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GENERALIZED DYADIC DERIVATIVE AND UNIFORM CONVERGENCE OF ITS WALSH–FOURIER SERIES

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Abstract. In the paper the notion of dyadic λ-derivative is introduced for nonnegative, nondecreasing and concave sequence $\{\lambda_n\}_{n=0}^{\infty}$. Analogues of Bernstein inequality for Walsh polynomials and of inverse approximation theorem are established. Also the uniform convergence of Walsh–Fourier series to this λ-derivative is studied.

1. Inroduction

Let us consider the function defined on the interval $[0,1)$ by $r_0(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}$, where χ_E is the indicator of a set E. We extend it to the real line by 1-periodicity and set $r_k(x) = r_0(2^k x)$, $k \in \mathbb{Z}_+ = \{0,1,\dots\}$, $x \in \mathbb{R}$. The functions $r_k(x)$ are called Rademacher functions.

Every number $n \in \mathbb{N} = \{1,2,\dots\}$ has a dyadic expansion $n = \sum_{i=0}^{k} \varepsilon_i 2^i$, where $\varepsilon_k = 1$ and ε_i are equal to 0 or 1 for $0 \le i \le k-1$. We set

$$w_n(x) = \prod_{i=0}^{k} (r_i(x))^{\varepsilon_i} = r_k(x) \prod_{i=0}^{k-1} (r_i(x))^{\varepsilon_i}$$

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in this case and $w_0(x) \equiv 1$. The system $\{w_n(x)\}_{n=o}^{\infty}$ is called Walsh system. It is well known that Walsh system is orthonormal and complete in $L^1[0,1)$, other its properties see in monographs [7] written by author, A. V. Efimov and V. A. Skvortsov and [14] written by F. Schipp, W. R. Wade and P. Simon. Also we note the paper of N. Fine [5].

For $f \in L^1[0,1)$ the Walsh-Fourier coefficients and partial sums are defined by

$$\widehat{f}(k) = \int_0^1 f(x)w_k(x)\,dx, \quad k \in \mathbb{Z}_+; \quad S_n(f)(x) = \sum_{k=0}^{n-1} \widehat{f}(k)w_k(x), \quad n \in \mathbb{N}.$$

The notions of strong and pointwise dyadic derivatives D_p and D (D_p is defined in $L^p[0,1)$) were introduced by P. L. Butzer and H. J. Wagner [3], [4]. They used a specific difference operator in these definitions and obtained the characteristic property $Dw_n = nw_n$, $n \in \mathbb{Z}_+$. Another approach of He Zelin [9] allows to define the derivative of arbitrary order $\alpha > 0$. Relations between different definitions of dyadic derivative and integral see in [8].

As usually, the space $L^p[0,1)$, $1 \le p < \infty$, consists of all measurable functions f such that $\|f\|_p^p = \int_0^1 |f(x)|^p\,dx < \infty$. Further we consider the space $C^*[0,1)$ of dyadically continuous functions as a completion of the set $\mathcal{P}$ of polynomials with respect to $\{w_n\}_{n=0}^{\infty}$ in the uniform norm $\|f\|_\infty = \sup_{x\in[0,1)} |f(x)|$ and $L^p[0,1) \equiv C^*[0,1)$ for $p = \infty$ (with the exception of Theorem 1.2).

Let $\mathcal{P}_n = \{f \in L^1[0,1) : \widehat{f}(k) = 0, k \ge n\}$, $n \in \mathbb{N}$. Then for $f \in L^p[0,1)$, $1 \le p \le \infty$, one can define the n-th best approximation by Walsh polynomials

$$E_n(f)_p = \inf\{\|f - t_n\|_p : t_n \in \mathcal{P}_n\}, \quad n \in \mathbb{N}.$$

It is known that best approximation by Walsh polynomials and dyadic modulus of continuity are equivalent in a certain sense (the corresponding C. Watari–A. V. Efimov result see in [14, Ch. 5, Theorem 2] and [7, Ch.10, Theorem 10.5.1]). Therefore we will use only the best approximation.

Let $\{\lambda_n\}_{n=0}^{\infty}$ be a nondecreasing and nonnegative sequence such that $\lim_{n\to\infty} \lambda_n = +\infty$. If $f \in L^p[0,1)$, $p \in [1,\infty]$, and the series $\sum_{n=0}^{\infty} \lambda_n \widehat{f}(n) w_n(x)$ is the Walsh-Fourier series of a function $\varphi \in L^p[0,1)$, then φ is called the λ-derivative of f in $L^p[0,1)$ (notation $\varphi = f_p^{(\lambda)}$). If $f^{(\lambda)}$ is independent of p, e.g., for Walsh polynomials, we write $f^{(\lambda)}$. Similar generalized derivatives in the trigonometric case were studied by A. I. Stepanets and his students (see, e.g., [16]). For $\lambda_n = n^\alpha$, $\alpha > 0$, λ-derivative reduces to fractional dyadic derivative of order α studied in a more general setting in [9].

Further one famous result will be used. Theorem 1.1 is an analogue of M. Riesz theorem (see [2, Ch. 8, Sect. 14 and 20]) and is due to R. E. A. C. Paley [12]. Its proof may be found in [7, Ch. 5] and in [14, Sect. 3.3]. A generalization of this result in the case of general Vilenkin systems was obtained independently by F. Schipp [13], P. Simon [15] and W.-S. Young [19].

Theorem 1.1. *Let $f \in L^p[0,1)$, $1 < p < \infty$. Then $\|f - S_n(f)\|_p \le CE_n(f)_p$, $n \in \mathbb{N}$. In particular, $\lim_{n\to\infty} \|f - S_n(f)\|_p = 0$.*

In [6] the fiollowing analogue of A. A. Konyushkov–S. B. Stechkin embedding theorem (see [11]) was obtained.

Theorem 1.2. *Let $f \in L^p[0,1)$, $1 \le p < q < \infty$ and the series $\sum_{n=1}^\infty n^{1/p-1/q-1} E_n(f)_p$ converges. Then $f \in L^q[0,1)$ and*

$$E_n(f)_q \le C(p,q)\left(n^{1/p-1/q} E_n(f)_p + \sum_{k=n+1}^\infty k^{1/p-1/q-1} E_k(f)_p\right), \quad n \in \mathbb{N}.$$

In the Lemma 2.3 below we extend this theorem on the case $q = \infty$.

In the present paper we study sufficient conditions for the continuity of $f_p^{(\lambda)}$ and uniform convergence of $\{S_n(f_p^{(\lambda)})\}_{n=1}^\infty$. Also the inverse approximation theorem is proved for λ-derivative in $L^p[0,1)$.

2. Auxiliary propositions

Lemma 2.1. *Let $\{\lambda_n\}_{n=0}^\infty$ be a nonnegative, nondecreasing and concave sequence, $1 \le p \le \infty$. Then for a polynomial $t_{2^r} = \sum_{k=0}^{2^r-1} c_k w_k \in \mathcal{P}_{2^r}$, $r \in \mathbb{Z}_+$, the inequality $\|t_{2^r}^{(\lambda)}\|_p \le C\lambda_{2^r} \|t_{2^r}\|_p$ holds.*

Proof. Let $D_n(x) := \sum_{k=0}^{n-1} w_k(x)$ and $F_n(x) := \sum_{k=1}^n D_k(x)/n$, $n \in \mathbb{N}$, $D_0(x) = 0$. Then summation by parts gives

$$\Lambda_r := \sum_{k=0}^{2^r-1} \lambda_k w_k = \sum_{k=1}^{2^r-1} (\lambda_{k-1} - \lambda_k) D_k + \lambda_{2^r-1} D_{2^r} =: I_1 + I_2.$$

Since $D_{2^k}(x) = 2^r \chi_{[0,1/2^r)}(x)$ (see [14, Sect. 1.2] or [7, Sect. 1.4]), one has

$$\|I_2\|_1 \le \lambda_{2^r-1} \le \lambda_{2^r}. \tag{2.1}$$

On the other hand, using summation by parts again we derive

$$I_1 = \sum_{k=0}^{2^r-2} \Delta^2\lambda_k(k+1)F_{k+1} + (\lambda_{2^r-1} - \lambda_{2^r})(2^r-1)F_{2^r-1},$$

where $\Delta^2\lambda_k = \lambda_k - 2\lambda_{k+1} + \lambda_{k+2}$. It is known that $\|F_k\|_1$ are bounded (see [14, Sect. 1.8]). Applying summation by parts we have

$$\sum_{k=0}^{2^r-1}(\lambda_k - \lambda_{k+1}) = \sum_{k=0}^{2^r-2}(k+1)\Delta^2\lambda_k + 2^r(\lambda_{2^r-1} - \lambda_{2^r})$$

or

$$\sum_{k=0}^{2^r-2}(k+1)\Delta^2\lambda_k + (2^r-1)(\lambda_{2^r-1} - \lambda_{2^r}) = \lambda_0 - \lambda_{2^r} - (\lambda_{2^r-1} - \lambda_{2^r}) = \lambda_0 - \lambda_{2^r-1}.$$

Therefore, by the concavity of $\{\lambda_n\}_{n=0}^\infty$

$$\|I_1\|_1 \le \sum_{k=0}^{2^r-2}(k+1)|\Delta^2\lambda_k|\|F_{k+1}\|_1 + (2^r-1)(\lambda_{2^r} - \lambda_{2^r-1})\|F_{2^r-1}\|_1 \le$$

$$\le C_1(\lambda_{2^r-1} - \lambda_0) \le C_1\lambda_{2^r}. \tag{2.2}$$

Thus, $\|\Lambda_r\|_1 \le (C_1+1)\lambda_{2^r}$ by (2.1) and (2.2). Finally, the equality $t_{2^r} * \Lambda_r(x) = = \sum_{k=0}^{2^r-1}\lambda_k c_k w_k(x)$ holds (see the definition of dyadic convolution in [14, Sect.1.3] and formula (45) for its Walsh–Fourier coefficients in the same place). Applying Lemma 1 from [14, Sect. 4.4] we obtain

$$\|t_{2^r}^{(\lambda)}\|_p = \|t_{2^r} * \Lambda_r\|_p \le \|t_{2^r}\|_p\|\Lambda_r\|_1 \le (C_1+1)\lambda_{2^r}\|t_{2^r}\|_p. \qquad \blacksquare$$

Remark 2.1. For $\lambda_n = n^\alpha$, $\alpha > 0$, the inequality of Lemma 2.1 is known in a more general setting (see [9, Lemma 1] and [18, Lemma 5]).

The following lemma is known at least in the case of concave functions (see [10, § 1, (1.20)]. The proof is given for the utility of a reader.

Lemma 2.2. *If $\{\lambda_n\}_{n=0}^\infty$ is a nonnegative, nondecreasing and concave sequence, then $\lambda_{2n} \le 2\lambda_n$, $n \in \mathbb{N}$.*

Proof. The concavity of $\{\lambda_n\}_{n=0}^\infty$ means that $\Delta^2\lambda_{n-1} \le 0$ for all $n \in \mathbb{N}$, whence

$$\lambda_{n+1} - \lambda_n \le \lambda_n - \lambda_{n-1} \le \cdots \le \lambda_k - \lambda_{k-1}, \quad k = 1, \ldots, n.$$

By summing similar inequalities one has

$$\sum_{k=n+1}^{2n} (\lambda_k - \lambda_{k-1}) \le \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}), \quad n \in \mathbb{N},$$

and $\lambda_{2n} \le 2\lambda_n - \lambda_0 \le 2\lambda_n$. ∎

Lemma 2.3 is a revision of Theorem 1.2.

Lemma 2.3. *Let $1 < p < \infty$, $f \in L^p[0,1)$, and the series $\sum_{n=1}^{\infty} n^{1/p-1} E_n(f)_p$ converges. Then f is equivalent to $f_0 \in C^*[0,1)$ (i.e. $f(x) = f_0(x)$ a.e. on $[0,1)$) and*

$$\|f_0 - S_n(f)\|_\infty \le C(p) \left(n^{1/p} E_n(f)_p + \sum_{k=n+1}^{\infty} k^{1/p-1} E_k(f)_p \right), \quad n \in \mathbb{N}. \tag{2.3}$$

In particular, there exists $\lim_{n\to\infty} \|f_0 - S_n(f)\|_\infty = 0$.

Proof. It is known the following Nikol'skii type inequality for Walsh system (see [1, Ch. 4, § 9, Lemma 1]):

$$\|t_n\|_\infty \le C_1 n^{1/p} \|t_n\|_p, \quad n \in \mathbb{N}, \quad t_n \in \mathcal{P}_n. \tag{2.4}$$

By Theorem 1.1 the equality

$$f = S_n(f) + \sum_{k=1}^{\infty} (S_{2^k n}(f) - S_{2^{k-1} n}(f)) \tag{2.5}$$

is valid, where the series converges in the space $L^p[0,1)$. From (2.4) it follows that

$$\sum_{k=1}^{\infty} \|S_{2^k n}(f) - S_{2^{k-1} n}(f)\|_\infty \le C_1 \sum_{k=1}^{\infty} (2^k n)^{1/p} \|S_{2^k n}(f) - S_{2^{k-1} n}(f)\|_p \le$$

(2.6)

$$\le 2C_1 \sum_{k=1}^{\infty} (2^k n)^{1/p} E_{2^{k-1} n}(f)_p \le C_2 \left(n^{1/p} E_n(f)_p + \sum_{j=n+1}^{\infty} j^{1/p-1} E_k(f)_p \right).$$

Since $S_k(f) \in C^*[0,1)$ for all $k \in \mathbb{N}$ and $C^*[0,1)$ with the norm $\|\cdot\|_\infty$ is a Banach space, the series in right-hand side of (2.5) converges uniformly to a function $f_0 \in C^*[0,1)$. But earlier it was proved that the series in right-hand side of (2.5) converges to the function f in $L^p[0,1)$. Therefore, $f(x) = f_0(x)$ a.e. on $[0,1)$. From (2.5) and (2.6) the inequality (2.3) follows. The last statement of Lemma 2.3 is proved as in Theorem 3.2. ∎

3. Main results

Theorem 3.1. *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *be a nonnegative, nondecreasing, tending to infinity and concave sequence,* $1 < p < \infty$*,* $f \in L^p[0,1)$ *and the series* $\sum_{k=1}^{\infty} k^{-1}\lambda_k E_k(f)_p$ *converges. Then there exists* $f_p^{(\lambda)}$ *and*

$$E_n(f_p^{(\lambda)})_p \le C\left(\lambda_n E_n(f)_p + \sum_{k=n+1}^{\infty} k^{-1}\lambda_k E_k(f)_p\right), \quad n \in \mathbb{N}.$$

Proof. Since $f \in L^p[0,1)$, $1 < p < \infty$, by Theorem 1.1 the series $S_n(f) + \sum_{k=1}^{\infty}(S_{2^k n}(f) - S_{2^{k-1}n}(f))$ converges in $L^p[0,1)$ to f. Let us consider the series

$$(S_n(f))^{(\lambda)} + \sum_{k=1}^{\infty}(S_{2^k n}(f) - S_{2^{k-1}n}(f))^{(\lambda)}. \tag{3.1}$$

By Lemmas 2.1 and 2.2 the estimate

$$\|(S_{2^k n}(f) - S_{2^{k-1}n}(f))^{(\lambda)}\|_p \le C_1\lambda_{2^k n}\|S_{2^k n}(f) - S_{2^{k-1}n}(f)\|_p \le$$

$$\le C_1\lambda_{2^k n}(\|f - S_{2^k n}(f)\|_p + \|f - S_{2^{k-1}n}(f)\|_p) \le 2C_1\lambda_{2^k n}E_{2^{k-1}n}(f)_p$$

holds. Since for $k \ge 2$

$$E_{2^{k-1}n}(f)_p \le C_2\sum_{i=2^{k-2}n+1}^{2^{k-1}n} i^{-1}E_i(f)_p,$$

we have

$$\sum_{k=1}^{\infty}\|(S_{2^k n}(f) - S_{2^{k-1}n}(f))^{(\lambda)}\|_p \le C_3\left(\lambda_n E_n(f)_p + \sum_{i=n+1}^{\infty} i^{-1}\lambda_i E_i(f)_p\right).$$

Therefore, the series (3.1) converges in $L^p[0,1)$ and its partial sum

$$(S_n(f))^{(\lambda)} + \sum_{k=1}^{N}(S_{2^k n}(f) - S_{2^{k-1}n}(f))^{(\lambda)} = (S_{2^N n}(f))^{(\lambda)}$$

has Walsh–Fourier coefficients $\lambda_j\widehat{f}(j)$ for $0 \le j \le 2^N n - 1$. Thus, there exists $\varphi \in L^p[0,1)$ such that $\lim_{N\to\infty}\|(S_{2^N n}(f))^{(\lambda)} - \varphi\|_p = 0$ and $\widehat{\varphi}(j) = \lambda_j\widehat{f}(j)$, $j \in \mathbb{Z}_+$. By definition, $\varphi = f_p^{(\lambda)}$ and

$$E_n(f_p^{(\lambda)})_p \le \|f_p^{(\lambda)} - (S_n(f))^{(\lambda)}\|_p \le \sum_{k=1}^{\infty} \|(S_{2^k n}(f) - S_{2^{k-1}n}(f))^{(\lambda)}\|_p \le$$

$$\le C_3 \left(\lambda_n E_n(f)_p + \sum_{i=n+1}^{\infty} i^{-1} \lambda_i E_i(f)_p \right). \qquad \blacksquare$$

Remark 3.1. Similar result for trigonometric case was obtained by A.I. Stepanets and E.I. Zhukina [16].

Theorem 3.2. *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *be a nonnegative, nondecreasing, tending to infinity and concave sequence,* $1 < p < \infty$, $f \in L^p[0,1)$ *and the series* $\sum_{k=1}^{\infty} k^{1/p-1} \lambda_k E_k(f)_p$ *converges. Then there exists* $f_p^{(\lambda)}$ *that is equivalent to* $f^{(\lambda)} \in C^*[0,1)$ *and* $\lim_{n\to\infty} \|f^{(\lambda)} - S_n(f^{(\lambda)})\|_\infty = 0$.

Proof. From the conditions of Theorem 3.2 it follows that $\sum_{k=1}^{\infty} k^{-1} \lambda_k E_k(f)_p < \infty$ and by Theorem 3.1 there exists $f_p^{(\lambda)} \in L^p[0,1)$. For the proof of convergence of the series $\sum_{n=1}^{\infty} n^{1/p-1} E_n(f_p^{(\lambda)})_p$ we use Theorem 3.1 and change the order of summation as follows $(1/p + 1/p' = 1)$

$$\sum_{n=1}^{\infty} n^{1/p-1} E_n(f_p^{(\lambda)})_p \le C_1 \sum_{n=1}^{\infty} n^{-1/p'} \lambda_n E_n(f)_p + C_1 \sum_{n=1}^{\infty} n^{-1/p'} \sum_{k=n}^{\infty} \frac{\lambda_k E_k(f)_p}{k}$$

$$= C_1 \sum_{n=1}^{\infty} n^{1/p-1} \lambda_n E_n(f)_p + C_1 \sum_{k=1}^{\infty} \sum_{n=1}^{k} n^{1/p-1} \frac{\lambda_k E_k(f)_p}{k} \le$$

$$\le C_2 \left(\sum_{n=1}^{\infty} n^{1/p-1} \lambda_n E_n(f)_p + \sum_{k=1}^{\infty} k^{1/p-1} \lambda_k E_k(f)_p \right) < \infty.$$

By Lemma 2.3 $f_p^{(\lambda)}$ is equivalent to $f^{(\lambda)} \in C^*[0,1)$. Using inequalities of Lemma 2.3 and Theorem 3.1 we obtain

$$\|f^{(\lambda)} - S_n(f^{(\lambda)})\|_\infty \le C_3 \left(n^{1/p} E_n(f_p^{(\lambda)})_p + \sum_{j=n+1}^{\infty} j^{1/p-1} E_j(f_p^{(\lambda)})_p \right) \le$$

$$\le C_3 n^{1/p} \left(\lambda_n E_n(f)_p + \sum_{k=n+1}^{\infty} k^{-1} \lambda_k E_k(f)_p \right) +$$

$$+ C_3 \sum_{j=n+1}^{\infty} j^{1/p-1} \left(\lambda_j E_j(f)_p + \sum_{i=j+1}^{\infty} i^{-1} \lambda_i E_i(f)_p \right) \le C_3 n^{1/p} \lambda_n E_n(f)_p +$$

$$+ 2C_3 \sum_{k=n+1}^{\infty} k^{1/p-1} \lambda_k E_k(f)_p + C_3 \sum_{j=n+1}^{\infty} j^{1/p-1} \sum_{i=j}^{\infty} i^{-1} \lambda_i E_i(f)_p. \tag{3.2}$$

Denote the last term in (3.2) by I. Then

$$I = C_3 \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{i} j^{1/p-1} i^{-1} \lambda_i E_i(f)_p \le C_4 \sum_{i=n+1}^{\infty} i^{1/p-1} \lambda_i E_i(f)_p.$$

Thus, we have

$$\|f^{(\lambda)} - S_n(f^{(\lambda)})\|_\infty \le C_5 \left(n^{1/p} \lambda_n E_n(f)_p + \sum_{k=n+1}^{\infty} k^{1/p-1} \lambda_k E_k(f)_p \right). \tag{3.3}$$

Due to Lemma 2.2 the estimate

$$n^{1/p} \lambda_n E_n(f)_p \le C_6 \sum_{k=[n/2]}^{n} k^{1/p-1} \lambda_k E_k(f)_p, \quad n \in \mathbb{N},$$

holds and the right-hand side of (3.3) tends to zero as $n \to \infty$. ■

For $\lambda_n = n^\alpha$, $\alpha > 0$, $n \in \mathbb{Z}_+$, denote $f_p^{(\lambda)}$ by $D_p^\alpha f$. Corollary 3.1 is also new.

Corollary 3.1. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $f \in L^p[0,1)$ and the series $\sum_{k=1}^{\infty} k^{\alpha+1/p-1} E_k(f)_p$ converges. Then there exists $D_p^\alpha f$ that is equivalent to $\varphi \in C^*[0,1)$ and $\lim_{n\to\infty} \|S_n(D_p^\alpha f) - \varphi\|_\infty = 0$.*

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