ON THE UNIFORM DISTRIBUTION AND
UNIFORM SUMMABILITY OF POSITIVE VALUED
MULTIPLICATIVE FUNCTIONS

Karl-Heinz Indlekofer (Paderborn, Germany)

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Abstract. Let \( n \mapsto g(n) \) be a positive valued arithmetic function which tends to infinity as \( n \to \infty \). Following [1], we shall say that the values of \( g \) are uniformly distributed in \((0, \infty)\) if there exists a positive \( c \) such that
\[
N(x, g) := \# \{ n : g(n) \leq x \} \sim cx
\]
as \( x \to \infty \).

In [4] we introduced the class \( \mathcal{L}^* \) of uniformly summable functions \( f \in \mathcal{L}^* \) in case
\[
\lim_{K \to \infty} \sup_{N \geq 1} \frac{1}{K} \sum_{n \leq N} |f(n)| < \infty.
\]
Here we investigate the asymptotic behaviour of \( N(x, g) \) as \( x \to \infty \) for multiplicative functions \( g \) such that the associated function \( n \mapsto n/g(n) \) is uniformly summable, and compare it with the behaviour of \( \sum_{n \leq x} n/g(n) \) as \( x \to \infty \).

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1. Introduction

Following Diamond and Erdös [1] we say that the values of a positive valued function $g$ are uniformly distributed in $(0,\infty)$ if $g(n)$ tends to infinity as $n \to \infty$, and if there exists a positive $c$ such that

$$N(x,g) := \sum_{g(n) \leq x} 1 = (c + o(1))x \text{ as } x \to \infty.$$ 

In [4] Indlekofer introduced the space $\mathcal{L}^*$ of uniformly summable functions. Here $f \in \mathcal{L}^*$ iff

$$\limsup_{x \to \infty} x^{-1} \sum_{n \leq x} |f(n)| < \infty$$

and

$$\lim_{K \to \infty} \sup_{N \geq 1} \frac{1}{N} \sum_{n \leq N \atop |f(n)| > K} |f(n)| = 0.$$ 

Putting

$$M(x,h) := \sum_{n \leq x} h(n)$$

for an arithmetical function $h : \mathbb{N} \to \mathbb{C}$ we define the mean-value $M(h)$ by

$$M(h) := \lim_{x \to \infty} \frac{1}{x} M(x,h)$$

if the limit exists.

In this paper $g$ always denotes a multiplicative function.

We observe that the generating function for the uniform distribution of values of $g$ is ($s = \sigma + it$ and $\sigma > 1$)

$$F_1(s) = \int_1^\infty x^{-s} dN(x,g) = \sum_{n=1}^\infty \frac{1}{g(n)^s} = \prod_p \left(1 + \sum_{k=1}^\infty \frac{1}{(g(p^k))^s}\right).$$ 

Define $h = id/g$ by $h(n) = n/g(n)$. Then the generating function for the mean value of the function $h$ is

$$F_2(s) = \int_1^\infty x^{-s} dM(x,h) = \sum_{n=1}^\infty \frac{h(n)}{n^s} = \sum_{n=1}^\infty \frac{1}{g(n)} \frac{1}{n^{s-1}} = \prod_p \left(1 + \sum_{k=1}^\infty \frac{1}{g(p^k)p^{k(s-1)}}\right).$$
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Obviously $F_1(s)$ and $F_2(s)$ are formally similar near $s = 1$.

In [1] Diamond and Erdős proved results which connects uniform distribution of the values of multiplicative function $g$ with the existence of the mean value for the associated function $h = id/g$. Their results are analogous to ones on mean values of multiplicative functions (cf [2], [3]) and their proofs are based on the analytic behaviour of the generating function $F_1(s)$ near $s = 1$.

In this paper we use elementary methods from [4], [7]. As a main result we determine the asymptotic behaviour, as $x \to \infty$, of $M(x, 1/f)$ ($f > 0$) and $N(x, g)$ ($g = id/f$) for uniformly summable multiplicative functions $1/f > 0$.

2. Results

Here $f, f^*$ and $g^* := id/f^*$ always denote positive-valued arithmetical functions.

**Theorem 1.** Let $g^*$ be completely multiplicative such that $g^*(p) > 1$ for all primes $p$ and $g^*(p) \sim p$ as $p \to \infty$. Then, as $x \to \infty$

$$N(x, g^*) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{g^*(p)}\right)^{-1}.$$ 

**Theorem 1’.** Let $f^*$ be completely multiplicative such that $f^*(p) > \frac{1}{2}$ for all primes $p$ and $f^*(p) \sim 1$ as $p \to \infty$. Then, as $x \to \infty$

$$M(x, 1/f^*) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{pf^*(p)}\right)^{-1}.$$ 

**Corollary 1.** Let $g^*$ as in Theorem 1. Then

$$N(x, g^*) \sim M(x, 1/f^*)$$

as $x \to \infty$.

where $g^* = id f^*$.

**Remark 1.** Suppose $g^*$ restricted to primes is a $1 - 1$ mapping of the primes. Then $g^*(\mathbb{N}) = \mathbb{N}$, and $g^*$ assumes each positive integer value exactly once, i.e. $g^*$ is uniformly distributed in $(0, \infty)$. Then Diamond and Erdős gave an example ([1], Example 2) such that $1/f^*$ does not have a mean-values.

Further, put, for example, $g^*(p) = p^2$ for all primes $p$. Then $g^*$ assumes each square integer value exactly once, i.e. $N(x, g^*) = x^{1/2} + O(1)$ but $M(x, 1/f^*) = \sum_{n \leq x} \frac{1}{n} = \log x + O(1)$. 


Next we assume

\[(2.1) \quad g(p) \sim p \text{ as } p \to \infty\]

and

\[(2.2) \quad \sum_{p, k \geq 2} \frac{1}{g(p^k)} < \infty.\]

Then we have

**Theorem 2.** Let $g$ be a multiplicative function satisfying (2.1) and (2.2). Then, as $x \to \infty$

$$N(x, g) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{g(p^k)}\right).$$

**Theorem 2’.** Let $f$ be a multiplicative function satisfying (2.1) and (2.2) for $g(p) = pf(p)$ and $g(p^k) = p^k f(p^k)$, respectively. Then, as $x \to \infty$

$$M(x, 1/f) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^k f(p^k)}\right).$$

**Corollary 2.** Let $g$ be as in Theorem 2. Then

$$N(x, g) \sim M(x, 1/f) \text{ as } x \to \infty$$

where $g = id f$.

The main result of this paper is

**Theorem 3.** Let $g = id f$ be multiplicative and assume $1/f \in \mathcal{L}^*$. Then, as $x \to \infty$

$$N(x, g) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{g(p^k)}\right).$$

As a well-known result we cite (see [4], [7])

**Theorem 3’.** Let $1/f \in \mathcal{L}^*$ be multiplicative. Then, as $x \to \infty$

$$M(x, 1/f) = \{1 + o(x)\} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^k f(p^k)}\right).$$

**Corollary 3’.** Let $g = id f$ as in Theorem 3. Then

$$N(x, g) \sim M(x, 1/f) \text{ as } x \to \infty.$$
3. Proofs of Theorem 1 and Theorem 1’

Assume that $g^*$ is completely multiplicative satisfying $g^*(p) > 1$ for all primes $p$ and $g^*(p) \sim p$ as $p \to \infty$. Put

$$F_1^*(s) = \sum_{n=1}^{\infty} \frac{1}{(g^*(n))^s} = \prod_p \left(1 - (g^*(p))^{-s}\right)^{-1}$$

where $s > 1$. Then

$$\log F_1^*(s) = \sum_p \log \left(\frac{1}{1 - (g^*(p))^{-s}}\right).$$

Differentiating with respect to $s$ and observing

$$\frac{d}{ds} \log \left(\frac{1}{1 - (g^*(p))^{-s}}\right) = -\frac{\log g^*(p)}{(g^*(p))^s - 1}$$

we conclude

$$-\frac{F_1'^*(s)}{F_1^*(s)} = \sum_p \frac{\log g^*(p)}{(g^*(p))^s - 1} = \sum_p \log g^*(p) \sum_{m=1}^{\infty} (g^*(p))^{-ms}.$$  \hspace{1cm} (3.1)

The double series in (3.1) is absolutely convergent when $s > 1$. Hence it may be written as

$$\sum_{p,m} (g^*(p))^{-ms} \log g^*(p) = \sum_n \Lambda^*(n) (g^*(n))^{-s},$$

where

$$\Lambda^*(n) = \begin{cases} \log g^*(p), & \text{if } n = p^m \\ 0, & \text{if } n \neq p^m \end{cases}$$

and

$$\sum_{g^*(n) \leq x} \log g^*(k) = \sum_{m,n \in \mathbb{N}, \ g^*(mn) \leq x} \Lambda(m) = \sum_{n \in \mathbb{N}, \ g^*(n) \leq x} \sum_{m \in \mathbb{N}, \ g^*(m) \leq \frac{x}{g^*(n)}} \Lambda(m) = \sum_{n \in \mathbb{N}, \ g^*(n) \leq x} H\left(\frac{x}{g^*(n)}\right).$$
Obviously,
\[ H(y) = \sum_{g^*(p) \leq x} \log g^*(p) + \sum_{p, k \geq 2, \frac{g^*(p)}{g^*(k)} \leq y^{1/k}} \log g^*(p) = \sum_1 + \sum_2. \]

Since \( g^*(p) > 1 \) and \( g^*(p) \sim p \) we conclude, as \( y \to \infty \),
\[ \sum_1 = \{1 + o(1)\} y \]
and
\[ \sum_2 = o(\sum_1) = o(y). \]

Therefore
\[ H(y) = y + o(y) \]
and
\[ \sum_k \log g^*(k) = \{1 + o(1)\} x \sum_{g^*(k) \leq x} \frac{1}{g^*(k)}. \]

Summation by parts yields
\[ \sum_{g^*(k) \leq x} 1 = \{1 + o(1)\} x \frac{\sum_{g^*(k) \leq x} \frac{1}{g^*(k)}}{\log x} = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{g^*(p)}\right)^{-1}. \]

The last equation holds, since \( c^{-1} \leq \frac{g^*(p)}{p} \leq c \) and
\[ \left| \sum_{g^*(p) \leq x} \frac{1}{g^*(p)} - \sum_{p \leq x} \frac{1}{g^*(p)} \right| \leq \sum_{\frac{c}{x} \leq p \leq cx} \frac{c}{p} = o(1) \]
as \( x \to \infty \).

Using the same method as in [4], pp. 266-267, one can show Theorem 1'.

The proof is left to the reader. \[ \blacksquare \]
4. Proofs of Theorem 3 and Theorem 3′

Let us come back to the positive valued multiplicative functions \(1/f \in \mathcal{L}^*\) (cf. [7]).

There exists \(w(p) : \mathbb{P} \to [9, \infty]\) such that \(w(p) \not\to \infty\) and

\[
\sum_p \frac{w(p)}{p} \left(\frac{1}{f(p)} - 1\right)^2 < \infty.
\]

Put

\[
E := \{p \in \mathbb{P} : \left(\frac{1}{f(p)} - 1\right)^2 > \frac{1}{w(p)}\}.
\]

Then

\[
\sum_{p \in E} \frac{1}{p} < \infty \quad \text{and} \quad \sum_{p \in E} \frac{1}{pf(p)} < \infty.
\]

Define \(f^*\) completely multiplicative by

\[
f^*(p) = \begin{cases} f(p), & \text{if } p \notin E \\ 1, & \text{if } p \in E. \end{cases}
\]

Then

\[
\frac{1}{f} = \frac{1}{f^*} \ast h
\]

and

\[
F_2(s) = \sum_{n=1}^{\infty} \frac{1}{f(n)n^s} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{1}{f(p^k)p^{ks}}\right) = \\
= \sum_{n=1}^{\infty} \frac{1}{f^*(n)n^s} \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \\
= \prod_p \left(1 - \frac{1}{f^*(p)p^s}\right)^{-1} \Pi_1(s) \Pi_2(s),
\]

where

\[
\Pi_1(s) = \prod_{p \in E} \left(1 - \frac{1}{p^s}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{f(p^k)p^{ks}}\right),
\]

\[
\Pi_2(s) = \prod_{p \notin E} \left(1 - \frac{1}{f(p)p^s}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{f(p^k)p^{ks}}\right).
\]
Observe that

\[(4.1) \quad \sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty\]

since

\[\sum_{p \in E \setminus \{p\}} \left( \frac{1}{p} + \frac{1}{p f(p)} \right) < \infty\]

and

\[(4.2) \quad \sum_{p,k \geq 2} \frac{1}{f(p^k)p^k} < \infty.\]

Then we obtain, by using the same method as in [4], pp. 266–267 (cf. [7]),

\[\sum_{n \leq x} \frac{1}{f^*(n)} = \{1 + o(1)\} x \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p f^*(p)} \right)^{-1}\]

as \(x \to \infty\). From this we conclude by (4.2)

\[\sum_{n \leq x} \frac{1}{f(n)} = \{1 + o(1)\} x \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \left( 1 + \sum_{k=1}^{\infty} \frac{1}{f(p^k)p^k} \right)^{-1}\]

which shows Theorem 3'.

Define \(g^*\) by

\[g^*(n) = nf^*(n) \quad (n \in \mathbb{N}).\]

Then

\[F_1(s) = \sum_{n=1}^{\infty} \frac{1}{(g(n))^s} = \prod_{p} \left( 1 + \sum_{k=1}^{\infty} \frac{1}{(g(p^k))^s} \right) = \prod_{p} \left( 1 - \frac{1}{(g^*(p))^s} \right)^{-1} \cdot \prod_{1}^{1} \prod_{2}(s)\]

where

\[\Pi_1'(s) = \prod_{p \in E} \left( 1 - \frac{1}{p^s} \right) \left( 1 + \sum_{k=1}^{\infty} \frac{1}{(g(p^k))^s} \right),\]

\[\Pi_2'(s) = \prod_{p \notin E} \left( 1 - \frac{1}{(g(p))^s} \right) \left( 1 + \sum_{k=1}^{\infty} \frac{1}{(g(p^k))^s} \right).\]

Obviously the products \(\Pi_1'(s)\) and \(\Pi_2'(s)\) are absolutely convergent for \(s = 1\).
Denote by $G$ the semigroup generated by

$$X_{p \in E \{1, p, pg(p) : k \geq 1\}} X_{p \notin E \{1, (g(p))^k, g(p)g(p^k) : k \geq 1\}}.$$ 

Then

$$\sum_{n=1}^{\infty} \frac{1}{(g(n))^s} = \sum_{n=1}^{\infty} \frac{1}{(g^*(n))^s} \sum_{a \in G} h'(a) \frac{1}{(a)^s}$$

with

$$(4.3) \quad \sum_{a \in G} \frac{|h'(a)|}{a} < \infty.$$ 

Therefore

$$\sum_{g(n) \leq x} 1 = \sum_{a \in G, m \in \mathbb{N}} h(a) =$$

$$\sum_{a \leq x} h(a) \sum_{g^*(m) \leq \frac{x}{a}} 1$$

and by $$(4.3),$$ and Theorem 1

$$\sum_{g(n) \leq x} 1 = \{1 + o(1)\} x \sum_{a \in G} h(a) \frac{1}{a} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{g^*(p)}\right).$$

Thus Theorem 3 holds.

References


**K.-H. Indlekofer**
Faculty of Computer Science
Electrical Engineering and Mathematics
University of Paderborn
Warburger Straße 100
D-33098 Paderborn
Germany
k-heinz@math.uni-paderborn.de