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# APPROXIMATING HOMOCLINIC ORBITS OF AUTONOMOUS SYSTEMS VIA DISCRETE DICHOTOMIES

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**Abstract.** This paper deals with the detection of transversal homoclinic orbits of implicit autonomous difference schemes using concepts and arguments of exponential dichotomy. The difference equations which are examined arise in the studying of first order autonomous systems by discretization. Based on the result in [3] a numerical method is developed for computing the homoclinic orbit of the discretized system. This method is applied to two examples, in which we show that a critical value of a system parameter Hopf bifurcation occurs and a homoclinic orbit borns. The homoclinic orbits are visualized by numerical simulations and by constructing a Matlab code.

## 1. Introduction

The theory of difference equations has received increasing attention because of its importance in various fields, such as numerical methods of differential equations, control theory and computer science. Difference equations appeared much earlier than differential equations, they were instrumental in paving the

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way for the development of the latter: difference equations also appear in the study of discretization methods for differential equations (cf. [1]). Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations (cf. [17]).

Changes in the qualitative behaviour of differential equations are often related to the appearance or disappearance of trajectories connecting one or two stationary points. They are either homoclinic or heteroclinic. Homoclinic trajectories arise typically as limiting cases of periodic orbits (cf. [4]). They appear in several systems which come from biological or physical models, for example at the study of bursting mechanism in excitable systems (cf. [21]) or determining the shape and speed of travelling waves in parabolic systems (cf. [9, 12]).

The subject of this paper is the numerical computation and visualization of homoclinic trajectories of implicit difference equations

$$g(x_{n+1}) = h(x_n) \qquad (n \in \mathbb{Z}) \tag{1.1}$$

with $\mathbb{C}^1$-diffeomorphisms $g, h : \mathbb{R}^d \to \mathbb{R}^d$ that come via discretization of suitable differential equations

$$\dot{x} = f(x) \tag{1.2}$$

with right-hand side $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfying the usual existence and uniqueness conditions, say $f \in \mathfrak{C}^1$, and having homoclinic trajectories. This mostly occurs by applying e.g. the backward Euler method or the trapezoidal rule.

The numerical detection of homoclinic orbits of system (1.2) is a widely investigated topic (cf. [3, 4, 5, 7, 10, 13, 22]). In [3] the authors apply an explicit numerical method for system (1.2), while in this paper we consider an implicit scheme instead of explicit ones. Implicit schemes have a lot of advantages over explicit schemes (cf. [23]). The main incentive why we use implicit methods is the fact that many of our numerical experiments show the following phenomenon: while using an explicit scheme for system (1.2), the bifurcation parameter (where a homoclinic orbit appears) in the difference system differs from the parameter of the continuous system, in contrast by applying an implicit scheme where the difference between the parameters in the two cases is essentially smaller.

The paper is organized as follows. In the next section we shortly review some basic tools from the theory of exponential dichotomies and we add a few results from those we mean to be important later on.

Section 3 contains the basic results of this paper. Firstly we introduce some new definitions corresponding to system (1.1) as a generalization of some generally known notions. Thereafter we give a method for the approximation of homoclinic orbits of system (1.1).

In the final section 4 we illustrate the results of section 3 by two examples using the trapezoidal rule.

## 2. Exponential dichotomy for difference equations

In this section, we shall consider a system of difference equations of the form

$$x_{n+1} = A_n x_n \qquad (n \in J) \tag{2.1}$$

where $x_n \in \mathfrak{B}$ and the operators $A_n$ belong to $L(\mathfrak{B})$, the space of bounded linear operators acting in the Banach space $\mathfrak{B}$ and $J \subset \mathbb{Z}$ is a discrete interval. Let us denote by $\|\cdot\|$ the norm on $\mathfrak{B}$ and the induced operator norm on $L(\mathfrak{B})$. Throughout the paper we assume that the operators in (2.1) are uniformly bounded, i.e.

$$\sup_{n \in J} \|A_n\| < +\infty$$

holds, furthermore we suppose the linear operators $A_n \in L(\mathfrak{B})$ $(n \in J)$ to be invertible (i.e. one-to-one and onto), then also their inverses $A_n^{-1}$ belong to the space $L(\mathfrak{B})$. In order to have non-degeneracy, additionally the condition

$$\sup_{n \in J} \left\|A_n^{-1}\right\| < +\infty$$

is required. Thus, every solution of (2.1) can be expressed in terms of the transition operator (or Cauchy operator) $\Lambda$ given by

$$\Lambda(n,m) := \begin{cases} A_n \cdot \ldots \cdot A_m & (n > m), \\ I & (n = m), \\ A_m^{-1} \cdot \ldots \cdot A_n^{-1} & (n < m) \end{cases} \qquad (n, m \in J),$$

where $I$ is the identity operator in $L(\mathfrak{B})$. This means that the solution of (2.1) initiated at $x_0 \in \mathfrak{B}$ takes the form

$$\varphi_n = \Lambda(n,0)x_0 \qquad (0 \le n \in J).$$

Without invertibility assumptions on $A_n$ the transition operator $\Lambda$ does not exist for $n < m$.

We introduce necessary definitions to state the results. First, we refer to the concept of exponential dichotomy in the sense of Coppel (cf. [8]) and it is defined as follows.

**Definition 2.1.** System (2.1) is said to posses an exponential dichotomy on the interval $J$ with constants $K_1, K_2, \alpha_1, \alpha_2 > 0$ if there exists a projection $P : \mathfrak{B} \to \mathfrak{B}$ such that for all $m, n \in J$

$$\|\Lambda(n,0) \cdot P \cdot \Lambda^{-1}(m,0)\| \leq K_1 e^{-\alpha_1(n-m)} \qquad (m \leq n), \tag{2.2}$$

$$\|\Lambda(n,0) \cdot (I-P) \cdot \Lambda^{-1}(m,0)\| \leq K_2 e^{-\alpha_2(m-n)} \qquad (m \geq n) \tag{2.3}$$

hold.

Although the approach provides the classical autonomous theory, the explicit expression for $\Lambda$ is hard to obtain in general (cf. [18]). In order to make sense of these inequalities we state an example in a finite dimensional case.

**Example 2.1.** The difference equation (2.1) with

$$A_n := A := \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \qquad (n \in \mathbb{Z})$$

possesses an exponential dichotomy on $\mathbb{Z}$ with constants $K := 1$, $\alpha := 1/2$ and projector $P = I$, because the Cauchy operator

$$\Lambda(n,m) := A^{n-m} = \begin{bmatrix} (1/2)^{n-m} & 0 \\ 0 & (-1/2)^{n-m} \end{bmatrix} \qquad (m, n \in \mathbb{Z})$$

fulfills

$$\begin{aligned} \|\Lambda(n,0)P\Lambda(0,m)\| &= \|A^{n-m}\| \leq Ke^{-\alpha(n-m)} \qquad (n \geq m \geq 0), \\ \|\Lambda(n,0)(I-P)\Lambda(0,m)\| &= \|O\| \leq Ke^{-\alpha(n-m)} \qquad (m \geq n \geq 0). \end{aligned}$$

Secondly, for the completeness of the treatment we recall a theorem of [2] (cf. [20]) which will be useful later on. With its help we will show how to relate the concept of a hyperbolic equilibrium of an autonomous system of difference equations to the exponential dichotomy of a suitable system with constant coefficients.

**Proposition 2.2.** *Suppose that the real Banach space $\mathfrak{B}$ is finite dimensional, i.e. $\mathfrak{B} := \mathbb{R}^d$, and let us consider equations of the form (2.1) with constant coefficients meaning that we are in the case $J = \mathbb{Z}$ and*

$$A_n := A \in \mathbb{R}^{d \times d} \qquad (n \in \mathbb{Z}).$$

*If the spectrum $\sigma(A)$ of the operator $A$ does not intersect the unit circle:*

$$\sigma(A) \cap \mathbb{T} = \emptyset \qquad \textit{where} \qquad \mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}, \tag{2.4}$$

*then equation (2.1) has an exponential dichotomy.*

**Example 2.3.** The spectrum of operator $A$ in Example 2.1 is the set

$$\sigma(A) = \{-1/2, 1/2\}$$

which clearly doesn't intersect the unit circle, therefore system (2.1) has an exponential dichotomy.

In order to investigate (1.1) in this section we introduce the concepts of hyperbolic fixed point and transversal homoclinic solution.

**Definition 2.2.** For $\mathbb{C}^1$-diffeomorphisms $g, h : \mathbb{R}^d \to \mathbb{R}^d$ a common fixed point $\xi \in \mathbb{R}^d$ of the maps $g$ and $h$ is called a hyperbolic fixed point of system (1.1) if the Jacobian $J_g(\xi)$ is invertible and the spectrum of

$$(J_g)^{-1}(\xi) \cdot J_h(\xi)$$

doesn't intersect the unit circle.

In the following let us denote $l_\infty(J, \mathbb{R}^d)$ the Banach space of bounded sequences on J, i.e. let

$$l_\infty(J, \mathbb{R}^d) := \{u_J := (u_n)_{n \in J} : \ u_n \in \mathbb{R}^d \ (n \in J), \ \|u_J\|_\infty := \sup_{n \in J} \|u_n\| < \infty\}$$

where $J \subset \mathbb{Z}$ is a discrete interval.

**Definition 2.3.** The sequence $\varphi_\mathbb{Z} \in S_\mathbb{Z}$ is called a homoclinic solution of system (1.1) with respect to the hyperbolic fixed point $\xi \in \mathbb{R}^d$ of (1.1) if

$$g(\varphi_{n+1}) \quad = \quad h(\varphi_n) \qquad (n \in \mathbb{Z}),$$

$$\lim_{n \to +\infty} \varphi_n = \xi \qquad \text{and} \qquad \lim_{n \to -\infty} \varphi_n = \xi.$$

A homoclinic solution $\varphi_\mathbb{Z} \in S_\mathbb{Z}$ of (1.1) is transversal if the linear system

$$x_{n+1} = J_f(\varphi_n) x_n \qquad (n \in \mathbb{Z})$$

possesses an exponential dichotomy on $\mathbb{Z}$ with $f := g^{-1} \circ h$.

Thus, for a given $\mathfrak{C}^1$-diffeomorphism $f$ we are able to connect the hyperbolic fixed points of system

$$x_{n+1} = f(x_n) \qquad (n \in \mathbb{Z}) \tag{2.5}$$

with the hyperbolic fixed points of (1.1), which is included in the following theorem.

**Theorem 2.4.** *If $\xi \in \mathbb{R}^d$ is a hyperbolic fixed point of system (1.1) and $f := g^{-1} \circ h$ then it is a hyperbolic fixed point of (2.5), too, which means that the Jacobian $J_f(\xi)$ has no eigenvalues on the unit circle. Furthermore, if the sequence $(\varphi_n)_{n\in\mathbb{Z}}$ is a homoclinic solution of (1.1) then it is a homoclinic solution of (2.5), too (with respect to the hyperbolic fixed point $\xi \in \mathbb{R}^d$).*

**Proof.** Due to the fact that the inverse of an isomorphism and every composition of diffeomorphisms are diffeomorphisms, the map $f := g^{-1} \circ h$ is well defined. Thus we have to show only two facts.

**Step 1.** If $\xi \in \mathbb{R}^d$ is a hyperbolic fixed point of system (1.1) then $\xi$, due to Definition 2.2, is a fixed point of $g$ and $h$ and a map defined by $g^{-1} \circ h$ too, because

$$(g^{-1} \circ h)(\xi) = g^{-1}(h(\xi)) = g^{-1}(\xi) = \xi$$

holds.

**Step 2.** Because of the invertibility of a composition the Jacobian of $g^{-1} \circ h$ is of the form

$$J_{g^{-1}\circ h}(x) = (J_g)^{-1}((g^{-1} \circ h)(x)) \cdot J_h(x) \qquad (x \in \mathbb{R}^d).$$

Thus the Jacobian of $f := g^{-1} \circ h$ at $\xi$ takes

$$J_f(\xi) = (J_g)^{-1}((g^{-1} \circ h)(\xi)) \cdot J_h(\xi) = (J_g)^{-1}(\xi) \cdot J_h(\xi).$$

By assumption $J_g^{-1}(\xi) \cdot J_h(\xi)$ has no eigenvalue on the unit circle, therefore $\xi$ is a hyperbolic fixed point of system (2.5).

**Step 3.** The second statement is a simple consequence of

$$\varphi_{n+1} = g^{-1}(h(\varphi_n)) = f(\varphi_n) \qquad (n \in \mathbb{Z}) \qquad \blacksquare$$

Assume that $\xi \in \mathbb{R}^d$ is a hyperbolic fixed point of system (1.1) and $f := := g^{-1} \circ h$. Then there exists $\mu_s, \mu_u \in \mathbb{R}$ such that $\mu_s < 1 < \mu_u$, furthermore each stable eigenvalue $\mu$ of $J_f(\xi)$ satisfies $|\mu| < \mu_s$ and each unstable one satisfies $|\mu| > |\mu_u|$. If $\mathcal{W}_s$ resp. $\mathcal{W}_u$ denotes the stable resp. unstable subspaces of $J_f(\xi)$, i.e.

$$\mathcal{W}_s := \left\{ x \in \mathbb{R}^d : \lim_{n\to\infty} \Lambda(n,0)x = 0 \right\},$$

resp.

$$\mathcal{W}_u := \left\{ x \in \mathbb{R}^d : \lim_{n\to-\infty} \Lambda(n,0)x = 0 \right\}$$

where

$$\Lambda(n,m) := J_f(\xi)^{n-m} \qquad (n,m \in \mathbb{Z})$$

is the solution operator of the linearized equation

$$u_{n+1} = J_f(\xi)u_n \qquad (n \in \mathbb{Z}), \tag{2.6}$$

then $\mathbb{R}^d = \mathcal{W}_s \oplus \mathcal{W}_u$ and for all $x_s \in \mathcal{W}_s$, resp. $x_u \in \mathcal{W}_u$ the estimates

$$\|J_f(\xi)^n x_s\| \le \mu_s^n \|x_s\| \quad (0 \le n \in \mathbb{N}),$$

resp.

$$\|J_f(\xi)^n x_u\| \le \mu_u^n \|x_u\| \quad (0 \le -n \in \mathbb{N})$$

hold (cf. [15]). This implies that (2.6) possesses an exponential dichotomy on any discrete interval $J \subset \mathbb{Z}$ with constants

$$K := 1, \qquad \alpha := \min\{-\ln(\mu_s), \ln(\mu_u)\}$$

and projector $P$ is onto $\mathcal{W}_s$ along $\mathcal{W}_u$.

## 3. Approximation of transversal homoclinic orbits

For the numerical computation of the homoclinic solution we have to find solution $(\varphi_n)$ of (1.1) for which

$$\lim_{n\to+\infty} \varphi_n = \xi \qquad \text{and} \qquad \lim_{n\to-\infty} \varphi_n = \xi \tag{3.1}$$

hold. Equation (1.1) may be written as an operator equation

$$\mathcal{S}(x_{\mathbb{Z}}) = 0,$$

where the $\mathfrak{C}^1$-operator $\mathcal{S} : l_\infty(\mathbb{Z}, \mathbb{R}^d) \to l_\infty(\mathbb{Z}, \mathbb{R}^d)$ is defined by

$$(\mathcal{S}(x_{\mathbb{Z}}))_n := g(x_{n+1}) - h(x_n) \qquad (x_{\mathbb{Z}} \in S_{\mathbb{Z}},\ n \in \mathbb{Z}).$$

We replace the problem solving (1.1) and (3.1) by a boundary-value problem in a finite interval

$$J := \{n \in \mathbb{Z} :\ n \in [n_-, n_+],\ n_- < 0 < n_+\}$$

with

$$\begin{aligned} g(x_{n+1}) &= h(x_n) \quad (n \in J), \\ b(x_{n_-}, x_{n_+}) &= 0, \end{aligned} \tag{3.2}$$

where $b : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a suitable $\mathfrak{C}^1$-mapping which defines a general set of boundary conditions. We will use periodic boundary conditions:

$$b(x_{n_-}, x_{n_+}) \equiv x_{n_-} - x_{n_+}.$$

Thus, we consider operator $\mathcal{S}$ on a finite interval $J$ and write equation (3.2) as

$$\mathcal{S}_J(x_J) = 0 \qquad (x_J \in l_\infty(J, \mathbb{R}^d)), \tag{3.3}$$

where $\mathcal{S}_J : l_\infty(J, \mathbb{R}^d) \to l_\infty(J, \mathbb{R}^d)$ is defined by

$$\mathcal{S}_J(x_J) = \begin{bmatrix} g(x_{n_-+1}) - h(x_{n-}) \\ \vdots \\ g(x_{n_+}) - h(x_{n_+-1}) \\ x_{n_-} - x_{n_+} \end{bmatrix}.$$

We have to solve a nonlinear equation (3.3). This is done by the Newton's method for which we need the Fréchet derivative of $\mathcal{S}_J$:

$$J_{\mathcal{S}_J}(x_J) = \begin{bmatrix} -J_h(x_{n_-}) & J_g(x_{n_-+1}) & O & \cdots & O \\ O & -J_h(x_{n_-+1}) & J_g(x_{n_-+2}) & \cdots & \vdots \\ \vdots & O & \ddots & \ddots & O \\ O & \vdots & \dots & -J_h(x_{n_+-1}) & J_g(x_{n_+}) \\ I & O & \cdots & O & -I \end{bmatrix},$$

where $I$ and $O$ denote the identity and the zero matrix in $\mathbb{R}^{d\times d}$, respectively.

Now, we formulate our main theorem as follows

**Theorem 3.1.** *Let $g, h : \mathbb{R}^d \to \mathbb{R}^d$ be $\mathfrak{C}^1$-diffeomorphisms and $\widehat{\varphi}_{\mathbb{Z}}$ be a transversal homoclinic solution with respect to the hyperbolic fixed point $\xi$ of* (1.1)*. Assume that $b \in \mathfrak{C}^1(\mathbb{R}^{2d}, \mathbb{R}^d)$ satisfies $b(\xi, \xi) = 0$ and the map $B \in L(\mathbb{R}^d, \mathbb{R}^d)$ defined by*

$$B(x_s + x_u) := \partial_1 b(\xi, \xi)x_s + \partial_2 b(\xi, \xi)x_u \qquad (x_s \in \mathcal{W}_s,\ x_u \in \mathcal{W}_s)$$

*is nonsingular. Then there exist constants $\delta, k, K > 0$ and $N \in \mathbb{N}$ such that* (3.3) *has a unique solution*

$$\varphi_J \in \mathcal{B}_\delta(\hat{\varphi}_{|J}) := \{\varphi \in l_\infty(J, \mathbb{R}^d) :\ \|\hat{\varphi}_{|J} - \varphi\|_\infty \le \delta\}$$

*for all intervals*

$$J := \{n \in \mathbb{Z} :\ n \in [n_-, n_+],\ N \le -n_-, n_+ \in \mathbb{N}\}$$

*and for all $\varphi_J \in \mathcal{B}_\delta(\hat{\varphi}_{|J})$ the following estimates hold:*

$$\|(\mathcal{S}_J)'(\varphi_J)^{-1}\| \le k, \qquad \textit{resp.} \qquad \|\hat{\varphi}_{|J} - \varphi_J\|_\infty \le K\|b(\widehat{\varphi}_{n_-}, \widehat{\varphi}_{n_+})\|.$$

**Proof.** Clearly, the map

$$f = g^{-1} \circ h : \mathbb{R}^d \to \mathbb{R}^d$$

is a $\mathfrak{C}^1$-diffeomorphism, thus in view of Theorem 2.4 and Definition 2.3 one can easily see that the conditions of Theorem 3.4. in [5] are fulfilled which gives the proof. ■

By applying the results in [6] we can leave the invertibility assumption on the righthand side function $h$ of the equation (1.1).

## 4. Applications

In the following two examples we compute approximate homoclinic orbits to the nonlinear system (3.3). We examine the given system of differential equations with trapezoidal rule. This is justified by Theorem 3.1. In both examples that follow we show that the suitable autonomous system of differential equations

$$\dot{u} = f(u) \tag{4.1}$$

has a homoclinic solution.

**4.1. Example 1.** For $c \in \mathbb{R}$ the map

$$\begin{aligned} f(x,y) &:= (f_1(x,y), f_2(x,y)) := \\ &:= \begin{bmatrix} 2y \\ 2x - 3x^2 - y(x^3 - x^2 + y^2 - c) \end{bmatrix} \quad (x, y \in \mathbb{R}) \end{aligned} \tag{4.2}$$

(cf. [11]) is a $\mathfrak{C}^\infty$-diffeomorphism and it has two fixed points for all values of the parameter $c$ given by

$$\xi_1 := (0,0), \qquad \xi_2 := (2/3, 0).$$

The Jacobians evaluated at these points are

$$J_f(\xi_1) = \begin{bmatrix} 0 & 2 \\ 2 & c \end{bmatrix} \qquad \text{and} \qquad J_f(\xi_2) = \begin{bmatrix} 0 & 2 \\ -2 & \dfrac{4}{27} + c \end{bmatrix}.$$

The eigenvalues of $\sigma(J_f(\xi_1))$ are contained in the set

$$\sigma(J_f(\xi_1)) = \left\{ \frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 + 4}, \frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + 4} \right\}.$$

Thus, the first one is an unstable hyperbolic equilibrium (a saddle) of system (4.1) and because of

$$\det(J_f(\xi_2)) = 4 > 0 \qquad \text{and} \qquad \mathrm{Tr}(J_f(\xi_2)) = \frac{4}{27} + c$$

the second equilibrium is asymptotically stable for $c < -4/27$ and it is unstable for $c > -4/27$.

**Theorem 4.1.** *If the parameter $c$ is increased at $c^* := -4/27$ then the equilibrium point $\xi_2$ undergoes a supercritical Poincaré–Andronov–Hopf bifurcation, i.e. system (4.1) has a branch of periodic solutions bifurcating from $\xi_2$ near $c = c^*$ and the bifurcating periodic solution is orbitally asymptotically stable.*

**Proof.** The eigenvalues of $J_f(\xi_2)$ are the roots of the characteristic polynomial

$$p(z) := z^2 - \mathrm{Tr}(J_f(\xi_2))z + \det(J_f(\xi_2)) \qquad (z \in \mathbb{C})$$

whose roots are $\pm\imath\omega$. Let us denote by $z(c)$ the root of $p$ that assumes the value $z(c^*) = \imath\omega$ and by

$$\mathcal{F}(z, c) :\equiv z^2 - \left(\frac{4}{27} + c\right) z + 4$$

the above characteristic polynomial as a function of parameter "$c$". Applying the implicit function theorem we get

$$z'(c^*) = -\frac{\partial_2\mathcal{F}(\imath\omega, c^*)}{\partial_1\mathcal{F}(\imath\omega, c^*)} = \frac{\imath\omega}{2\imath\omega} = \frac{1}{2} \neq 0$$

for the derivative. Thus, system (4.1) has a periodic solution near $c^*$. To prove the supercriticality of the bifurcating solution, we have to compute the sign of the first Lyapunov coefficient $l_1$ which can be calculated by

$$\text{(4.3)} \qquad \left.\begin{aligned} l_1 \;=\; & \frac{1}{2\omega} \cdot \Re\Big[\langle p, C(q, q, \overline{q})\rangle - 2\left\langle p, B\left(q, \mathfrak{A}^{-1}B(q, \overline{q})\right)\right\rangle + \\ & +\Big\langle p, B\Big(\overline{q}, (2\imath\omega I_2 - \mathfrak{A})^{-1} B(q, q)\Big)\Big\rangle\Big] \end{aligned}\right\}$$

where $\mathfrak{I}_2$ denotes the $2\times 2$ identity matrix, the bilinear function $B : \mathbb{R}^2 \times \mathbb{R}^2 \to$ $\to \mathbb{R}^2$ is given by

$$B_i(u, v) := \sum_{j,k=1}^{2} \left.\frac{\partial^2 f_i(\zeta, c^*)}{\partial\zeta_j\partial\zeta_k}\right|_{\zeta=\xi_2} u_j v_k, \quad (i \in \{1, 2\})$$

while the function $C : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$C_i(u, v, w) := \sum_{j,k,l=1}^{2} \left. \frac{\partial^3 f_i(\zeta, c^*)}{\partial\zeta_j \partial\zeta_k \partial\zeta_l} \right|_{\zeta=\xi_2} u_j v_k w_l \quad (i \in \{1, 2\}),$$

furthermore $p, q \in \mathbb{C}^2$ are left and right eigenvectors of $\mathfrak{A} := J_f(\xi_2; c^*)$ corresponding to the eigenvalues $\imath\omega$ and $-\imath\omega$, respectively, i.e. satisfying

$$\mathfrak{A}q = \imath\omega q, \qquad \mathfrak{A}^T p = -\imath\omega p \tag{4.4}$$

and are normalized by setting $\langle p, q\rangle = 1$ where $\langle\cdot, \cdot\rangle$ is the standard scalar product in $\mathbb{C}^2$, antilinear in the first argument (c.f. [19], resp. [16]). A straightforward calculation shows that $l_1 = -2 < 0$ which completes the proof. ■

We are now considering the limit case $c \longrightarrow 0$. It is easy to show that the Lyapunov-like function

$$V(x, y) := x^3 - x^2 + y^2 \qquad ((x, y) \in \mathbb{R}^2)$$

satisfies

$$\dot{V}(x, y) \equiv 3x^2\dot{x} - 2x\dot{x} + 2y\dot{y} \equiv -2y^2(V(x, y) - c).$$

For the solution $\varphi = (\varphi_1, \varphi_2)$ of (4.2)

$$V(\varphi_1(t), \varphi_2(t)) - c = (V(\varphi_1(0), \varphi_2(0)) - c) \exp\left(-2\int_0^t \varphi_2^2(s)\,\mathrm{d}s\right) \qquad (t \geq 0)$$

holds. This means that as $t \to +\infty$ either $\varphi_2(t) \longrightarrow 0$ or $V(\varphi_1(t), \varphi_2(t)) \longrightarrow c$. The first case corresponds to one of the equilibrium solutions, or to the stable manifold of the saddle equilibrium $\xi_1$. As $c \longrightarrow 0$ the level curve

$$V(x, y) = 0, \qquad x \geq 0 \tag{4.5}$$

is a homoclinic orbit of system (4.5) that tends to $\xi_1$. To see this, it is enough to show that the vector field $f$ in (4.2) with $c = 0$ is tangent to the curve (4.5) in all nonequilibrium points, which means that $f$ is orthogonal along the curve to the normal vector to the curve. It is easy to see that at $c = 0$

$$\langle \operatorname{grad} V(x, y), f(x, y)\rangle \equiv -2y^2 V(x, y) \equiv 0$$

holds along the curve (4.5).

Now we apply trapezoidal rule for the discrete system corresponding to (4.2) and get the equation

$$g(x_{n+1}, y_{n+1}) = h(x_n, y_n) \qquad (n \in \mathbb{Z}) \tag{4.6}$$

with

$$g(x,y) = \begin{bmatrix} x-\delta y \\ y-\frac{\delta}{2}(2x-3x^2-y(x^3-x^2+y^2-c)) \end{bmatrix},$$

$$h(x,y) = \begin{bmatrix} x+\delta y \\ y+\frac{\delta}{2}(2x-3x^2-y(x^3-x^2+y^2-c)) \end{bmatrix},$$

where $\delta > 0$ denotes the step size in the trapezoidal rule.

Our goal is to apply the method which we have defined in Section 3 for discrete system (4.6) and to find an orbit which is homoclinic to the origin. That is why we check first whether the origin is a hyperbolic equilibrium point of the system. For this investigation we notice that the matrix

$$(J_g)(0,0) = \begin{bmatrix} 1 & -\delta \\ -\delta & 1-\frac{\delta}{2}c \end{bmatrix}$$

is regular if and only if $\delta \neq \frac{1}{2}\cdot(\frac{c}{2} \pm \sqrt{\frac{c^2}{2}+4})$. Then we compute the matrix

$$M_c := (J_g)^{-1}(0,0)\cdot J_h(0,0) = \frac{1}{1-\delta\frac{c}{2}-\delta^2}\begin{bmatrix} 1-\frac{\delta}{2}c+\delta^2 & 2\delta \\ 2\delta & 1+\frac{\delta}{2}c+\delta^2 \end{bmatrix}.$$

Henceforward we treat only the $c=0$ case, because in the continuous system (4.2) a homoclinic orbit appears with this parameter. The matrix $M_c$ with the assumption $c=0$ has the following eigenvalues:

$$\lambda_{1,2} = \frac{1+\delta^2 \pm \sqrt{2(1+\delta^2)}}{1-\delta^2},$$

which satisfies the condition $|\lambda_{1,2}| \neq 1$ for $\delta$ small enough (cf. Figure 1), thus the origin is hyperbolic.

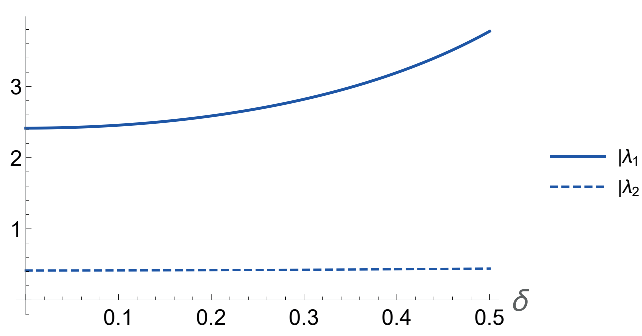


*Figure* 1. The eigenvalues of the matrix $M_c$ with $c=0$ and $\delta \in [0, 0.5]$.

After this examination we apply our new method for the system (4.6). One result can be found in Figure 2, where the bigger circles are certain points of the homoclinic orbit of the continuous system (4.2). Therefore we find the homoclinic orbit of the continuous system (4.2) in the discrete system (4.6) with the same bifurcation parameter in contrast to [3], where the parameter differs in the cases of the discrete and the continuous systems.

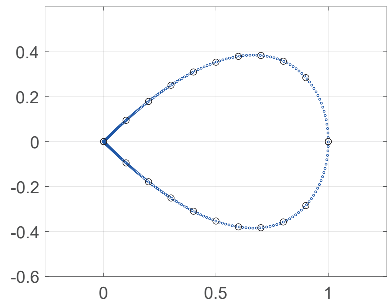


*Figure* 2. The approximation of the homoclinic orbit in system (4.6).

**4.2. Example 2.** For $\alpha, \lambda \in \mathbb{R}$: $\alpha > 0$ the map

$$f(x,y) := (f_1(x,y), f_2(x,y)) := \begin{bmatrix} y \\ x - x^2 + \lambda y + \alpha xy \end{bmatrix} \qquad (x, y \in \mathbb{R}) \tag{4.7}$$

(cf. [3]) is a $\mathfrak{C}^\infty$-diffeomorphism and the corresponding system of differential equations has two fixed points for all values of the parameters $\alpha, \lambda$ given by

$$\xi_1 := (0,0), \qquad \xi_2 := (1,0).$$

The Jacobians evaluated at these points are

$$J_f(\xi_1) = \begin{bmatrix} 0 & 1 \\ 1 & \lambda \end{bmatrix} \qquad \text{and} \qquad J_f(\xi_2) = \begin{bmatrix} 0 & 1 \\ -1 & \lambda + \alpha \end{bmatrix}.$$

The eigenvalues of $J_f(\xi_1)$ are contained in the set

$$\sigma(J_f(\xi_1)) = \left\{ \frac{\lambda}{2} - \sqrt{\left(\frac{\lambda}{2}\right)^2 + 1}, \frac{\lambda}{2} + \sqrt{\left(\frac{\lambda}{2}\right)^2 + 1} \right\}.$$

Thus, the first one is an unstable hyperbolic equilibrium (a saddle) of system (4.1) and because of

$$\det(J_f(\xi_2)) = 1 > 0 \qquad \text{and} \qquad \mathrm{Tr}(J_f(\xi_2)) = \lambda + \alpha$$

the second equilibrium is asymptotically stable for $\lambda < -\alpha$ and it is unstable for $\lambda > -\alpha$.

Similarly, as in the case of Example 1. one can prove that the following theorem holds.

**Theorem 4.2.** *Let us choose $\alpha > 0.4$. If the parameter $\lambda$ is increased at $\lambda^* := -\alpha$ then the equilibrium point $\xi_2$ undergoes a supercritical Poincaré–Andronov–Hopf bifurcation, i.e. system (4.1) has a branch of periodic solutions bifurcating from $\xi_2$ near $\lambda = \lambda^*$ and because of $l_1 = (2-5\alpha)/12 < 0$ the bifurcating periodic solution is orbitally asymptotically stable.*

Letting

$$\lambda \to \varepsilon\overline{\lambda} \qquad \text{and} \qquad \alpha \to \varepsilon\overline{\alpha}$$

system (4.1) takes the form of

$$\dot{u} = \Phi(u) + \varepsilon\Psi(u, \lambda, \alpha), \tag{4.8}$$

where for $x, y \in \mathbb{R}$

$$\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y)) := \begin{bmatrix} y \\ x - x^2 \end{bmatrix}$$

and

$$\Psi(x,y,\lambda,\alpha) = (\Psi_1(x,y), \Psi_2(x,y,\lambda,\alpha)) := \begin{bmatrix} 0 \\ \lambda y + \alpha xy \end{bmatrix}$$

(for convenience the bars are dropped). For $\varepsilon = 0$, (4.1) is a Hamiltonian system and its first integral has the form

$$V(x,y) := \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^3}{3} \qquad ((x,y) \in \mathbb{R}^2).$$

The phase portrait of (4.1) with $\varepsilon = 0$ is shown in Figure 3.

The separatrix cycle $\Gamma_0 \cup \{0\}$ shown in Figure 3 corresponds to $V(x,y) = 0$, i.e. it is represented by motions on the curves defined by

$$y_\pm(x) = \pm\frac{x}{3}\sqrt{3-2x} \qquad (x \in [0, 3/2])\,.$$

Let $\varphi^0 = (\varphi_1^0, \varphi_2^0)$ be the solution on $\Gamma_0$. We prove the existence of homoclinic orbits using Melnikov's method. We compute the Melnikov function

$$M(\lambda,\alpha) := \int_{-\infty}^{+\infty} \Phi(\varphi^0(t)) \wedge \Psi(\varphi^0(t), \lambda, \alpha)\,\mathrm{d}t,$$

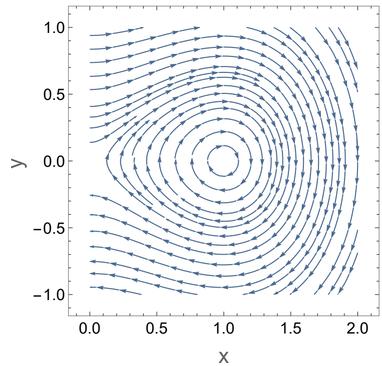


*Figure* 3. The phase portrait of (4.8) with $\varepsilon = 0$.

where the wedge product of two vectors $u, v \in \mathbb{R}^2$ is defined as $u \wedge v := \langle u, v^\perp \rangle$. Thus, using the fact that along trajectories of (4.8) $\mathrm{d}t = \mathrm{d}x/\dot{x} = \mathrm{d}x/\Phi_1(\varphi^0)$ we have

$$
\begin{aligned}
M(\lambda, \alpha) &= \int_{-\infty}^{+\infty} \Phi_1(\varphi^0(t))\Psi_2(\varphi^0(t), \lambda, \alpha)\,\mathrm{d}t = \\
&= \int_0^{3/2} \Psi_2(x, y_+(x), \lambda, \alpha)\,\mathrm{d}x - \int_0^{3/2} \Psi_2(x, y_-(x), \lambda, \alpha)\,\mathrm{d}x = \\
&= \int_0^{3/2} (\lambda + \alpha x)\left\{\Psi_2(x, y_+(x), \lambda, \alpha) - \Psi_2(x, y_-(x), \lambda, \alpha)\right\}\mathrm{d}x = \\
&= \frac{2}{3}\int_0^{3/2} (\lambda + \alpha x)x\sqrt{3 - 2x}\,\mathrm{d}x = \frac{2\sqrt{3}}{35}(7\lambda + 6\alpha),
\end{aligned}
$$

which means (cf. [14], Theorem 4.5.3) that for fixed $\alpha > 0.4$ and for all sufficiently small $\varepsilon \neq 0$, there is a $\lambda_\varepsilon := -6\alpha/7 + \mathcal{O}(\varepsilon)$ such that system (4.8) with $\lambda = \lambda^* = -6\alpha/7 + \mathcal{O}(\varepsilon)$ has a homoclinic orbit $\Gamma_\varepsilon$ with the saddle point at the origin in an $\varepsilon$-neighborhood of $\Gamma_0$.

Let us consider the continuous system corresponding to (4.7) for $\alpha = 1/2$. In the second example we use a difference equation with using trapezoidal rule, too. Therefore we treat the system

$$
g(x_{n+1},\ y_{n+1}) = h(x_n,\ y_n) \qquad (n \in \mathbb{Z}), \tag{4.9}
$$

where

$$g(x,\ y) \quad := \quad \begin{bmatrix} x - \frac{\delta}{2}y \\ (1-\frac{\delta}{2}\lambda)y - \frac{\delta}{2}(x-x^2) - \frac{\delta}{4}xy \end{bmatrix} \qquad ((x,y) \in \mathbb{R}^2),$$

$$h(x,\ y) \quad := \quad \begin{bmatrix} x + \frac{\delta}{2}y \\ (1+\frac{\delta}{2}\lambda)y + \frac{\delta}{2}(x-x^2) + \frac{\delta}{4}xy \end{bmatrix} \qquad ((x,y) \in \mathbb{R}^2).$$

We apply our new method to approximate (and find) a homoclinic solution in system (4.9). Since we show with Melnikov's method that the continuous system (4.7) has a homoclinic orbit with $\lambda = -3/7 \approx -0.428571$, we consider the implicit difference system (4.9) with the parameter $\hat{\lambda} := -0.428571$.

Similarly to the previous example we verify that the origin is a hyperbolic equilibrium point of system (4.9). For this investigation we have to compute the eigenvalues of the following matrix

(4.10)

$$M := (J_g(0,0))^{-1}J_h(0,0) = \frac{1}{1-\frac{\delta}{2}\hat{\lambda}-\frac{\delta^2}{4}} \begin{bmatrix} 1-\frac{\delta}{2}\hat{\lambda}+\frac{\delta^2}{4} & \delta \\ \delta & 1+\frac{\delta}{2}\hat{\lambda}+\frac{\delta^2}{4} \end{bmatrix}.$$

The eigenvalues of the above matrix (4.10) are

$$z_{1,2} = \frac{\mathrm{Tr}(M) \pm \sqrt{(\mathrm{Tr}(M))^2 - 4\det(M)}}{2}.$$

Because $z_1 < 1 < z_2$ with $\lambda = \hat{\lambda}$ and $\delta \in (0,\ 0.2]$ (cf. Figure 4.), the origin is a hyperbolic node.

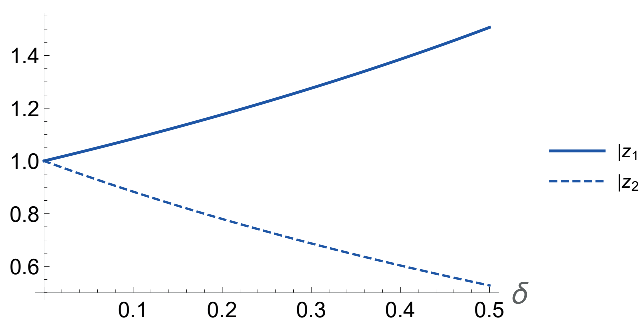


*Figure* 4. The eigenvalues of matrix (4.10) with $\lambda = \hat{\lambda}$, $\delta \in (0,\ 0.2)$.

Finally with $\lambda = \hat{\lambda}$ and $\delta = 0.02$ we apply our method for system (4.9) on the interval $J = [-500,\ 500]$. The result can be seen in Figure 5. By comparing the homoclinic orbit on Figure 5 and Figure 3 it can be observed that the orbits go on almost the same curve. Moreover again we find a homoclinic orbit in the discrete system (4.9) with the same parameter as in the continuous system (4.7).

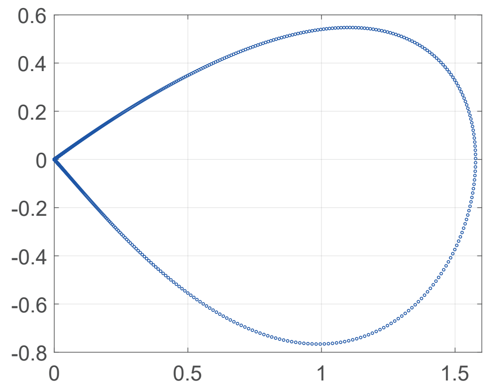


*Figure* 5. The approximation of the homoclinic orbit in system (4.9).

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