PROBLEMS AND RESULTS ON GENERALIZED QUASI-ARITHMETIC MEANS

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Abstract. Let $M : I^2 \to I$ be a non-symmetric generalized quasi-arithmetic mean. We investigate the function $N : I^2 \to \mathbb{R}$ defined by

$$N(x, y) := \alpha x + \beta y + \gamma M(x, y) \quad (x, y \in I),$$

where $\alpha \beta \gamma \neq 0$, $\alpha \neq \beta$.

We answer the following question: Under what conditions is the function $N$ symmetric?

1. Introduction

Let $I \subset \mathbb{R}$ be non-empty open interval. A well-known generalization of the arithmetic mean

$$A(x, y) := \frac{x + y}{2} \quad (x, y \in I)$$

is the quasi-arithmetic mean

$$Af(x, y) := f^{-1}\left(\frac{f(x) + f(y)}{2}\right) \quad (x, y \in I),$$

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where \( f : I \to \mathbb{R} \) is a continuous strictly monotone function. Its generalization is the weighted quasi-arithmetic mean
\[
A_f(x, y; q) := f^{-1} \left( qf(x) + (1 - q)f(y) \right) \quad (x, y \in I),
\]
where \( f : I \to \mathbb{R} \) is again a continuous strictly monotone function and \( 0 < q < 1 \), \( q \neq \frac{1}{2} \). Unlike the first two means, the weighted quasi-arithmetic mean is a non-symmetric quasi-arithmetic mean.

A common generalization of these means is the generalized quasi-arithmetic mean introduced by J. Matkowski [2].

Let \( f, g : I \to \mathbb{R} \) be continuous strictly increasing or decreasing functions. Then the function
\[
A_{f,g}(x, y) := (f + g)^{-1} (f(x) + g(y)) \quad (x, y \in I),
\]
is called generalized quasi-arithmetic mean (see Matkowski [2] and Matkowski-Páles [3]). One can see easily that
\[
\min\{x, y\} < A_{f,g}(x, y) < \max\{x, y\}
\]
is satisfied for all \( x \neq y \) and this family of quasi-arithmetic means contains all the previous cases.

For example, if \( 0 < q < 1 \) with the notations \( f(x) := qh(x), g(x) := (1 - q)h(x), \) where \( h : I \to \mathbb{R} \) is a continuous strictly monotone function, we get that
\[
A_{f,g}(x, y) = (qh + (1 - q)h)^{-1} (qh(x) + (1 - q)h(y)) = h^{-1} (qh(x) + (1 - q)h(y)) = A_h(x, y; q) \quad (x, y \in I).
\]

The symmetry of \( A_{f,g} \) means that
\[
(f + g)^{-1} (f(x) + g(y)) = (f + g)^{-1} (f(y) + g(x))
\]
or
\[
(f - g)(x) = (f - g)(y) \quad (x, y \in I).
\]
From this one can see that if \( A_{f,g} \) is non-symmetric (that is \( A_{f,g}(x, y) \neq A_{f,g}(y, x) \) for all \( x \neq y \)), then \( f - g \) is strictly monotone and reversely, if \( f - g \) is strictly monotone, then \( A_{f,g}(x, y) \) is non-symmetric.

Our investigation starts with the function
\[
N(x, y) := \alpha x + \beta y + \gamma M(x, y) \quad (x, y \in I),
\]
where \( \alpha \beta \gamma \neq 0, \alpha \neq \beta \) and \( M : I^2 \to I \) is a non-symmetric generalized quasi-arithmetic mean. \( N : I^2 \to \mathbb{R} \) is the sum of two non-symmetric functions \( \alpha x + \beta y \) and \( \gamma M(x, y) \).

We examine the following question: Under what conditions is the function \( N \) symmetric?
2. The differentiable case

We need the following.

**Definition 1.** Let $I \subset \mathbb{R}$ be non-empty open interval. Let $D^1(I)$ denote the pairs of functions $(f,g)$ for which the functions are differentiable on $I$ and $f'(x)g'(x) > 0$ for all $x \in I$.

If $(f,g) \in D^1(I)$, then $f$ and $g$ are continuous strictly increasing or decreasing functions.

**Lemma 1.** If $(f,g) \in D^1(I)$, then
\[
\partial_1 A_{f,g}(x,x) = \frac{f''(x)}{f'(x) + g'(x)} \quad (x \in I).
\]

**Proof.** Differentiating the equation
\[
f(x) + g(y) = (f + g)(A_{f,g}(x,y))
\]
with respect to $x$ we obtain
\[
f'(x) = (f + g)'(A_{f,g}(x,y)) \partial_1 A_{f,g}(x,y).
\]
With the substitution $y = x$ we get our statement.  

**Theorem 1.** Suppose that $(f,g) \in D^1(I)$ and $f-g$ is strictly monotone. Moreover let $\alpha \beta \gamma \neq 0$ and $\alpha \neq \beta$. If function
\[
(2) \quad N(x,y) := \alpha x + \beta y + \gamma A_{f,g}(x,y) \quad (x,y \in I),
\]
satisfies symmetry equation
\[
(3) \quad N(x,y) = N(y,x) \quad (x,y \in I),
\]
then
\[
f(x) = \frac{1 + A}{1 - A} g(x) + B \quad (x \in I),
\]
where for the constant $A := \frac{\beta - \alpha}{\gamma}$, we have $A \in ]-1,1[ \setminus \{0\}$ and $B$ is an arbitrary constant.

**Proof.** From $A_{f,g}(y,x) = A_{g,f}(x,y)$ and symmetry equation (3) we get that
\[
(4) \quad A_{f,g}(x,y) - A_{g,f}(x,y) = A(x-y) \quad (x,y \in I).
Since the generalized quasi-arithmetic mean is a strict mean, inequality $|A| < 1$ holds and $A \neq 0$, that is, $A \in ]1, 1[ \setminus \{0\}$.

Differentiating equation (4) with respect to $x$ and using Lemma 1 we get, that
\[
\frac{f'(x)}{f'(x) + g'(x)} - \frac{g'(x)}{g'(x) + f'(x)} = A \quad (x \in I).
\]
Hence
\[
f'(x) = \frac{1 + A}{1 - A} g'(x) \quad (x \in I),
\]
therefore by integration
\[
f(x) = \frac{1 + A}{1 - A} g(x) + B \quad (x \in I),
\]
with a constant $B$.

**Theorem 2.** Suppose that $(f, g) \in D^1(I)$ and $f - g$ is strictly monotone. Moreover let $\alpha \beta \gamma \neq 0$ and $\alpha \neq \beta$. If function
\[
N(x, y) := \alpha x + \beta y + \gamma A_{f, g}(x, y) \quad (x, y \in I)
\]
is symmetric (that is equation (3) holds), then
\[
A_{f, g}(x, y) = h^{-1}(q h(x) + (1 - q) h(y)) = A_h(x, y; q) \quad (x, y \in I)
\]
where $h : I \rightarrow \mathbb{R}$ defined by
\[
h(x) := \frac{2 g(x)}{1 - A} + B \quad (x \in I),
\]
is continuous and strictly monotone, $A = \frac{\beta - \alpha}{\gamma}$, for the constant $q := \frac{1 + A}{2}$ we have $q \in ]0, 1[ \setminus \{\frac{1}{2}\}$ and $B$ is an arbitrary constant.

**Proof.** According to Theorem 1
\[
N(x, y) = \alpha x + \beta y + \gamma A_{f, g}(x, y) =
\]
\[
= \alpha x + \beta y + \gamma \left( \frac{1 + A}{1 - A} g + g \right)^{-1} \left( \frac{1 + A}{1 - A} g(x) + g(y) \right) =
\]
\[
= \alpha x + \beta y + \gamma \left( \frac{2 g}{1 - A} + B \right)^{-1} \left( \frac{1 + A}{1 - A} g(x) + g(y) + B \right) =
\]
\[
= \alpha x + \beta y + \gamma h^{-1} \left( \frac{(1 + A)(1 - A)(h(x) - B)}{(1 - A)^2} + \frac{(h(y) - B)(1 - A)}{2} + B \right) =
\]
\[ = \alpha x + \beta y + \gamma h^{-1} \left( \frac{1 + A}{2} h(x) + \frac{1 - A}{2} h(y) \right) = \]
\[ = \alpha x + \beta y + \gamma h^{-1} (qh(x) + (1 - q) h(y)) = \alpha x + \beta y + \gamma A_h(x, y; q). \]

Here \( A \in ]-1, 1[ \) \{0\} and hence \( q \in ]0, 1[ \) \{1/2\}.

Summarizing Theorems 1 and 2, we can say that (in the case of \((f, g) \in D^1(I)\)), if the function
\[ N(x, y) := \alpha x + \beta y + \gamma A_{f,g}(x, y) \quad (x, y \in I) \]
is symmetric then its last term (the mean \( A_{f,g} \)) is necessarily a weighted quasi-arithmetic mean.

**Problem and Conjecture:** If for the pair \((f, g)\) holds that \(f, g : I \to \mathbb{R}\) are continuous strictly increasing (or decreasing) functions and function \(N : I^2 \to \mathbb{R}\) defined in (2) (with \(\alpha \beta \gamma \neq 0\) and \(\alpha \neq \beta\)) is symmetric, then in case of the strict monotonicity of function \(f - g\), \(A_{f,g}\) is a non-symmetric weighted quasi-arithmetic mean. To prove this (with the previously used notations) it would be enough to show that the solutions of the functional equation
\[ A_{f,g}(x, y) - A_{g,f}(x, y) = A(x - y) \quad (x, y \in I), \]
satisfy \((f, g) \in D^1(I)\). In this case the statement would be a consequence of Theorem 2.

**Problem 1.** If functional equation (4) is satisfied, where \(f, g : I \to \mathbb{R}\) are continuous strictly increasing (or decreasing) functions, \(f - g\) is strictly monotone and \(A \in ]-1, 1[ \) \{0\}, then there exists a continuous strictly monotone function \(h : I \to \mathbb{R}\) such that
\[ A_{f,g}(x, y) = A_h(x, y; q), \]
where \(q = \frac{1 + A}{2} \in ]0, 1[ \) \{1/2\}.

**Problem 2.** If \(f\) and \(g\) satisfy assumptions in Problem 1, then \((f, g) \in D^1(I)\).

We need the following definition.

**Definition 2.** Let \(\mathcal{H}(I)\) be the family of function \(h : I \to \mathbb{R}\) which have one of the following forms:

(i) \(h(x) = ax + b, \ (x \in I)\), where \(a \neq 0\) and \(b\) are arbitrary constants;

(ii) \(h(x) = a\sqrt{x + t} + b, \ (x \in I)\) if there exists a \(t \in T_+(I) := \{t | I + t \subset \mathbb{R}_+\}\), where \(a \neq 0\) and \(b\) are arbitrary constants;
(iii) \( h(x) = a\sqrt{-x} + t + b \), \((x \in I)\) if there exists a \( t \in T_-(I) := \{ t \mid t - I + t \subset \mathbb{R}_+ \}\), where \( a \neq 0 \) and \( b \) are arbitrary constants.

If \( T_+(I) \cup T_-(I) = \emptyset \), that is, \( I = \mathbb{R} \), then only the case (i) is possible.

Now we can state our theorem which closes the symmetric case (see Daróczy [1]).

**Theorem 3.** Let \( I \subset \mathbb{R} \) be non-empty open interval and \( \alpha \beta \gamma \neq 0 \) and \( \alpha \neq \beta \). Suppose that \((f,g) \in D^1(I)\) and \( f - g \) is strictly monotone. Then the function
\[
N(x,y) := \alpha x + \beta y + \gamma A_{f,g}(x,y) \quad (x,y \in I)
\]
is symmetric, i.e. satisfies symmetry equation (3) if and only if
\[
q := \frac{\gamma + \beta - \alpha}{2\gamma} \in ]0,1[ \setminus \{ \frac{1}{2} \}
\]
and there exists \( h \in \mathcal{H}(I) \) such that
\[
(5) \quad f(x) = qh(x) \quad \text{and} \quad g(x) = (1-q)h(x) \quad (x \in I).
\]

**Remark 1.** If the answers for Problems 1 and 2 are affirmative, then instead of \((f,g) \in D^1(I)\) it is enough to suppose that \( A_{f,g} \) in (2) is a non-symmetric generalized quasi-arithmetic mean.

**References**


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