Annales Univ. Sci. Budapest., Sect. Comp. 49 (2019) 179–185

FUNCTIONAL EQUATIONS ON AN INFINITE HYPERGROUP JOIN

Żywilla Fechner (Łódź, Poland)
László Székelyhidi (Debrecen, Hungary)

Communicated by Antal Járai

(Received April 16, 2019; accepted August 8, 2019)

Abstract. In the paper we deal with some basic functional equations on an infinite hypergroup join. The special property of this join is that every non-identity element is of infinite order, i.e. no power of it is the identity, but there is no nonzero additive function on this hypergroup. This cannot happen in the group case, as it is well-known. As a consequence, exponential polynomials on this hypergroup have a particularly simple form.

1. Introduction

A comprehensive monograph on hypergroups is [1] and a detailed study on functional equations on hypergroups can be found in [2]. Notation and terminology here will be used according to these works. By $\mathbb{C}$ we denote the set of complex numbers, $\mathbb{N}$ is the set of all non-negative integers.

A particular hypergroup which plays a basic role in this paper is the two-element hypergroup $D(\theta)$ (see [1, 2]). For the sake of completeness, we recall here the definition.

The two-element hypergroup $D(\theta)$ on the set $\{o, a\}$ with $a \neq o$ and θ in $(0, 1]$ is defined as follows: o is the identity element, involution is the identity

Key words and phrases: Hypergroup, exponential polynomial.
2010 Mathematics Subject Classification: 20N20, 43A62, 39B99.

mapping, and convolution is defined by the equations

$$\begin{aligned}\delta_o * \delta_o &= \delta_o,\\ \delta_o * \delta_a &= \delta_a * \delta_o = \delta_a,\\ \delta_a * \delta_a &= \theta\delta_o + (1-\theta)\delta_a.\end{aligned}$$

We note that in the case $\theta = 1$, $D(\theta)$ is the two-element group $\mathbb{Z}_2$.

It is easy to check that the normalized Haar measure on $D(\theta)$ is

$$\omega_{D(\theta)} = \frac{\theta}{\theta+1}\delta_o + \frac{1}{\theta+1}\delta_a.$$

We note that on discrete hypergroups we shall use the Haar measure which is normalized in the way that the singleton $\{o\}$ has measure 1. This will be called the *unit-normalized* Haar measure.

2. Basic functions

Let K denote the hypergroup introduced in the previous section. Recall that $M\colon \mathbb{N} \to \mathbb{C}$ is an *exponential* on K if and only if for each m, n in $\mathbb{N}$ we have

$$\int_K M(k)d(\delta_m * \delta_n)(k) = M(m)M(n).$$

On a general hypergroup a *semi-character* is an exponential M satisfying $M(x^*) = \overline{M(x)}$, where x^* is the *involution* of x and overline denotes the complex conjugation. Bounded semi-characters are called *characters*. Linear combinations of characters are called *trigonometric polynomials* (see [1]). On Hermitian hypergroups semi-characters are exactly the real valued exponentials. Now we describe all exponentials on K. We need the following simple proposition.

Proposition 2.1. *We have for each* $n = 1, 2, \ldots$

$$\sum_{k=0}^{n-1} \frac{\theta_{k+1}\theta_{k+2}\cdots\theta_n}{(\theta_k+1)(\theta_{k+1}+1)\cdots(\theta_{n-1}+1)} = \theta_n.$$

Proof. We prove by induction on n. The statement for $n = 1$ is

$$\sum_{k=0}^{0} \frac{\theta_1}{\theta_0+1} = \frac{\theta_1}{0+1} = \theta_1.$$

Suppose that the statement is true for $n \geq 1$, and we prove it for $n+1$: we have

$$\sum_{k=0}^{n} \frac{\theta_{k+1}\theta_{k+2}\cdots\theta_{n+1}}{(\theta_k+1)(\theta_{k+1}+1)\cdots(\theta_n+1)} =$$

$$= \sum_{k=0}^{n-1} \left[\frac{\theta_{k+1}\theta_{k+2}\cdots\theta_{n}}{(\theta_k+1)(\theta_{k+1}+1)\cdots(\theta_{n-1}+1)}\right] \frac{\theta_{n+1}}{1+\theta_n} + \frac{\theta_{n+1}}{1+\theta_n}.$$

In the square bracket we have θ_n, by the induction hypothesis, hence

$$\sum_{k=0}^{n-1} \left[\frac{\theta_{k+1}\theta_{k+2}\cdots\theta_{n}}{(\theta_k+1)(\theta_{k+1}+1)\cdots(\theta_{n-1}+1)}\right] \frac{\theta_{n+1}}{1+\theta_n} + \frac{\theta_{n+1}}{1+\theta_n} =$$

$$= \frac{\theta_n\theta_{n+1}}{1+\theta_n} + \frac{\theta_{n+1}}{1+\theta_n} = \theta_{n+1}. \qquad \blacksquare$$

Proposition 2.2. *Assume that $M\colon K \to \mathbb{C}$ is an exponential. If $M(n) = 0$ for some n in $\mathbb{N}$, then $M(k) = 0$ for $k = n, n+1, \ldots$.*

Proof. Let $k > n$, then $\delta_k * \delta_n = \delta_k$, hence

$$M(k) = M(\delta_k) = M(\delta_k * \delta_n) = M(\delta_k)M(\delta_n) = M(k)M(n) = 0. \qquad \blacksquare$$

Proposition 2.3. *Assume that $M\colon K \to \mathbb{C}$ is an exponential. If $M(k) = 1$ for $k = 0, 1, \ldots, n-1$ for some $n \geq 1$ and $M(n) \neq 1$, then either $M(n) = 0$, or $M(n) = -\theta_n$, and $M(n+1) = 0$.*

Proof. Suppose that $M(n) \neq 0$. Then

$$M(n)^2 = M(n)M(n) = M(\delta_n * \delta_n) =$$

$$= \sum_{k=0}^{n-1} \frac{\theta_{k+1}\theta_{k+2}\cdots\theta_{n}}{(\theta_k+1)(\theta_{k+1}+1)\cdots(\theta_{n-1}+1)} M(\delta_k) + (1-\theta_n)M(\delta_n) =$$

$$= \sum_{k=0}^{n-1} \frac{\theta_{k+1}\theta_{k+2}\cdots\theta_{n}}{(\theta_k+1)(\theta_{k+1}+1)\cdots(\theta_{n-1}+1)} + (1-\theta_n)M(\delta_n) =$$

$$= \theta_n + (1-\theta_n)M(\delta_n) = \theta_n + (1-\theta_n)M(n),$$

which implies $M(n) = -\theta_n$. $\blacksquare$

Theorem 2.4. *The function $M\colon K \to \mathbb{C}$ is an exponential if and only if one of the following conditions holds:*

1. $M \equiv 1$;
2. *There exists a positive integer* n *such that* $M(k) = 1$ *for* $k = 1, 2, \dots, n-1$ *and* $M(n) = -\theta_n$, *and* $M(k) = 0$ *for* $k = n+1, n+2, \dots$.

Proof. This follows from the previous propositions. ∎

Corollary 2.1. *Every exponential on* K *is a character.*

3. Exponential monomials and polynomials

Exponential polynomials were defined in [3] (see also [4]). Here we recall the definition. In what follows K will denote the hypergroup defined above. Let $M\colon K \to \mathbb{C}$ be an exponential and y an element in K. Then

$$\Delta_{M;y} = \delta_{\check{y}} - M(y)\delta_0$$

is called the *modified difference* associated with M with increment y. In our case $\check{y} = y$, so the above formula is of the following form:

$$\Delta_{M;y} = \delta_y - M(y)\delta_0.$$

The iterates of the modified differences are denoted as follows: for each natural number n we define

$$\Delta_{M;y_1,y_2,\dots,y_{n+1}} = \prod_{k=1}^{n+1} \Delta_{M;y_k} = \Delta_{M;y_1} * \cdots * \Delta_{M;y_{n+1}},$$

where the right side is understood as a convolution product. If all increments coincide, i.e. $y_1 = y_2 = \cdots = y_{n+1} = y$ then we simply write $\Delta^{n+1}_{M;y}$ for this product.

The function $\varphi\colon K \to \mathbb{C}$ is a *generalized exponential monomial* on K, if there exists an exponential M on K and a natural number n such that

$$\Delta_{M;y_1,y_2,\dots,y_{n+1}} * \varphi(x) = 0$$

holds for each $x, y_1, y_2, \dots, y_{n+1}$ in K. It is known that if $\varphi \neq 0$, then M is unique, and the smallest n with the above property is called the *degree of* φ. In this case we may call φ a *generalized M-exponential monomial*. Linear combinations of generalized exponential monomials are called *generalized exponential polynomials*. We omit the adjective "generalized" if φ is included in a finite dimensional translation invariant linear space. Clearly, every exponential is an exponential monomial of degree 0. The next simplest exponential monomials are the M-sine functions. Given an exponential M the function φ is called an *M-sine function*, if it is a generalized exponential monomial of degree at most 1 and $\varphi(o) = 0$. It is easy to see that φ is an M-sine function if and only if it satisfies

$$f(x * y) = f(x)M(y) + f(y)M(x) \tag{3.1}$$

for each x, y in K. Obviously, every nonzero M-sine function is an M-exponential monomial of degree 1.

Now we show that on K the generalized exponential monomials are exactly the constant multiples of exponentials.

Theorem 3.1. *Every generalized exponential monomial on K is a constant multiple of an exponential.*

Proof. First we prove that every M-sine function is identically zero on K. Indeed, every K_n is a subhypergroup of K, hence the restriction of every exponential function, resp. every M-sine function on K is an exponential function, resp. M-sine function on K_n for $n = 1, 2, \ldots$. However, as K_n is finite, every M-sine function is zero on K_n (see [5]).

Now we show that every generalized exponential monomial φ of degree at most 1 is a constant multiple of an exponential on K, i.e. it is of degree 0. The function $\varphi\colon K \to \mathbb{C}$ satisfies

$$0 = \Delta_{M;y,z} * \varphi(x) = \varphi(x*y*z) - M(y)\varphi(x*z) - M(z)\varphi(x*y) + M(y)M(z)\varphi(x)$$

for each x, y, z in K. Putting $x = 0$ we have

$$\varphi(y * z) = \varphi(y)M(z) + \varphi(z)M(y) - \varphi(0)M(y)M(z),$$

or equivalently

$$\varphi(y * z) - \varphi(0)M(y)M(z) = \varphi(y)M(z) + \varphi(z)M(y) - 2\varphi(0)M(y)M(z)$$

for each y, z in K. This can be written in the form

$$\varphi(y * z) - \varphi(0)M(y * z) = [\varphi(y) - \varphi(0)M(y)]M(z) + [\varphi(z) - \varphi(0)M(z)]M(y),$$

that is, $\varphi - \varphi(0)M$ is an M-sine function. It follows that $\varphi - \varphi(0)M = 0$, hence φ is a constant multiple of an exponential. It follows that there is no generalized exponential monomial on K, which is of degree 1.

Suppose now that $\varphi\colon K \to \mathbb{C}$ is a generalized exponential monomial of degree $n \geq 2$. Then we have

$$\Delta_{M;y_1,y_2,\ldots,y_{n+1}} * \varphi(x) = 0$$

holds for each $x, y_1, y_2, \ldots, y_{n+1}$ in K, further there exist $z, z_1, z_2, \ldots, z_n$ in K such that

$$\Delta_{M;z_1,z_2,\ldots,z_n} * \varphi(z) \neq 0.$$

It follows that the function $\psi = \Delta_{M;z_1,z_2,\ldots,z_{n-1}} * \varphi$ is nonzero, and

$$\Delta_{M;y_1,y_2} * \psi(x) = \Delta_{M;y_1,y_2,z_1,\ldots,z_{n-1}} * \varphi(x) = 0,$$

which is a contradiction. It follows that there is no nonzero generalized exponential monomial on K which is of degree at least 2. ■

Corollary 3.1. *Every exponential polynomial on K is a trigonometric polynomial.*

Recall that a function $A : K \to \mathbb{C}$ is called *additive* if

$$A(x * y) = A(x) + A(y) \tag{3.2}$$

holds for each x, y in K. In particular, additive functions are 1-sine functions, where 1 stands for the exponential identically 1. Hence we have the corollary:

Corollary 3.2. *Every additive function on K is zero.*

It follows that $\mathrm{Hom}(K, \mathbb{R})$, the linear space of all real additive functions on K is trivial. We recall that the dimension of $\mathrm{Hom}(G, \mathbb{R})$ for an Abelian group G is equal to the *torsion free rank* of G, which is always positive unless G is a *torsion group*, that is, for every element g of G there is a positive integer n such that $n \cdot g = 0$, or, equivalently $\delta_g^n = \delta_0$. In the case of commutative hypergroups the situation is completely different: for every element $x \neq 0$ in K no positive convolution power of δ_x is equal to δ_0: in fact, the support of δ_x^n is $\{0, 1, \ldots, n\}$.

References

[1] **Bloom, W.R. and H. Heyer,** *Harmonic Analysis of Probability Measures on Hypergroups*, volume 20 of *de Gruyter Studies in Mathematics*, Walter de Gruyter & Co., Berlin, 1995.

[2] **Székelyhidi, L.,** *Functional Equations on Hypergroups*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[3] **Székelyhidi, L.,** Exponential polynomials on commutative hypergroups, *Arch. Math.* (*Basel*), **101(4)**, (2013), 341–347.

[4] **Székelyhidi, L.,** Characterization of exponential polynomials on commutative hypergroups, *Ann. Funct. Anal.*, **5(2)**, (2014) 53–60.

[5] **Voit, M.,** Sine functions on compact commutative hypergroups, *Arch. Math.* (*Basel*), **107(3)**, (2016), 259–263.

Ż. Fechner
Institute of Mathematics
Lodz University of Technology
Łódź
Poland
zfechner@gmail.com

L. Székelyhidi
Institute of Mathematics
University of Debrecen
Debrecen
Hungary
lszekelyhidi@gmail.com