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ON THE UNIFORM DISTRIBUTION AND UNIFORM SUMMABILITY OF POSITIVE VALUED MULTIPLICATIVE FUNCTIONS

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Abstract. Let $n \mapsto g(n)$ be a positive valued arithmetic function which tends to infinity as $n \to \infty$. Following [1], we shall say that the values of g are uniformly distributed in $(0, \infty)$ if there exists a positive c such that

$$N(x, g) := \#\{n : g(n) \leq x\} \sim cx$$

as $x \to \infty$.

In [4] we introduced the class $\mathcal{L}^*$ of uniformly summable functions $f \in \mathcal{L}^*$ in case

$$\lim_{K\to\infty} \sup_{N\geq 1} \frac{1}{N} \sum_{n\leq N} |f(n)| < \infty.$$

Here we investigate the asymptotic behaviour of $N(x, g)$ as $x \to \infty$ for multiplicative functions g such that the associated function $n \mapsto n/g(n)$ is uniformly summable, and compare it with the behaviour of $\sum_{n\leq x} n/g(n)$ as $x \to \infty$.

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1. Introduction

Following Diamond and Erdös [1] we say that the values of a positive valued function g are *uniformly distributed* in $(0,\infty)$ if $g(n)$ tends to infinity as $n\to\infty$, and if there exists a positive c such that

$$N(x,g):=\sum_{\substack{n\\ g(n)\le x}} 1=\big(c+o(1)\big)x \quad as \quad x\to\infty.$$

In [4] Indlekofer introduced the space $\mathcal{L}^*$ of *uniformly summable functions*. Here $f\in\mathcal{L}^*$ iff

$$\limsup_{x\to\infty} x^{-1}\sum_{n\le x}|f(n)|<\infty$$

and

$$\lim_{K\to\infty}\sup_{N\ge 1}\frac{1}{N}\sum_{\substack{n\le N\\ |f(n)|>K}}|f(n)|=0.$$

Putting

$$M(x,h):=\sum_{n\le x}h(n)$$

for an arithmetical function $h:\mathbb{N}\to\mathbb{C}$ we define the *mean-value* $M(h)$ by

$$M(h):=\lim_{x\to\infty}\frac{1}{x}M(x,h)$$

if the limit exists.

In this paper g always denotes a multiplicative function.

We observe that the generating function for the uniform distribution of values of g is ($s=\sigma+it$ and $\sigma>1$)

$$F_1(s)=\int_1^\infty x^{-s}dN(x,g)=\sum_{n=1}^\infty\frac{1}{g(n)^s}=\prod_p\left(1+\sum_{k=1}^\infty\frac{1}{(g(p^k))^s}\right).$$

Define $h=id/g$ by $h(n)=n/g(n)$. Then the generating function for the mean value of the function h is

$$F_2(s)=\int_1^\infty x^{-s}dM(x,h)=\sum_{n=1}^\infty\frac{h(n)}{n^s}=\sum_{n=1}^\infty\frac{1}{g(n)}\frac{1}{n^{s-1}}=$$
$$=\prod_p\left(1+\sum_{k=1}^\infty\frac{1}{g(p^k)p^{k(s-1)}}\right).$$

Obviously $F_1(s)$ and $F_2(s)$ are formally similar near $s = 1$.

In [1] Diamond and Erdös proved results which connects uniform distribution of the values of *multiplicative function* g with the existence of the mean value for the associated function $h = id/g$. Their results are analogous to ones on mean values of multiplicative functions (cf [2], [3]) and their proofs are based on the analytic behaviour of the generating function $F_1(s)$ near $s = 1$.

In this paper we use elementary methods from [4], [7]. As a main result we determine the asymptotic behaviour, as $x \to \infty$, of $M(x, 1/f)$ $(f > 0)$ and $N(x, g)$ $(g = id\ f)$ for uniformly summable multiplicative functions $1/f > 0$.

2. Results

Here f, f^* and $g^* := id\ f^*$ always denote positive-valued arithmetical functions.

Theorem 1. *Let g^* be completely multiplicative such that $g^*(p) > 1$ for all primes p and $g^*(p) \sim p$ as $p \to \infty$. Then, as $x \to \infty$*

$$N(x, g^*) = \{1 + o(1)\}x \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{g^*(p)}\right)^{-1}.$$

Theorem 1$'$. *Let f^* be completely multiplicative such that $f^*(p) > \frac{1}{2}$ for all primes p and $f^*(p) \sim 1$ as $p \to \infty$. Then, as $x \to \infty$*

$$M(x, 1/f^*) = \{1 + o(1)\}x \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{pf^*(p)}\right)^{-1}.$$

Corollary 1. *Let g^* as in Theorem 1. Then*

$$N(x, g^*) \sim M(x, 1/f^*) \text{ as } x \to \infty$$

where $g^* = id\ f^*$.

Remark 1. Suppose g^* restricted to primes is a $1 - 1$ mapping of the primes. Then $g^*(\mathbb{N}) = \mathbb{N}$, and g^* assumes each positive integer value exactly once, i.e. g^* is uniformly distributed in $(0, \infty)$. Then Diamond and Erdös gave an example ([1], Example 2) such that $1/f^*$ does not have a mean-values.

Further, put, for example, $g^*(p) = p^2$ for all primes p. Then g^* assumes each square integer value exactly once, i.e. $N(x, g^*) = x^{1/2} + O(1)$ but $M(x, 1/f^*) = \sum_{n \le x} \frac{1}{n} = \log x + O(1)$.

Next we assume

$$g(p) \sim p \quad \text{as} \quad p \to \infty \tag{2.1}$$

and

$$\sum_{p,\, k \geq 2} \frac{1}{g(p^k)} < \infty. \tag{2.2}$$

Then we have

Theorem 2. *Let g be a multiplicative function satisfying (2.1) and (2.2). Then, as $x \to \infty$*

$$N(x, g) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{g(p^k)}\right).$$

Theorem 2′. *Let f be a multiplicative function satisfying (2.1) and (2.2) for $g(p) = pf(p)$ and $g(p^k) = p^k f(p^k)$, respectively. Then, as $x \to \infty$*

$$M(x, 1/f) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^k f(p^k)}\right).$$

Corollary 2. Let g be as in Theorem 2. Then

$$N(x, g) \sim M(x, 1/f) \quad \textit{as} \quad x \to \infty$$

where $g = id\, f$.

The main result of this paper is

Theorem 3. *Let $g = id\, f$ be multiplicative and assume $1/f \in \mathcal{L}^*$. Then, as $x \to \infty$*

$$N(x, g) = \{1 + o(1)\} x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{g(p^k)}\right).$$

As a well-known result we cite (see [4], [7])

Theorem 3′. *Let $1/f \in \mathcal{L}^*$ be multiplicative. Then, as $x \to \infty$*

$$M(x, 1/f) = \{1 + o(x)\} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^k f(p^k)}\right).$$

Corollary 3′. Let $g = id\, f$ as in Theorem 3. Then

$$N(x, g) \sim M(x, 1/f) \quad \textit{as} \quad x \to \infty.$$

3. Proofs of Theorem 1 and Theorem 1′

Assume that g^* is completely multiplicative satisfying $g^*(p) > 1$ for all primes p and $g^*(p) \sim p$ as $p \to \infty$. Put

$$F_1^*(s) = \sum_{n=1}^{\infty} \frac{1}{(g^*(n))^s} = \prod_p \left(1 - (g^*(p))^{-s}\right)^{-1}$$

where $s > 1$. Then

$$\log F_1^*(s) = \sum_p \log \frac{1}{1 - (g^*(p))^{-s}}.$$

Differentiating with respect to s and observing

$$\frac{d}{ds} \log \frac{1}{1 - (g^*(p))^{-s}} = -\frac{\log g^*(p)}{(g^*(p))^s - 1}$$

we conclude

$$(3.1)\qquad \begin{aligned} -\frac{F_1^{*'}(s)}{F_1^*(s)} &= \sum_p \frac{\log g^*(p)}{(g^*(p))^s - 1} = \\ &= \sum_p \log g^*(p) \sum_{m=1}^{\infty} (g^*(p))^{-ms}. \end{aligned}$$

The double series in (3.1) is absolutely convergent when $s > 1$. Hence it may be written as

$$\sum_{p,m} (g^*(p))^{-ms} \log g^*(p) = \sum_n \Lambda^*(n)(g^*(n))^{-s},$$

where

$$\Lambda^*(n) = \begin{cases} \log g^*(p), & \text{if } n = p^m \\ 0, & \text{if } n \neq p^m \end{cases}$$

and

$$\begin{aligned} \sum_{\substack{k \\ g^*(n) \le x}} \log g^*(k) &= \sum_{\substack{m,n \in \mathbb{N} \\ g^*(mn) \le x}} \Lambda(m) = \\ &= \sum_{\substack{n \in \mathbb{N} \\ g^*(n) \le x}} \sum_{\substack{m \in \mathbb{N} \\ g^*(m) \le \frac{x}{g^*(n)}}} \Lambda(m) = \\ &=: \sum_{\substack{n \\ g^*(n) \le x}} H\left(\frac{x}{g^*(n)}\right). \end{aligned}$$

Obviously,

$$\begin{aligned} H(y) &= \sum_{\substack{p \\ g^*(p)\le x}} \log g^*(p) + \sum_{\substack{p,k\ge 2 \\ g^*(p)\le y^{1/k}}} \log g^*(p) = \\ &= \sum_1 + \sum_2 . \end{aligned}$$

Since $g^*(p) > 1$ and $g^*(p) \sim p$ we conclude, as $y \to \infty$,

$$\sum_1 = \{1 + o(1)\}y$$

and

$$\sum_2 = o(\sum_1) = o(y).$$

Therefore

$$H(y) = y + o(y)$$

and

$$\sum_{\substack{k \\ g^*(k)\le x}} \log g^*(k) = \{1 + o(1)\}x \sum_{g^*(k)\le x} \frac{1}{g^*(k)}.$$

Summation by parts yields

$$\begin{aligned} \sum_{\substack{k \\ g^*(k)\le x}} 1 &= \{1 + o(1)\}x \frac{\sum_{g^*(k)\le x} \frac{1}{g^*(k)}}{\log x} = \\ &= \{1 + o(1)\}x \prod_{p\le x} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{g^*(p)}\right)^{-1}. \end{aligned}$$

The last equation holds, since $c^{-1} \le \frac{g^*(p)}{p} \le c$ and

$$|\sum_{g^*(p)\le x} \frac{1}{g^*(p)} - \sum_{p\le x} \frac{1}{g^*(p)}| \le \sum_{\frac{x}{c}\le p\le cx} \frac{c}{p} = o(1)$$

as $x \to \infty$.

Using the same method as in [4], pp. 266-267, one can show Theorem $1'$. The proof is left to the reader. ■

4. Proofs of Theorem 3 and Theorem 3′

Let us come back to the positive valued multiplicative functions $1/f \in \mathcal{L}^*$ (cf. [7]).

There exists $w(p) : \mathbb{P} \to [9, \infty]$ such that $w(p) \nearrow \infty$ and

$$\sum_p \frac{w(p)}{p} \left(\frac{1}{f(p)} - 1 \right)^2 < \infty.$$

Put

$$E := \Big\{ p \in \mathbb{P} : \Big(\frac{1}{f(p)} - 1 \Big)^2 > \frac{1}{w(p)} \Big\}.$$

Then

$$\sum_{p \in E} \frac{1}{p} < \infty \quad and \quad \sum_{p \in E} \frac{1}{pf(p)} < \infty.$$

Define f^* completely multiplicative by

$$f^*(p) = \begin{cases} f(p), & \text{if } p \notin E \\ 1, & p \in E. \end{cases}$$

Then

$$\frac{1}{f} = \frac{1}{f^*} \star h$$

and

$$\begin{aligned} F_2(s) &= \sum_{n=1}^{\infty} \frac{1}{f(n)n^s} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{1}{f(p^k)p^{ks}} \right) = \\ &= \sum_{n=1}^{\infty} \frac{1}{f^*(n)n^s} \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \\ &= \prod_p \left(1 - \frac{1}{f^*(p)p^s} \right)^{-1} \prod_1 (s) \prod_2 (s), \end{aligned}$$

where

$$\begin{aligned} \prod_1 (s) &= \prod_{p \in E} \left(1 - \frac{1}{p^s} \right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{f(p^k)p^{ks}} \right), \\ \prod_2 (s) &= \prod_{p \notin E} \left(1 - \frac{1}{f(p)p^s} \right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{f(p^k)p^{ks}} \right). \end{aligned}$$

Observe that

$$\sum_{n=1}^{\infty} \frac{|h(n)|}{n} < \infty \tag{4.1}$$

since

$$\sum_{p\in E} \left(\frac{1}{p} + \frac{1}{pf(p)}\right) < \infty$$

and

$$\sum_{p,k\geq 2} \frac{1}{f(p^k)p^k} < \infty. \tag{4.2}$$

Then we obtain, by using the same method as in [4], pp. 266–267 (cf. [7]),

$$\sum_{n\leq x} \frac{1}{f^*(n)} = \{1+o(1)\}x \prod_{p\leq x} \left(1-\frac{1}{p}\right)\left(1-\frac{1}{pf^*(p)}\right)^{-1}$$

as $x \to \infty$. From this we conclude by (4.2)

$$\sum_{n\leq x} \frac{1}{f(n)} = \{1+o(1)\}x \prod_{p\leq x} \left(1-\frac{1}{p}\right)\left(1+\sum_{k=1}^{\infty} \frac{1}{f(p^k)p^k}\right)$$

which shows Theorem $3'$. ■

Define g^* by

$$g^*(n) = nf^*(n) \quad (n \in \mathbb{N}).$$

Then

$$\begin{aligned} F_1(s) &= \sum_{n=1}^{\infty} \frac{1}{(g(n))^s} = \prod_p \left(1+\sum_{k=1}^{\infty} \frac{1}{(g(p^k))^s}\right) = \\ &= \prod_p \left(1-\frac{1}{(g^*(p))^s}\right)^{-1} \cdot \prod_1{}'(s) \prod_2{}'(s), \end{aligned}$$

where

$$\prod_1{}'(s) = \prod_{p\in E} \left(1-\frac{1}{p^s}\right)\left(1+\sum_{k=1}^{\infty} \frac{1}{(g(p^k))^s}\right),$$

$$\prod_2{}'(s) = \prod_{p\notin E} \left(1-\frac{1}{(g(p))^s}\right)\left(1+\sum_{k=1}^{\infty} \frac{1}{(g(p^k))^s}\right).$$

Obviously the products $\prod_1'(s)$ and $\prod_2'(s)$ are absolutely convergent for $s=1$.

Denote by G the semigroup generated by

$$\mathsf{X}_{p\in E}\{1, p^k, g(p^k), pg(p^k) : k \geq 1\}\mathsf{X}_{p\notin E}\{1, (g(p))^k, g(p^k), g(p)g(p^k) : k \geq 1\}.$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{(g(n))^s} = \sum_{n=1}^{\infty} \frac{1}{(g^*(n))^s} \sum_{a\in G} \frac{h'(a)}{(a)^s}$$

with

$$\sum_{a\in G} \frac{|h'(a)|}{a} < \infty. \tag{4.3}$$

Therefore

$$\sum_{\substack{n \\ g(n)\leq x}} 1 = \sum_{\substack{a\in G, m\in\mathbb{N} \\ ag^*(m)\leq x}} h(a) = \tag{4.4}$$

$$= \sum_{a\leq x} h(a) \sum_{\substack{m \\ g^*(m)\leq \frac{x}{a}}} 1 \tag{4.5}$$

(4.6)

and by (4.3), and Theorem 1

$$\sum_{\substack{n \\ g(n)\leq x}} 1 = \{1 + o(1)\}x \sum_{a\in G} \frac{h(a)}{a} \prod_{p\leq x} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{g^*(p)}\right).$$

Thus Theorem 3 holds. ■

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