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PROBLEMS AND RESULTS ON GENERALIZED QUASI-ARITHMETIC MEANS

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Abstract. Let $M : I^2 \to I$ be a non-symmetric generalized quasi-arithmetic mean. We investigate the function $N : I^2 \to \mathbb{R}$ defined by

$$N(x,y) := \alpha x + \beta y + \gamma M(x,y) \qquad (x,y \in I),$$

where $\alpha\beta\gamma \neq 0$, $\alpha \neq \beta$.
We answer the following question: Under what conditions is the function N symmetric?

1. Introduction

Let $I \subset \mathbb{R}$ be non-empty open interval. A well-known generalization of the arithmetic mean

$$A(x,y) := \frac{x+y}{2} \qquad (x,y \in I)$$

is the quasi-arithmetic mean

$$A_f(x,y) := f^{-1}\left(\frac{f(x)+f(y)}{2}\right) \qquad (x,y \in I),$$

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where $f : I \to \mathbb{R}$ is a continuous strictly monotone function. Its generalization is the weighted quasi-arithmetic mean

$$A_f(x, y; q) := f^{-1}\left(qf(x) + (1-q)f(y)\right) \qquad (x, y \in I),$$

where $f : I \to \mathbb{R}$ is again a continuous strictly monotone function and $0 < q < 1$, $q \neq \frac{1}{2}$. Unlike the first two means, the weighted quasi-arithmetic mean is a non-symmetric quasi-arithmetic mean.

A common generalization of these means is the generalized quasi-arithmetic mean introduced by J. Matkowski [2].

Let $f, g : I \to \mathbb{R}$ be continuous strictly increasing or decreasing functions. Then the function

$$A_{f,g}(x, y) := (f+g)^{-1}\left(f(x) + g(y)\right) \qquad (x, y \in I),$$

is called generalized quasi-arithmetic mean (see Matkowski [2] and Matkowski-Páles [3]). One can see easily that

$$\min\{x, y\} < A_{f,g}(x, y) < \max\{x, y\}$$

is satisfied for all $x \neq y$ and this family of quasi-arithmetic means contains all the previous cases.

For example, if $0 < q < 1$ with the notations $f(x) := qh(x)$, $g(x) := (1-q)h(x)$, where $h : I \to \mathbb{R}$ is a continuous strictly monotone function, we get that

$$\begin{aligned} A_{f,g}(x, y) &= (qh + (1-q)h)^{-1}\left(qh(x) + (1-q)h(y)\right) = \\ &= h^{-1}\left(qh(x) + (1-q)h(y)\right) = A_h(x, y; q) \qquad (x, y \in I). \end{aligned}$$

The symmetry of $A_{f,g}$ means that

$$(f+g)^{-1}\left(f(x) + g(y)\right) = (f+g)^{-1}\left(f(y) + g(x)\right)$$

or

$$(f-g)(x) = (f-g)(y) \ (x, y \in I).$$

From this one can see that if $A_{f,g}$ is non-symmetric (that is $A_{f,g}(x, y) \neq A_{f,g}(y, x)$ for all $x \neq y$), then $f - g$ is strictly monotone and reversely, if $f - g$ is strictly monotone, then $A_{f,g}(x, y)$ is non-symmetric.

Our investigation starts with the function

$$N(x, y) := \alpha x + \beta y + \gamma M(x, y) \qquad (x, y \in I), \tag{1}$$

where $\alpha\beta\gamma \neq 0$, $\alpha \neq \beta$ and $M : I^2 \to I$ is a non-symmetric generalized quasi-arithmetic mean. $N : I^2 \to \mathbb{R}$ is the sum of two non-symmetric functions $\alpha x + \beta y$ and $\gamma M(x, y)$.

We examine the following question: Under what conditions is the function N symmetric?

2. The differentiable case

We need the following.

Definition 1. Let $I \subset \mathbb{R}$ be non-empty open interval. Let $D^1(I)$ denote the pairs of functions (f, g) for which the functions are differentiable on I and $f'(x)g'(x) > 0$ for all $x \in I$.

If $(f, g) \in D^1(I)$, then f and g are continuous strictly increasing or decreasing functions.

Lemma 1. *If* $(f, g) \in D^1(I)$, *then*

$$\partial_1 A_{f,g}(x, x) = \frac{f'(x)}{f'(x) + g'(x)} \qquad (x \in I).$$

Proof. Differentiating the equation

$$f(x) + g(y) = (f + g)\left(A_{f,g}(x, y)\right)$$

with respect to x we obtain

$$f'(x) = (f + g)'\left(A_{f,g}(x, y)\right) \partial_1 A_{f,g}(x, y).$$

With the substitution $y = x$ we get our statement. ∎

Theorem 1. *Suppose that* $(f, g) \in D^1(I)$ *and* $f - g$ *is strictly monotone. Moreover let* $\alpha\beta\gamma \neq 0$ *and* $\alpha \neq \beta$. *If function*

$$N(x, y) := \alpha x + \beta y + \gamma A_{f,g}(x, y) \qquad (x, y \in I), \tag{2}$$

satisfies symmetry equation

$$N(x, y) = N(y, x) \qquad (x, y \in I), \tag{3}$$

then

$$f(x) = \frac{1 + A}{1 - A} g(x) + B \qquad (x \in I),$$

where for the constant $A := \frac{\beta - \alpha}{\gamma}$, *we have* $A \in]-1, 1[\setminus \{0\}$ *and* B *is an arbitrary constant.*

Proof. From $A_{f,g}(y, x) = A_{g,f}(x, y)$ and symmetry equation (3) we get that

$$A_{f,g}(x, y) - A_{g,f}(x, y) = A(x - y) \qquad (x, y \in I). \tag{4}$$

Since the generalized quasi-arithmetic mean is a strict mean, inequality $|A| < 1$ holds and $A \neq 0$, that is, $A \in]-1, 1[\setminus \{0\}$.

Differentiating equation (4) with respect to x and using Lemma 1 we get, that

$$\frac{f'(x)}{f'(x)+g'(x)} - \frac{g'(x)}{g'(x)+f'(x)} = A \qquad (x \in I).$$

Hence

$$f'(x) = \frac{1+A}{1-A} g'(x) \qquad (x \in I),$$

therefore by integration

$$f(x) = \frac{1+A}{1-A} g(x) + B \qquad (x \in I),$$

with a constant B. ■

Theorem 2. *Suppose that $(f, g) \in D^1(I)$ and $f - g$ is strictly monotone. Moreover let $\alpha\beta\gamma \neq 0$ and $\alpha \neq \beta$. If function*

$$N(x, y) := \alpha x + \beta y + \gamma A_{f,g}(x, y) \qquad (x, y \in I)$$

is symmetric (that is equation (3) holds), then

$$A_{f,g}(x, y) = h^{-1}\left(qh(x) + (1-q)h(y)\right) = A_h(x, y; q) \qquad (x, y \in I)$$

where $h : I \to \mathbb{R}$ defined by

$$h(x) := \frac{2g(x)}{1-A} + B \qquad (x \in I),$$

is continuous and strictly monotone, $A = \frac{\beta-\alpha}{\gamma}$, for the constant $q := \frac{1+A}{2}$ we have $q \in]0, 1[\setminus \left\{\frac{1}{2}\right\}$ and B is an arbitrary constant.

Proof. According to Theorem 1

$$N(x, y) = \alpha x + \beta y + \gamma A_{f,g}(x, y) =$$

$$= \alpha x + \beta y + \gamma \left(\frac{1+A}{1-A} g + B + g\right)^{-1} \left(\frac{1+A}{1-A} g(x) + B + g(y)\right) =$$

$$= \alpha x + \beta y + \gamma \left(\frac{2g}{1-A} + B\right)^{-1} \left(\frac{1+A}{1-A} g(x) + g(y) + B\right) =$$

$$= \alpha x + \beta y + \gamma h^{-1}\left(\frac{(1+A)(1-A)(h(x)-B)}{(1-A)\,2} + \frac{(h(y)-B)(1-A)}{2} + B\right) =$$

$$= \alpha x + \beta y + \gamma h^{-1}\left(\frac{1+A}{2}h(x) + \frac{1-A}{2}h(y)\right) =$$

$$= \alpha x + \beta y + \gamma h^{-1}\left(qh(x) + (1-q)\,h\,(y)\right) = \alpha x + \beta y + \gamma A_h\,(x,y;q)\,.$$

Here $A \in]-1,1[\setminus\{0\}$ and hence $q \in]0,1[\setminus\left\{\frac{1}{2}\right\}$. ■

Summarizing Theorems 1 and 2, we can say that (in the case of $(f,g) \in$ $\in D^1(I)$), if the function

$$N(x,y) := \alpha x + \beta y + \gamma A_{f,g}(x,y) \qquad (x,y \in I)$$

is symmetric then its last term (the mean $A_{f,g}$) is necessarily a weighted quasi-arithmetic mean.

Problem and Conjecture: If for the pair (f,g) holds that $f,g : I \to \mathbb{R}$ are continuous strictly increasing (or decreasing) functions and function $N : I^2 \to$ $\to \mathbb{R}$ defined in (2) (with $\alpha\beta\gamma \neq 0$ and $\alpha \neq \beta$) is symmetric, then in case of the strict monotonicity of function $f-g$, $A_{f,g}$ is a non-symmetric weighted quasi-arithmetic mean. To prove this (with the previously used notations) it would be enough to show that the solutions of the functional equation

$$A_{f,g}(x,y) - A_{g,f}(x,y) = A(x-y) \qquad (x,y \in I),$$

satisfy $(f,g) \in D^1(I)$. In this case the statement would be a consequence of Theorem 2.

Problem 1. If functional equation (4) is satisfied, where $f,g : I \to \mathbb{R}$ are continuous strictly increasing (or decreasing) functions, $f-g$ is strictly monotone and $A \in]-1,1[\setminus\{0\}$, then there exists a continuous strictly monotone function $h : I \to \mathbb{R}$ such that

$$A_{f,g}(x,y) = A_h(x,y;q),$$

where $q = \frac{1+A}{2} \in]0,1[\setminus\left\{\frac{1}{2}\right\}$.

Problem 2. If f and g satisfy assumptions in Problem 1, then $(f,g) \in D^1(I)$.

We need the following definition.

Definition 2. Let $\mathcal{H}(I)$ be the family of function $h : I \to \mathbb{R}$ which have one of the following forms:

(i) $h(x) = ax + b,\ (x \in I)$, where $a \neq 0$ and b are arbitrary constants;

(ii) $h(x) = a\sqrt{x+t} + b,\ (x \in I)$ if there exists a $t \in T_+(I) := \{t | I + t \subset \mathbb{R}_+\}$, where $a \neq 0$ and b are arbitrary constants;

(iii) $h(x) = a\sqrt{-x+t} + b$, $(x \in I)$ if there exists a $t \in T_-(I) := \{t| -I + t \subset \subset \mathbb{R}_+\}$, where $a \neq 0$ and b are arbitrary constants.

If $T_+(I) \cup T_-(I) = \emptyset$, that is, $I = \mathbb{R}$, then only the case (i) is possible.

Now we can state our theorem which closes the symmetric case (see Daróczy [1]).

Theorem 3. *Let $I \subset \mathbb{R}$ be non-empty open interval and $\alpha\beta\gamma \neq 0$ and $\alpha \neq \beta$. Suppose that $(f, g) \in D^1(I)$ and $f - g$ is strictly monotone. Then the function*

$$N(x,y) := \alpha x + \beta y + \gamma A_{f,g}(x,y) \qquad (x, y \in I)$$

is symmetric, i.e. satisfies symmetry equation (3) if and only if

$$q := \frac{\gamma + \beta - \alpha}{2\gamma} \in]0,1[\setminus \left\{\frac{1}{2}\right\}$$

and there exists $h \in \mathcal{H}(I)$ such that

$$f(x) = qh(x) \quad \text{and} \quad g(x) = (1-q)h(x) \qquad (x \in I). \tag{5}$$

Remark 1. If the answers for Problems 1 and 2 are affirmative, then instead of $(f, g) \in D^1(I)$ it is enough to suppose that $A_{f,g}$ in (2) is a non-symmetric generalized quasi-arithmetic mean.

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