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SYMMETRIC POLYNOMIAL-LIKE BOOLEAN FUNCTIONS

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Abstract. Polynomial-like Boolean functions form a class of the Boolean functions invariant with respect to a special transform of the linear space of the two-valued logical functions. Another special set of the Boolean-functions are the set of the symmetric functions. In this article we introduce the class of the symmetric polynomial-like Boolean functions and investigate some elementary properties of such functions.

In this article disjunction and logical sum, conjunction and logical product, exclusive or and modulo two sum, as well as complementation and negation are used in the same sense and they are denoted respectively by $\vee$, $\wedge$, $\oplus$ and $\bar{\ }$. The elements of the field with two elements and the elements of the Boolean algebra with two elements are denoted by the same signs, namely by 0 and 1; $\mathbf{N}$ denotes the non-negative integers, and $\mathbf{N}^+$ the positive ones.

1. Introduction

Logical functions and especially the two-valued ones have important role in our everyday life, so it is easy to understand why they are widely investigated. A scope of the investigations is the representations of these functions and the transforms from one representation to another ([3], [4], [5]). Another area of the examinations is the search of special classes of the set of these functions.

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Post determined the closed classes of the switching functions [10], but there are a lot of another classes of the Boolean functions invariant with respect to some property. Such properties can be for example linear transforms. In [6] the author of the present paper introduced a class of the Boolean functions invariant under a special linear transform. The functions of that class are called polynomial-like Boolean functions.

1.1. Represantations of a Boolean function

It is well-known that an arbitrary two-valued logical function of n variables can be written in the uniquely determined canonical disjunctive normal form, i.e. as a logical sum whose members are pairwise distinct logical products of n factors, where each of such logical products contains every logical variable exactly once, either negated or not negated exclusively. Clearly, there exist exactly 2^n such products. Supposing that the variables are indexed by the integers $0 \leq j < n$ and the variable indexed by j is denoted by x_j, these products can be numbered by the numbers $0 \leq i < 2^n$ in such a way that we consider the non-negative integer containing 0 in the j-th position of its binary expansion if the j-th variable of the given product is negated, and 1 in the other case. Of course, this is a one to one correspondence between the 2^n distinct products and the integers of the interval $[0..2^n - 1]$, and if $i = \sum_{j=0}^{n-1} a_j^{(i)} 2^j$, where $a_j^{(i)}$ is either 0 or 1, then the product corresponding to it is

$$m_i^{(n)} = \bigwedge_{j=0}^{2^n-1} x_j^{\left(a_j^{(i)}\right)}, \tag{1.1}$$

where $x^{(0)} = \overline{x} = \overline{0} \oplus x$ and $x^{(1)} = x = \overline{1} \oplus x$. Such a product is called *minterm* (of n variables).

With the numbering given above we numbered the Boolean functions of n variables, too. A Boolean function is uniquely determined by the minterms contained in its canonical disjunctive normal form, so a Boolean function is uniquely determined by a 2^n long sequence of 0-s and 1-s, where a 0 in the j-th position (now $0 \leq j < 2^n$) means that $m_j^{(n)}$ doesn't occur in that function, and 1 means that the canonical disjunctive normal form of the function contains the minterm of the index j (this sequence is the spectrum of the canonical disjunctive normal form of the function, and similarly will be defined the spectra with respect to other representations of the function), i.e. for $l = \sum_{i=0}^{2^n-1} \alpha_i^{(l)} 2^i$ with $\alpha_i^{(l)} \in \{0, 1\}$

$$f_l^{(n)} = \bigvee_{i=0}^{2^n-1} \left(\alpha_i^{(l)} \wedge m_i^{(n)}\right). \tag{1.2}$$

Now $f_l^{(n)}$ denotes the l-th Boolean function of n variables.

Another possibility for giving a Boolean function is the so-called Zhegalkin-polynomial. Let $S_i^{(n)} = \wedge_{j=0}^{n-1} x_j^{a_j^{(i)}}$, where $x^0 = 1 = \overline{0} \vee x$, $x^1 = x = \overline{1} \vee x$ and $i = \sum_{j=0}^{n-1} a_j^{(i)} 2^j$ again. This product contains only non-negated variables, and the j-th variable is contained in it if and only if the j-th digit is 1 in the binary expansion of i. There exist exactly 2^n such products which are pairwise distinct. Now any Boolean function of n variables can be written as a modulo two sum of such terms, and the members occurring in the sum are uniquely determined by the function. That means that we can give the function by a 2^n-long 0 - 1 sequence, and if the i-th member of such a sequence is k_i then

$$f^{(n)} = \bigoplus_{i=0}^{2^n-1} \left(k_i \wedge S_i^{(n)}\right). \tag{1.3}$$

But this polynomial can be considered as a polynomial over the field of two elements, and in this case we write the polynomial in the following form:

$$f^{(n)} = \sum_{i=0}^{2^n-1} k_i S_i^{(n)}, \tag{1.4}$$

where now $S_i^{(n)} = \prod_{j=0}^{n-1} x_j^{a_j^{(i)}}$, and the sum, the product and the exponentiation are the operations of the field.

Between the first and the second representation of the same Boolean function there is a very simple linear algebraic transform. Considering the coefficients of the canonical disjunctive normal form of a Boolean function of n variables and the coefficients of the Zhegalkin polynomial of a function of n variables, respectively, as the components of an element of a 2^n-dimensional linear space over the field of two elements, denoted by $\mathbf{F}_2$, the relation between the vectors belonging to the two representations of the same Boolean function of n variables can be given by $\underline{k} = \mathbf{A}^{(n)}\underline{\alpha}$. Here $\underline{k}$ is the vector containing the components of the Zhegalkin polynomial, $\underline{\alpha}$ is the vector, composed of the coefficients of the disjunctive representation of the given function, and $\mathbf{A}^{(n)}$ is the matrix of the transform in the natural basis.

For the matrix of the transform it is true that

$$\mathbf{A}^{(n)} = \begin{cases} (1) & \text{if } n = 0 \\ \begin{pmatrix} \mathbf{A}^{(n-1)} & \mathbf{0}^{(n-1)} \\ \mathbf{A}^{(n-1)} & \mathbf{A}^{(n-1)} \end{pmatrix} & \text{if } n \in \mathbf{N}^+ \end{cases} \tag{1.5}$$

(this form of the matrix shows that for every $n \in \mathbf{N}$, $\mathbf{A}^{(n)}$ is the n-th power of the two-order $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ regular quadratic matrix, if the operation is the Kronecker-product).

From the previous results immediately follows that

$$\left(\mathbf{A}^{(n+1)}\right)^2 = \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{0}^{(n)} \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{0}^{(n)} \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \end{pmatrix} =$$

$$= \begin{pmatrix} \left(\mathbf{A}^{(n)}\right)^2 & \mathbf{0}^{(n)} \\ \mathbf{0}^{(n)} & \left(\mathbf{A}^{(n)}\right)^2 \end{pmatrix} \tag{1.6}$$

and as $\left(\mathbf{A}^{(0)}\right)^2 = (1)$, so we get by induction that

$$\left(\mathbf{A}^{(n+1)}\right)^2 = \mathbf{I}^{(n+1)}, \tag{1.7}$$

where $\mathbf{I}^{(n)}$ denotes the n-order identity matrix.

1.2. Polynomial-like Boolean functions

Let us consider again the transform between the canonical disjunctive normal form and the Zhegalkin polynomial of the same function. If $\underline{\alpha}$ is the spectrum of the canonical disjunctive normal form of the function, and $\underline{k}$ is the spectrum of the Zhegalkin polynomial of the function, then $\underline{k} = \mathbf{A}^{(n)}\underline{\alpha}$. In the special case when $\underline{\alpha} = \underline{k}$, the corresponding function is a *polynomial-like Boolean function* [6]. As $\mathbf{A}^{(0)} = (1)$, so each of the two zero variable Boolean functions is polynomial-like. Now let $\underline{u} = \underline{u}_0\underline{u}_1$ be the spectrum of the canonical disjunctive normal form of a Boolean function f of $n+1$ variables, where n is a nonnegative integer. Then

$$\begin{pmatrix} \underline{u}_0 \\ \underline{u}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{(n)} & \mathbf{0}^{(n)} \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n)} \end{pmatrix} \begin{pmatrix} \underline{u}_0 \\ \underline{u}_1 \end{pmatrix} \tag{1.8}$$

if and only if $\underline{u}_0 = \mathbf{A}^{(n)}\underline{u}_0$ and $\underline{u}_1 = \mathbf{A}^{(n)}\underline{u}_0 + \mathbf{A}^{(n)}\underline{u}_1 = \underline{u}_0 + \mathbf{A}^{(n)}\underline{u}_1$, that is f is polynomial-like if and only if $\underline{u}_0 = \left(\mathbf{A}^{(n)} + \mathbf{I}^{(n)}\right)\underline{u}_1$, where $\underline{u}_1$ is the spectrum of the canonical disjunctive normal form of an arbitrary Boolean function of n variables. As a consequence we get that the number of the $n+1$ variable polynomial-like Boolean functions is equal to 2^{2^n}. It is easy to see, too, that the spectra of the canonical disjunctive normal forms of the polynomial-like Boolean functions of $n+1$ variables make up a 2^n-dimensional subspace of the 2^{n+1}-dimensional linear space of the spectra of the canonical disjunctive normal forms of all of the $n+1$ variable Boolean functions. This space is spanned by the columns of the following matrix:

$$\begin{pmatrix} \mathbf{A}^{(n)} + \mathbf{I}^{(n)} \\ \mathbf{I}^{(n)} \end{pmatrix}. \tag{1.9}$$

1.3. Symmetric functions and symmetric polynomials

Definition 1.1. Let $n \in \mathbb{N}$, let X and Y be sets, $f : X^n \to\ Y$ and π an arbitrary element of the symmetric group S_n. The function f is symmetric, if for any $(u_0, \cdots, u_i, \cdots, u_{n-1}) \in X^n$

$$f\left(u_0, \cdots, u_i, \cdots, u_{n-1}\right) = f\left(u_{\pi(0)}, \cdots, u_{\pi(i)}, \cdots, u_{\pi(n-1)}\right). \tag{1.10}$$

If $\mathcal{K}$ is a field, and $p \in K\left[x_0, \cdots, x_i, \cdots, x_{n-1}\right]$, then p is a symmetric polynomial over $\mathcal{K}$, if

$$p = p \circ \left(x_{\pi(0)}, \cdots, x_{\pi(i)}, \cdots, x_{\pi(n-1)}\right), \tag{1.11}$$

where $\circ$ denotes the composition.

Theorem 1.1. *The Boolean function f is symmetric if and only if its Zhegalkin-polynomial is symmetric.*

Proof. Let n be a nonnegative integer and p a symmetric polynomial in n indeterminates over the field $\mathcal{K}$, furthermore let $\hat{p}$ be the polynomial function belonging to p. If π is a permutation of the set $\{0, \cdots, n-1\}$, then for any element $u_0 \cdots\ u_{n-1}$ of the set K^n

$$\begin{aligned}
\hat{p}\left(u_{0,} \cdots, u_{n-1}\right) &= p \circ \left(u_0, \cdots, u_{n-1}\right) = \\
&= \left(p \circ \left(x_0, \cdots, x_{n-1}\right)\right) \circ \left(u_0, \cdots, u_{n-1}\right) = \\
&= \left(p \circ \left(x_{\pi^{-1}(0)}, \cdots, x_{\pi^{-1}(n-1)}\right)\right) \circ \left(u_0, \cdots, u_{n-1}\right) = \\
&= p \circ \left(\left(x_{\pi^{-1}(0)}, \cdots, x_{\pi^{-1}(n-1)}\right) \circ \left(u_0, \cdots, u_{n-1}\right)\right) = \\
&= p \circ \left(\left(x_0, \cdots, x_{n-1}\right) \circ \left(u_{\pi(0)}, \cdots, u_{\pi(n-1)}\right)\right) = \\
&= p \circ \left(u_{\pi(0)}, \cdots, u_{\pi(n-1)}\right) = \\
&= \hat{p}\left(u_{\pi(0)}, \cdots, u_{\pi(n-1)}\right),
\end{aligned} \tag{1.12}$$

that is, if the polynomial p is symmetric, then so is the polynomial function determined by p, too.

Now let $\mathcal{K}$ be a field of q elements and $\varphi : K^n \to\ K$ a symmetric function. Then there exists one and only one polynomial p of degree at most $q-1$ in every indeterminates over that field that $\hat{p} = \varphi$, namely

$$p = \sum_{u_0 \cdots u_{n-1} \in K^n} \varphi\left(u_0, \cdots, u_{n-1}\right) \prod_{i=0}^{n-1} \left(e - \left(x_i - u_i\right)^{q-1}\right). \tag{1.13}$$

Then

$$
\begin{aligned}
p[x_0,\cdots,x_{n-1}] &= \sum_{u_0\cdots u_{n-1}\in K^n} \varphi\left(u_0,\cdots,u_{n-1}\right)\prod_{i=0}^{n-1}\left(e-\left(x_i-u_i\right)^{q-1}\right) = \\
&= \sum_{u_0\cdots u_{n-1}\in K^n} \varphi\left(u_0,\cdots,u_{n-1}\right)\prod_{i=0}^{n-1}\left(e-\left(x_{\pi(i)}-u_{\pi(i)}\right)^{q-1}\right) = \\
&= \sum_{u_0\cdots u_{n-1}\in K^n} \varphi\left(u_{\pi(0)},\cdots,u_{\pi(n-1)}\right)\prod_{i=0}^{n-1}\left(e-\left(x_{\pi(i)}-u_{\pi(i)}\right)^{q-1}\right) = \qquad (1.14)\\
&= \sum_{u_0\cdots u_{n-1}\in K^n} \varphi\left(u_0,\cdots,u_{n-1}\right)\prod_{i=0}^{n-1}\left(e-\left(x_{\pi(i)}-u_i\right)^{q-1}\right) \\
&= p[x_{\pi(0)},\cdots,x_{\pi(n-1)}],
\end{aligned}
$$

so also the polynomial p is symmetric.

Both parts of the above proof obviously apply even when $\mathcal{K}$ is a field of two elements, and then $\varphi = \hat{p}$ is a Boolean function and p is the corresponding Zhegalkin polynomial. ■

Remark 1.1. The polynomial function for a symmetric polynomial is a symmetric function, but the converse is not necessarily true. There are infinitely many polynomials with the same polynomial function over a finite field, most of which are not symmetric even when the corresponding polynomial function is symmetric. For example, the polynomial function for the polynomials $p^{(1)} = x_0x_1^2$ and $p^{(2)} = x_0x_1$ over the field of two elements is the symmetric function $\hat{p} = x_0x_1$, but $p^{(1)}$ is not a symmetric polynomial, since $p^{(1)} = x_0x_1^2 \neq x_0^2x_1 = p^{(3)}$.

It is worth to mention the following fact.

Theorem 1.2. *Let k be a nonnegative integer and $k \leq n \in \mathbf{N}$. The k-degree homogeneous symmetric Zhegalkin-polynomial in n indeterminates is the k-degree elementary symmetric polynomial in n indeterminates over $\mathbf{F}_2$.*

Proof. A Zhegalkin-polynomial is a polynomial of at most 1-degree in every indeterminates over $\mathbf{F}_2$, so every monomial of such a polynomial is a product of some distinct indeterminates of the polynomial. That means that the degree of a term is equal to the number of the indeterminates occuring in that term. If the polynomial is homogeneous and symmetric, and the degree of one of its terms is k then the polynomial is the sum of the k-degree monomial and only of these monomial. But now each of the k-degree monomial is a product of k

distinct indeterminates and the polynomial is the sum of every such term, and this is the k-degree elementary symmetric polinomial in n indeterminates. ■

By this result we can determine a homogeneous symmetric Zhegalkin-polynomial by fixing the number of the indeterminates and the degree of the polynomial, that is by the ordered pair of $(n; k)$ where n is the number of the indeterminates and k is the degree of the monomials occuring in the polynomial. Similarly, if A is a set of nonnegative integers not greater than n then $(n; A)$ determines a symmetric Zhegalkin-polynomial containing the k-degree monomial if and only if $k \in A$.

Let n be a nonnegative integer, $n \geq k \in \mathbf{N}$ and A is a subset of the nonnegative integers not greater than n. Then $p^{(n;k)}$ is the k-degree homogeneous symmetric Zhegalkin-polynomial in n indeterminates and $p^{(n;A)} = \sum_{k \in A} p^{(n;k)}$.

The condition that $k \leq n$ is not necessary, if we consider $p^{(n;k)}$ as the zero-polynomial in the case when k is not a nonnegative integer not greater than n.

As $2a = 0$ for any $a \in \mathbf{F}_2$, so $p^{(n;A_1)} + p^{(n;A_2)} = p^{(n;A_1 \Delta A_2)}$, where Δ denotes the symmetric difference, that is, $A_1 \Delta A_2 = (A_1 \cap \bar{A}_2) \cup (\bar{A}_1 \cap A_2)$.

If $p^{(n;A)}$ is polynomial-like, then the Boolean-function f belonging to that polynomial is the logical sum of the minterms containing exactly $n-k$ negated variables, as the spectra of the function and the polynomial are identical.

2. New results

By Proposition 7. in [6] if $f(x_0, \cdots, x_{n-1})$ is an n-variable polynomial-like Boolean function, and π is in S_n, that is in the symmetric group of n elements, then $f\left(x_{\pi(0)}, \cdots, x_{\pi(n-1)}\right)$ is also a polynomial-like Boolean function. As the modulo two sum of polynomial-like Boolean functions is again a polynomial-like Boolean function, so if f is an n-variable polynomial-like Boolean function, then

$$g(x_0, \cdots, x_{n-1}) = \oplus_{\pi \in S_n} f(x_{\pi(0)}, \cdots, x_{\pi(n-1)}) \tag{2.1}$$

is evidently a symmetric polynomial-like Boolean function. By this result the following theorem is not very surprising.

Theorem 2.1. *For every nonengative integer n there are symmetric polynomial-like Boolean functions.*

Proof. The 0-function of n-variables is polynomial-like and symmetric for every nonnegative integer n, and this is true for the Boolean function belonging to the monomial of $p = \prod_{i=0}^{n-1} x_i$ that is for the n-variable $f = \bigwedge_{i=0}^{n-1} x_i$ AND-function. ■

Remark 2.1. In the case when $n = 0$ then $p = 1$ is the constant 1-polynomial, while the corresponding f function is the TRUTH-function.

Theorem 2.2. *For any $n \in \mathbf{N}$ the spectra of the symmetric polynomial-like Boolean functions of n-variables form a linear space.*

Proof. For a given nonnegative integer n the sum of symmetric functions of n variables is a symmetric function, and over the field of two elements this is enough to be a linear space. Similarly, the set of the polynomial-like Boolean functions of n variables is a linear space, but then the intersection of the two spaces is a linear space, too. ■

Theorem 2.3. *If n is a positive integer, then the negated function of a symmetric polynomial-like Boolean function of n variables is not a symmetric polynomial-like Boolean function.*

Proof. The index of a polynomial-like Boolean function of at least one variable is an even number (see [7], Proposition 3.), so the negated function can not be polynomial-like, as then $2^{2^n} - 1 - k$ is an odd number. ■

Corollary 2.1. *If n is a positive integer then at most half of the n-variable symmetric functions is polynomial-like, so the space of the symmetric polynomial-like Boolean functions of n-variables is a proper subspace of the space of the n-variable symmetric Boolean functions.*

Proof. The complement of a symmetric Boolean function is symmetric, and it is different from the original one. ■

Theorem 2.4. *For any $3 \leq n \in \mathbf{N}$ the collection of the symmetric polynomial-like Boolean functions is a proper subspace of the space of the polynomial-like Boolean functions.*

Proof. The number of the n-variable symmetric Boolean functions is equal to 2^{n+1}, and exactly half of the indeces of that functions are even, so for a positive integer n the number of the symmetric polynomial-like Boolean functions is at most 2^n. For that n the cardinality of the set of the polynomial-like Boolean functions of n variables is equal to $2^{2^{n-1}}$. If $n \geq 3$ then $2^{n-1} > n$, and then $2^{2^{n-1}} > 2^n$, the statement is true. ■

The bound given above is sharp. Really, the zero- and the one-variable Boolean functions are obviously symmetric. The number of the two-variable polynomial-like Boolean functions is $2^{2^{n-1}} = 2^{2^{2-1}} = 2^{2^1} = 2^2 = 4$, and these functions are the 0-function, the AND and the OR function, finally the EXCLUSIVE OR function, and each of them is a symmetric function.

Theorem 2.5. *If f_k is an n-variable symmetric polynomial-like Boolean function, where n is a nonnegative integer, then $f_{2^{2^n}-2-k}$ is an n-variable symmetric polynomial-like Boolean function, too.*

Proof. If f_k is polynomial-like then $f_{2^{2^n}-2-k}$ is polynomial-like, too (see [7], Proposition 5.), and then p_k and $p_{2^{2^n}-2-k}$ are the Zhegalkin polynomials of the Boolean functions f_k and $f_{2^{2^n}-2-k}$ respectively. In the case when p_k is symmetric, then $p_{2^{2^n}-1-k}$ is symmetric, too, and symmetric also the polynomial $p_0 \equiv 1$. But then symmetric is the polynomial $p_{2^{2^n}-2-k} = p_{2^{2^n}-1-k} + p_0$, too. So, if f_k is an n-variable symmetric polinomial-like Boolean-function, then $f_{2^{2^n}-2-k}$ is a symmetric polynomial-like Boolean function, too. ■

Corollary 2.2. *The n-variable* OR*-function is a symmetric polynomial-like Boolean function for any $n \in \mathbf{N}$.*

Proof. For any nonnegative integer n $f_0 \equiv 0$ is an n-variable symmetric polynomial-like Boolean function and then $f_{2^{2^n}-2-0} = f_{2^{2^n}-2}$ has the same properties. But

$$f^{(n)}_{2^{2^n}-2} = \bigvee_{i=1}^{2^n-1} m_i^{(n)}, \tag{2.2}$$

and this function is the n-variable OR-function. ■

We have seen that the zero function, the AND function, and the OR function are symmetric polynomial-like Boolean functions for all nonnegative integers n. By our notation the polynomials determined by these functions are $p^{(n;\{\})}$, $p^{(n;n)}$ and $p^{(n;\{1,\dots,n\})}$. Let's see some more examples, namely the polynomials of the symmetric polynomial-like Boolean functions of less than 6 variables:

$p^{(0;\{\})}, p^{(0;0)}$
$p^{(1;\{\})}, p^{(1;1)}$
$p^{(2;\{\})}, p^{(2;1)}, p^{(2;2)}, p^{(2;\{1,2\})}$
$p^{(3;\{\})}, p^{(3;\{1,2\})}, p^{(3;3)}, p^{(3;\{1,2,3\})}$
$p^{(4;\{\})}, p^{(4;\{1,2\})}, p^{(4;3)}, p^{(4;\{1,2,3\})}, p^{(4;4)}, p^{(4;\{1,2,4\})}, p^{(4;\{3,4\})}, p^{(4;\{1,2,3,4\})}$
$p^{(5;\{\})}, p^{(5;3)}, p^{(5;\{1,2,4\})}, p^{(5;\{1,2,3,4\})}, p^{(5;5)}, p^{(5;\{3,5\})}, p^{(5;\{1,2,4,5\})}, p^{(5;\{1,2,3,4,5\})}$

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