + k \to \infty \quad \text{as } n, m \to \infty,
\]
we infer from (3.3) that
\[
A = \lim_{n, m \to \infty} \left[ A \left( 1 + \frac{k}{n^2 + m^2 + k} \right)^2 + \frac{B_{n^2 + m^2 + k}}{(n^2 + m^2 + k)^2} \right] = \lim_{n, m \to \infty} \left[ (A + \frac{-k + B_n + B_m + K}{n^2 + m^2 + k})^2 \right] = A^2.
\]
Therefore, we have \( A \in \{0, 1\} \). Lemma 3 is proved. ■

**Lemma 4.** Assume that \( A = 1 \), i.e.
\[
F(\ell) = f^2(\ell) = \ell^2 + B_\ell \quad \text{for every } \ell \in \mathbb{N},
\]
then
\[
B_m = 0, \quad f(m) = \pm m \quad \text{and} \quad f(n^2 + m^2 + k) = n^2 + m^2 + k
\]
for every \( n, m \in \mathbb{N} \).

**Proof.** From (3.3) we obtain that
\[
(3.4) \quad (n^2 + m^2 + k)^2 + B_{n^2 + m^2 + k} - \left( (n^2 + m^2) + B_n + B_m + K \right)^2 = 2(k - B_n - B_m - K)n^2 + W(n, m) = 0,
\]
holds for every \( n, m \in \mathbb{N} \), where
\[
(3.5) \quad W(n, m) := B_{n^2 + m^2 + k} + (m^2 + k)^2 - (m^2 + B_n + B_m + K)^2.
\]
Now let $m \in \mathbb{N}$ be fixed, $n \in \mathbb{N}$, $n \equiv a \pmod{60}$ with some $a \in \mathbb{N}$, $0 \leq a < 60$. Since the sequence $\{B_r\}_{r=0}^{\infty}$ is periodic $\pmod{60}$, we have

$$|W(n,m)| = |W(a,m)| < \infty$$

and so we obtain from (3.4) that

$$k - B_a - K - B_m = \lim_{n \equiv a \pmod{60}} \frac{-W(n,m)}{2n^2} = 0$$

and

$$W(a,m) = 0$$

hold for each $a, m \in \mathbb{N}$. It follows from these that

$$B_m = k - B_a - K = c \quad \text{for every} \quad m \in \mathbb{N}$$

and

$$W(a,m) := c + (m^2 + k)^2 - (m^2 + 2c + K)^2 = 0 \quad \text{for every} \quad m \in \mathbb{N},$$

where $c \in \mathbb{C}$ is some fixed constant. These show that

$$(3.6) \quad k - K = 2c \quad \text{and} \quad c + k^2 - (2c + K)^2 = 0.$$ 

Finally, we obtain from the last relations that

$$k = 2c + K \quad \text{and} \quad 0 = c + k^2 - (2c + K)^2 = c + k^2 - k^2 = c.$$ 

Thus we have

$$B_m = c = 0 \quad \text{and} \quad F(m) = m^2 \quad \text{for every} \quad m \in \mathbb{N}$$

and from (3.6) we have $k = K$. Thus, in the case $A = 1$, we proved that

$$f^2(m) = m^2, f(n) = \pm m$$

and

$$f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K = n^2 + m^2 + k$$

for every $n, m \in \mathbb{N}$. Lemma 4 is proved.

**Lemma 5.** Assume that $A = 0$. Then

$$(3.7) \quad \Gamma_4 = 0,$$

where $\Gamma_4$ is defined in Lemma 2.
Proof. We infer from the fact $A = 0$ and from (3.1), (3.2) that

\[(3.8) \quad B_{n^2+m^2+k} = (B_n + B_m + K)^2 \quad \text{for every} \quad n,m \in \mathbb{N}.\]

Since $\{B_\ell\}_{0}^{\infty}$ is periodic (mod 60), therefore

\[B_{2^2+2^2+k} = B_{8^2+2^2+k} \quad \text{and} \quad B_{2^2+2^2+k} = B_{8^2+8^2+k},\]

which with (3.8) imply that

\[(B_2 + B_2 + K)^2 = (B_8 + B_2 + K)^2 \quad \text{and} \quad (B_2 + B_2 + K)^2 = (B_8 + B_8 + K)^2.\]

Consequently

\[(3.9) \quad (B_2 - B_8)(3B_2 + B_8 + 2K) = 0\]

and

\[(3.10) \quad 4(B_2 - B_8)(B_2 + B_8 + K) = 0.\]

Since

\[B_2 = \Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma \quad \text{and} \quad B_8 = \Gamma_3 - \Gamma_4 - \Gamma_5 + \Gamma,\]

we have $\Gamma_4 = 0$ if $B_2 = B_8$. Assume now that $B_2 - B_8 \neq 0$. Then we infer from (3.9) and (3.10) that

\[3B_2 + B_8 + 2K = 0 \quad \text{and} \quad B_2 + B_8 + K = 0,\]

consequently

\[B_2 = B_8.\]

This contradicts to the assumption $B_2 - B_8 \neq 0$. Thus, $\Gamma_4 = 0$ follows and the proof of Lemma 5 is completed. \qed

Lemma 6. Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 \neq 0$. Then $K = -1$, furthermore either

\[2 \nmid k, \quad f(n) = \pm \chi_2(n) \quad \text{and} \quad f(n^2 + m^2 + k) = \chi_2(n) + \chi_2(m) - 1\]

or

\[2 \mid k, \quad f(n) = \pm \chi_2(n+1) \quad \text{and} \quad f(n^2 + m^2 + k) = \chi_2(n+1) + \chi_2(m+1) - 1\]

for every $n,m \in \mathbb{N}$.\]
Proof. Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 \neq 0$. Then

(3.11) \[ B_k := \Gamma_2\chi_2(k) + \Gamma_3\chi_3(k) + \Gamma_5\chi_5(k) + \Gamma \quad \text{for every} \quad k \in \mathbb{N}, \]
and

(3.12) \[ B_n = B_m \quad \text{if} \quad n \equiv m \pmod{30}. \]

It is clear from (3.12) that if $n^2 + m^2 \equiv u^2 + v^2 \pmod{30}$, then

\[ n^2 + m^2 + k \equiv u^2 + v^2 + k \pmod{30} \quad \text{and} \quad B_{n^2+m^2+k} = B_{u^2+v^2+k}. \]

Let

(3.13) \[ E(n, m, u, v) := (B_n + B_m + K)^2 - (B_u + B_v + K)^2 = (B_n + B_m - B_u - B_v)(B_n + B_m + B_u + B_v + 2K). \]

We obtain from the above relations and from (3.8) that

(3.14) \[ \text{If} \quad n^2 + m^2 \equiv u^2 + v^2 \pmod{30}, \quad \text{then} \quad E(n, m, u, v) = 0. \]

With the help of a computer and Maple program, we find those positive integers $n, m, u, v$ such that $n^2 + m^2 \equiv u^2 + v^2 \pmod{30}$, therefore we obtain the equations $E(n, m, u, v) = 0$.

It is clear from (3.11) that

\[
\begin{align*}
B_1 &= \Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma, \\
B_2 &= \Gamma_3 - \Gamma_5 + \Gamma, \\
B_3 &= \Gamma_2 - \Gamma_5 + \Gamma, \\
B_4 &= \Gamma_3 + \Gamma_5 + \Gamma, \\
B_5 &= \Gamma_2 + \Gamma_3 + \Gamma, \\
B_6 &= \Gamma_5 + \Gamma.
\end{align*}
\]

By using (3.13), we infer from these and from the following relations

\[
\begin{align*}
E(1, 1, 4, 4) &= 4\Gamma_2(2\Gamma + 2\Gamma_5 + 2\Gamma_3 + \Gamma_2 + K) = 0, \\
E(1, 5, 4, 10) &= 4\Gamma_2(2\Gamma + \Gamma_5 + 2\Gamma_3 + \Gamma_2 + K) = 0, \\
E(1, 3, 2, 6) &= 4\Gamma_2(2\Gamma + \Gamma_3 + \Gamma_2 + K) = 0
\end{align*}
\]

that

\[
\begin{align*}
2\Gamma + 2\Gamma_5 + 2\Gamma_3 + \Gamma_2 + K &= 0, \\
2\Gamma + \Gamma_5 + 2\Gamma_3 + \Gamma_2 + K &= 0, \\
2\Gamma + \Gamma_3 + \Gamma_2 + K &= 0.
\end{align*}
\]
Solve these equations to get
\[ \Gamma_3 = 0, \Gamma_5 = 0 \quad \text{and} \quad \Gamma_2 = -(2\Gamma + K). \]

Therefore the sequence \( B_n = \Gamma_2 \chi_2(n) + \Gamma \) is periodic \pmod{2} and
\[ B_1 = \Gamma_2 + \Gamma = -(\Gamma + K) \quad \text{and} \quad B_2 = \Gamma. \]

On the other hand, we obtain from (3.8) that
\[ B_k = (2\Gamma + K)^2 \quad \text{and} \quad B_{k+1} = 0. \]

Thus, two possibilities exist: either
\[
\begin{align*}
(a) \quad & B_1 = B_k \quad \text{and} \quad B_2 = B_{k+1} = 0 \\
\text{or} \quad & B_1 = B_{k+1} = 0 \quad \text{and} \quad B_2 = B_k.
\end{align*}
\]

**Case (a).** Assume that \( B_k = B_1 \) and \( B_{k+1} = B_2 \). Then \( k \equiv 1 \pmod{2} \) and
\[
\begin{cases}
B_k - B_1 = (2\Gamma + K)^2 + (\Gamma + K) = 0, \\
B_{k+1} - B_2 = 0 - \Gamma = 0.
\end{cases}
\]

Since \( \Gamma_2 = -(2\Gamma + K) \neq 0 \), the above system implies that \( \Gamma = 0 \) and \( K = -1 \).
Consequently
\[ B_1 = 1, \quad B_2 = 0 \quad \text{and} \quad F(n) = B_n = \chi_2(n) \]
and
\[ f(n^2 + m^2 + k) = \chi_2(n) + \chi_2(m) - 1, \quad k \equiv 1 \pmod{2}. \]

**Case (b).** Assume that \( B_k = B_2 \) and \( B_{k+1} = B_1 \). Then
\[
\begin{cases}
B_k - B_2 = (2\Gamma + K)^2 - \Gamma = 0, \\
B_{k+1} - B_1 = 0 + (\Gamma + K) = 0.
\end{cases}
\]

These imply that \( \Gamma = -K \) and \( 0 = (2\Gamma + K)^2 - \Gamma = K^2 + K \). Since
\[ \Gamma_2 = -(2\Gamma + K) = -(-2K + K) = K \neq 0, \]
the relation \( K^2 + K = 0 \) gives \( K = -1 \), and so \( \Gamma = -1, \Gamma_2 = -(2\Gamma + K) = -(2 - 1) = -1. \) Thus, we have proved that
\[ B_1 = -(\Gamma + K) = 0, \quad B_2 = \Gamma = 1 \quad \text{and} \quad F(n) = B_n = \chi_2(n + 1). \]

Thus all assertions of Lemma 6 are proved. \[ \blacksquare \]
Lemma 7. Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 = 0$. Then one of the following assertions holds:

(A) $f(n) = \varepsilon(n)\frac{1 + \sqrt{-8K + 1}}{4}$,

(B) $f(n) = \varepsilon(n)\frac{a(n)\sqrt{-8\chi_3(n) + 9}}{4}$ and $K = -\frac{7}{8}$ for $k \equiv 1 \pmod{3}$,

(C) $f(n) = \varepsilon(n)b(n)\chi_3(n)$ and $K = -1$ for $k \equiv 2 \pmod{3}$,

(D) $f(n) = \varepsilon(n)\frac{b(n)\sqrt{-8\chi_3(n) + 9}}{4}$ and $K = -\frac{3}{8}$ for $k \equiv 0 \pmod{3}$,

where $\chi_3(n) \pmod{3}$ is a principal Dirichlet character, $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(n^2 + m^2 + k) = 1$ for every $n, m \in \mathbb{N}$, $a(n)$, $b(n)$ are periodic sequences (mod 3) with $a(1) = 1$, $a(2) = a(3) = 1$, $b(1) = b(3) = 1$, $b(2) = -1$.

Proof. Assume that $\Gamma_2 = 0$ and $\Gamma_4 = 0$. Then the sequence

$$B_k = \Gamma_3\chi_3(k) + \Gamma_5\chi_5(k) + \Gamma$$

for every $k \in \mathbb{N}$ is periodic (mod 15). The elements of sequence are:

$$
\begin{align*}
B_1 &= B_4 = B_{11} = B_{14} = \Gamma_3 + \Gamma_5 + \Gamma \\
B_2 &= B_7 = B_8 = B_{13} = \Gamma_3 - \Gamma_5 + \Gamma \\
B_3 &= B_{12} = -\Gamma_5 + \Gamma \\
B_5 &= B_{10} = \Gamma_3 + \Gamma \\
B_6 &= B_9 = \Gamma_5 + \Gamma \\
B_{15} &= \Gamma.
\end{align*}
$$

(3.15)

By using (3.8) and (3.15) we have

$$
\begin{align*}
B_{k+1} &= B_{1^2 + 15^2 + k} = (\Gamma_3 + \Gamma_5 + 2\Gamma + K)^2 \\
B_{k+2} &= B_{1^2 + 1^2 + k} = (2\Gamma_3 + 2\Gamma_5 + 2\Gamma + K)^2 \\
B_{k+3} &= B_{3^2 + 3^2 + k} = (-2\Gamma_5 + 2\Gamma + K)^2 \\
B_{k+4} &= B_{2^2 + 15^2 + k} = (\Gamma_3 - \Gamma_5 + 2\Gamma + K)^2 \\
B_{k+5} &= B_{1^2 + 2^2 + k} = (2\Gamma_3 + 2\Gamma + K)^2 \\
B_{k+6} &= B_{15^2 + 6^2 + k} = (\Gamma_5 + 2\Gamma + K)^2
\end{align*}
$$

(3.16)

Since the sequence $\{B_k\}_0^\infty$ is periodic (mod 15), we have

$$B_{16} = B_1, B_{17} = B_2, B_{18} = B_3, B_{19} = B_4, B_{20} = B_5.$$

Let $I$ be the set of 15 different 6 consecutive elements of sequence $\{B_k\}_0^\infty$, i.e

$$I := \{(B_1, B_2, B_3, B_4, B_5, B_6), (B_2, B_3, B_4, B_5, B_6, B_7), \ldots, (B_{15}, B_1, B_2, B_3, B_4, B_5)\}.$$
It is obvious that

\[(B_{k+1}, B_{k+2}, B_{k+3}, B_{k+4}, B_{k+5}, B_{k+6}) \in I,\]

which with (3.15) and (3.16) give 15 system of equations. By using the computations with Maple, we solve these systems and we have 4 solutions:

- **(A)** \(\Gamma_3 = 0, \Gamma_5 = 0\) and \(\Gamma = \frac{-4K + 1 \pm \sqrt{-8K + 1}}{8}\) for every \(n \in \mathbb{N}\),
- **(B)** \(\Gamma_3 = -\frac{1}{2}, \Gamma_5 = 0, \Gamma = \frac{9}{16}\) and \(K = -\frac{7}{8}\) for \(k \equiv 1\) (mod 3),
- **(C)** \(\Gamma_3 = 1, \Gamma_5 = 0, \Gamma = 0\) and \(K = -1\) for \(k \equiv 2\) (mod 3),
- **(D)** \(\Gamma_3 = -\frac{1}{2}, \Gamma_5 = 0, \Gamma = \frac{9}{16}\) and \(K = -\frac{3}{8}\) for \(k \equiv 0\) (mod 3).

In the case (A) we infer from

\[\Gamma = \frac{-4K + 1 \pm \sqrt{-8K + 1}}{8} = \left(\frac{1 \pm \sqrt{-8K + 1}}{4}\right)^2,\]

and

\[f^2(n) = B_n = \Gamma = \left(\frac{1 \pm \sqrt{-8K + 1}}{4}\right)^2\]

that

\[f(n) = \pm \left(\frac{1 \pm \sqrt{-8K + 1}}{4}\right),\]

and so

\[f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K = 2\Gamma + K = \frac{1 \pm \sqrt{-8K + 1}}{4}.\]

Thus in the cases (A)-(D) we have

- **(A)** \(f(n) = \varepsilon(n)\frac{1 \pm \sqrt{-8K + 1}}{4}\),
- **(B)** \(f(n) = \frac{\varepsilon(n)}{4}a(n)\sqrt{-8\chi_3(n) + 9}\) and \(K = -\frac{7}{8}\) for \(k \equiv 1\) (mod 3),
- **(C)** \(f(n) = \varepsilon(n)b(n)\chi_3(n)\) and \(K = -1\) for \(k \equiv 2\) (mod 3),
- **(D)** \(f(n) = \frac{\varepsilon(n)}{4}b(n)\sqrt{-8\chi_3(n) + 9}\) and \(K = -\frac{3}{8}\) for \(k \equiv 0\) (mod 3).

Thus, Lemma 7 is proved.
**Proof of the Theorem.** Our theorem follows from Lemma 4, Lemma 6 and Lemma 7.

**Remarks.** I. Kátai and B. M. Phong posed the following conjecture:

**Conjecture.** (I. Kátai and B. M. Phong [2]) Assume that the number $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation

$$f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$  

Then one of the following assertions holds:

a) $f(n) = 0$ for every $n \in \mathbb{N}$,

b) $f(n) = \frac{\epsilon(n)}{D + 1}$ for every $n \in \mathbb{N}$,

c) $f(n) = \epsilon(n)n$ for every $n \in \mathbb{N}$,

where $E := \{n^2 + Dm^2 | n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$.

This conjecture is proved in our paper [4]. Some special cases of this result have been proven in [3], [5], [6] and [7].

In a next article we determine all functions $f, g : \mathbb{N} \to \mathbb{C}$ for which for which $f(n^2 + Dm^2 + k) = g^2(n) + Dg^2(m) + K$ is satisfied for all $n, m \in \mathbb{N}$, where a non-negative integer $k$, $D \in \mathbb{N}$ and $K \in \mathbb{C}$ are given.

**References**


B.M.M. Khanh
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest
Pázmány Péter sétány 1/C
Hungary
mbuiminh@yahoo.com