POISSON LIMIT THEOREMS FOR THE GENERALIZED ALLOCATION SCHEME

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Abstract. Poisson limit theorems are studied for discrete probability models. Consider the allocation of n balls into N boxes. Let $\mu_r(N, K, n)$ denote the number of those boxes from the first K boxes which contain r balls. Convergence of $\mu_r(N, K, n)$ to a Poisson distribution is proved as $K, n \to \infty$. Analogous results are obtained for the generalized allocation scheme and also for several discrete probabilistic models.

1. Introduction

Poisson approximation is a good tool to estimate the probability of rare events. The well-known approximation theorem of the binomial distribution by Poisson distribution has lot of extensions and refinements. One of the most famous results is Le Cam's theorem. Let $X_1, \dots, X_n$ be independent Bernoulli variables with $\mathbb{P}(X_i = 1) = p_i$, $i = 1, \dots, n$, $S_n = X_1 + \cdots + X_n$ and $\lambda = p_1 + \cdots + p_n$. Then

$$\sum_{k=0}^{\infty} \left| \mathbb{P}\left(S_n = k\right) - \frac{\lambda^k}{k!} e^{-\lambda} \right| < 2 \sum_{k=1}^{n} p_k^2.$$

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This result has many versions. One can find an overview of the Poisson approximation theorems in the monograph [1]. In [1] the Stein-Chen method and coupling are used to find upper bounds for the distance of the distribution studied and an appropriate Poisson distribution. However, in this paper we do not study rate of convergence, so here we mention only some usual convergence theorems.

The generalized allocation scheme was introduced by V. F. Kolchin in [6]. Several models of discrete probability theory, such as random permutations, random forests, random partitions, urn schemes are particular cases of the generalized allocation scheme, see [7]. There are papers and books presenting Poisson limit theorems for the number of boxes containing r balls (where r is a fixed number), see e.g. [17], [18], [19], [20] and the references therein. Poisson limit theorems for the number of boxes containing fixed number of balls in the model of allocation of distinct balls into boxes were presented by Kolchin, Sevast'yanov and Chistyakov in the monograph [11]. In [5] the usual allocation scheme was studied and limit theorems were obtained for the number of empty boxes in a fixed set of boxes.

In this paper we will study the generalized allocation scheme. We shall obtain Poisson limit theorems for the number of those boxes from the first K boxes which contain r balls, Theorems 2.2 and 2.3. Refinements of these theorems are obtained for certain particular cases. Theorem 2.4 concerns the uniform allocation scheme of n distinguishable balls into N boxes. Theorem 2.5 is devoted to the homogeneous allocation scheme of n indistinguishable balls into N boxes. Theorem 2.6 deals with random forests. Poisson limit is obtained for the number of those trees in the set of the first K trees which have r non-root vertices. In Theorems 2.7 and 2.8 the multicolour urn scheme is studied. We consider the number of those colours among the first K colours from which r balls are chosen, and we prove that this quantity is asymptotically Poisson. The novelty of our results is that we study the first K boxes and not the whole set of boxes. The proofs are based on a Poisson limit theorem for exchangeable events, see Proposition 2.1. We also use Le Cam's inequality in our proofs.

2. Main results for the generalized allocation scheme

Throughout the paper $\stackrel{d}{\to}$ will denote convergence in distribution, $\pi(\beta)$ will denote a Poisson random variable with parameter β, $0 \le \beta < \infty$. Let $\eta'_i = \eta'_{Ki}$, $1 \le i \le K$, $K = 1, 2, \ldots$, be an array of row-wise independent identically distributed non-negative integer valued random variables. Let r be a non-negative integer. Let $A_i = A_{Ki} = \{\eta'_i = r\}$ be the event that η'_i has fixed value r and let $\mu_r(K) = \sum_{i=1}^{K} \mathbb{I}_{A_i}$, where $\mathbb{I}_A$ is the indicator of the set A. By the simplest version of the Poisson limit theorem we have

$$\mu_r(K) \xrightarrow{d} \pi(\beta) \quad \text{as} \quad K\mathbb{P}(A_i) \to \beta. \tag{2.1}$$

In this paper we obtain analogues of this result in the case when $\eta'_1, \dots, \eta'_N$ are certain dependent random variables playing important role in combinatorial probability theory.

Recall that the random variables $\eta'_1, \dots, \eta'_K$ are called exchangeable if the distribution of $(\eta'_1, \dots, \eta'_K)$ coincides with the distribution of $(\eta'_{i_1}, \dots, \eta'_{i_K})$ for any permutation $(i_1, \dots, i_K)$ of $(1, \dots, K)$.

The following known elementary limit theorem will play fundamental role in our paper (see Theorem II in [4]; we mention that Benczúr presented a slightly more general result without proof, see Theorem 1 in [2]).

Proposition 2.1. *Let the array of random variables $\eta_i = \eta_{Ki}$, $1 \le i \le K$, $K = 1, 2, \dots$, be row-wise exchangeable. Let $A_i = A_{Ki} = \{\eta_{Ki} = r\}$, where r is fixed and let $\mu_r(K) = \sum_{i=1}^{K} \mathbb{I}_{A_i}$. Suppose that the following condition is valid.*

There exists β $(0 \le \beta < \infty)$ such that for any $k = 1, 2, \dots$

$$K^k \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_k) \to \beta^k \quad \textit{as} \quad K \to \infty. \tag{2.2}$$

Then

$$\mu_r(K) \xrightarrow{d} \pi(\beta) \quad \textit{as} \quad K \to \infty. \tag{2.3}$$

The generalized allocation scheme was introduced by V. F. Kolchin in [6]. Let $\xi_1, \xi_2, \dots, \xi_N$ be independent identically distributed non-negative integer valued random variables. Denote by

$$p_l = \mathbb{P}\{\xi_i = l\}, \quad l = 0, 1, \dots$$

their distribution. We say that the random variables $\eta_1, \dots, \eta_N$ satisfy the generalized allocation scheme of allocation of n balls into N boxes, if their joint distribution is of the form

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \mathbb{P}\left\{\xi_1 = k_1, \dots, \xi_N = k_N \;\middle|\; \sum_{i=1}^{N} \xi_i = n\right\}, \tag{2.4}$$

for all non-negative integer numbers $k_1, k_2, \dots, k_N$ such that $k_1 + k_2 + \dots + k_N = n$. Various models of discrete probability theory, such as random permutations, random forests, random partitions, urn schemes are particular cases of the generalized allocation scheme, see [7].

Let $0 < K \leq N$. In the generalized allocation scheme, we will denote by $\mu_r(N, K, n)$ the number of those boxes from the first K boxes which contain r balls. That is

$$\mu_r(N, K, n) = \sum_{i=1}^{K} \mathbb{I}_{A_i},$$

where $A_i = \{\eta_i = r\}$.

The aim of this paper is to prove that under appropriate conditions

$$\mu_r(N, K, n) \xrightarrow{d} \pi(\beta), \tag{2.5}$$

where $\pi(\beta)$ is a Poisson random variable with parameter β. Observe that $\eta_1, \ldots, \eta_N$ are exchangeable random variables. Let

$$\zeta_l = \xi_1 + \xi_2 + \cdots + \xi_l, \quad l = 1, 2, \ldots, N.$$

From (2.4) it follows that

$$\begin{aligned} &\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \\ &= (p_r)^k \frac{\mathbb{P}\{\zeta_{N-k} = n - kr\}}{\mathbb{P}\{\zeta_N = n\}}, \text{ if } n - kr \geq 0,\ N - k \geq 1, \end{aligned} \tag{2.6}$$

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \frac{(p_r)^N}{\mathbb{P}\{\zeta_N = n\}}, \text{ if } n = kr,\ N = k,$$

and

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = 0$$

in all other cases.

Remark 2.1. We have the following representation of the distribution of $\mu_r(N, K, n)$:

$$\mathbb{P}\left(\mu_r(N, K, n) = k\right) = p_r^k (1 - p_r)^{K-k} \frac{\mathbb{P}\{\zeta_{N-k}^{\{r\}} = n - kr\}}{\mathbb{P}\{\zeta_N = n\}}, \quad 0 \leq k \leq K, \tag{2.7}$$

where $\zeta_N = \xi_1 + \xi_2 + \cdots + \xi_N$,

$$\zeta_{N-k}^{\{r\}} = \xi_1^{\{r\}} + \cdots + \xi_{K-k}^{\{r\}} + \xi_{K+1} + \cdots + \xi_N,$$

the random variables $\xi_1^{\{r\}}, \ldots, \xi_{K-k}^{\{r\}}, \xi_{K+1}, \ldots, \xi_N$ are independent, and the random variables $\xi_i^{\{r\}}$, $i = 1, \ldots, K - k$, have the following distribution

$$\mathbb{P}\{\xi_i^{\{r\}} = j\} = \mathbb{P}\{\xi_i = j \mid \xi_i \neq r\}, \quad j = 0, 1, 2, \ldots.$$

The proof of this formula is similar to the proofs of some well-known analogous formulae (see, for example, Lemma 1.2.1 in [7]). But in (2.7) we have conditional probabilities, therefore it is more difficult than (2.6). Therefore in this paper we shall use Proposition 2.1 and (2.6).

Usually, in the generalized allocation scheme, the distribution of ξ_i is of power law. In this paper we will assume this, so $\xi_i = \xi_i(\alpha)$, $1 \le i \le N$, have the following distribution

$$p_k = p_k(\alpha) = \mathbb{P}\{\xi_i = k\} = \frac{b_k \alpha^k}{k! B(\alpha)}, \quad k = 0, 1, 2, \dots, \tag{2.8}$$

where $b_k \ge 0$ for each k, and the power series $B(\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \alpha^k$ has positive radius of convergence $R > 0$. We shall use the following condition for $l \ge 1$

$$(Al): \qquad b_0 > 0, b_l > 0 \ \text{ and } \ b_i = 0 \ \text{ for } \ 0 < i < l.$$

Condition $(A1)$ was introduced in [8], while condition (Al) appeared first in [9]. Integral and local limit theorems for sums of $\xi_i(\alpha)$ with the property $(A1)$ were obtained in [8]. Generalizations of those results for random variables $\xi_i(\alpha)$ with condition (Al) were obtained in [9] and [10].

The expectation $m = m(\alpha)$, the variance $\sigma^2 = \sigma^2(\alpha)$ and the characteristic function $\phi_\alpha(t)$ of $\xi_i = \xi_i(\alpha)$ have the representation

$$m(\alpha) = \frac{\alpha B'(\alpha)}{B(\alpha)}, \quad \sigma^2(\alpha) = \alpha m'(\alpha), \quad \phi_\alpha(t) = \frac{B(\alpha e^{it})}{B(\alpha)}. \tag{2.9}$$

Therefore $m = m(\alpha)$ is an increasing continuous function (see [7]).

Lemma 2.1. (A.V. Kolchin, [8], Theorem 5.) *Let $(A1)$ be valid. Let $\alpha = \alpha(N)$ be such that $\alpha \to 0$ and $\alpha N \to \infty$ if $N \to \infty$. Then we have*

$$\sigma\sqrt{N}\mathbb{P}\{\zeta_N = n\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(n-Nm)^2}{2\sigma^2 N}} \to 0 \quad \text{as} \quad N \to \infty$$

uniformly on $n = 0, 1, 2, \dots$.

Let l, l_1, $0 \le l < l_1$, be integer numbers. We will also use the following condition

$$Al(l_1): \qquad b_l > 0, b_{l_1} > 0 \ \text{ and } \ b_i = 0 \ \text{ if either } \ i < l \ \text{ or } \ l < i < l_1.$$

Observe that if $B(\alpha)$ is not a constant, then $Al(l_1)$ is valid for some l, l_1. From $Al(l_1)$ it follows that

$$m = \left(l + \frac{b_{l_1} l!}{b_l (l_1 - 1)!} \alpha^{l_1 - l} \right) (1 + \mathrm{o}(1))$$

as $\alpha \to 0$.

Observe also that $(A1)$ coincides with $A0(1)$. Therefore $Al(l+1)$ is a generalization of $(A1)$ and following lemma is a generalization of Lemma 2.1.

Lemma 2.2. *Let $Al(l+1)$ be valid. Let $\alpha = \alpha(N)$ be such that $\alpha \to 0$ and $\alpha N \to \infty$ as $N \to \infty$. Then we have*

$$\sigma\sqrt{N}\mathbb{P}\{\zeta_N = n\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(n-Nm)^2}{2\sigma^2 N}} \to 0 \quad as \quad N \to \infty \tag{2.10}$$

uniformly for $n = 0, 1, 2, \dots$.

Lemma 2.3. (A.V. Kolchin, [3], Theorem 2 for the case of $l = 1$.) *Assume that (Al) is true. Let $\alpha = \alpha(N)$ be such that $\alpha \to 0$ and $Np_l(\alpha) \to \lambda$ as $N \to \infty$, where $0 \le \lambda < \infty$. Then we have*

$$\zeta_N \xrightarrow{d} l\pi(\lambda) \quad as \quad N \to \infty.$$

The following stronger result than Lemma 2.3 is also true.

Lemma 2.4. *Assume that (Al) is true. Let B be a subset of non-negative integer numbers. Then we have*

$$|\mathbb{P}(\zeta_N \in B) - \mathbb{P}(l\pi(Np_l(\alpha)) \in B)| \le N\mathbb{P}(\xi_i(\alpha) \ge l+1) + N(p_l(\alpha))^2. \tag{2.11}$$

Corollary 2.1. *Assume that (Al) is true. Let n be a non-negative integer number. Then we have*

$$\mathbb{P}(\zeta_N = ln) = \mathbb{P}(\pi(N\alpha) = n) + \mathrm{O}(N\alpha^{l+1}),$$

where

$$\mathrm{O}(N\alpha^{l+1}) \le CN\alpha^{l+1}, \quad C = \left(\frac{b_l}{l!B(\alpha)}\right)^2 \alpha^{l-1} + \frac{1}{B(\alpha)} \sum_{k=l+1}^{\infty} \frac{b_k \alpha^{k-l-1}}{k!}, \quad \alpha \le 1.$$

Using Lemma 2.2 to estimate the fractional in (2.6), we obtain the following theorem.

Theorem 2.2. *Assume that $Al(l+1)$ is true. Let $r \ge l+2$ and assume that r belongs to a bounded set. Suppose that $n, N, K \to \infty$ and $\alpha \to 0$ such that $m(\alpha) = \frac{n}{N}$ and $Kp_r \to \beta$, where $0 < \beta < \infty$. Then we have*

$$\mu_r(N, K, n) \xrightarrow{d} \pi(\beta).$$

Using Lemma 2.4 to estimate the fraction in (2.6), we obtain the following theorem.

Theorem 2.3. *Let $r = 1$ and assume that $(A1)$ is valid. Let $\alpha = \frac{n}{N}$. Suppose that $n, K \to \infty$ such that $\frac{n^{2,5}}{N} \to 0$ and $Kp_1(\alpha) \to \beta$. Then*

$$\mu_1(N, K, n) \xrightarrow{d} \pi(\beta).$$

In most models of discrete probability theory the probability of $A_{i_1} \cap A_{i_2} \cap \cap \cdots \cap A_{i_k}$ has a nice representation. This allows us to refine Theorems 2.2 and 2.3 for particular cases.

First we consider allocations of distinct balls. The uniform (equidistributed) allocation scheme of n distinct balls into N different boxes is the random vector $\eta_1, \dots, \eta_N$, with the following distribution

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \frac{n!}{k_1!k_2!\cdots k_N!}\left(\frac{1}{N}\right)^n, \tag{2.12}$$

where for $k_1, k_2, \dots, k_N$ are non-negative numbers such that $k_1 + k_2 + \dots + +k_N = n$. Limit theorems for this model were obtained in lot of papers (see the monograph [11] by Kolchin, Sevast'yanov and Chistyakov). We mention that in Theorem 7 of [3] Poisson approximation is obtained for the number of boxes containing at least two balls. The conditions of that theorem are in accordance with our conditions in Theorem 2.4. We also mention that section 6.2 in [1] is devoted to the Poisson approximation for the non-uniform allocation scheme.

If $\xi_1, \xi_2, \dots, \xi_N$ are independent identically distributed Poisson random variables, then the generalized allocation scheme of occupation n balls into N boxes gives the above model (2.12). Therefore we can apply Theorem 2.2 in this case. However, we can obtain the following refinement of Theorem 2.2 for this particular model. In the following theorem $\alpha = \frac{n}{N}$ can converge to 0 or to ∞ moreover r can be 0 or 1.

Theorem 2.4. *Consider the equidistributed allocation scheme (2.12) of n distinct balls into N different boxes. Suppose that $n, K \to \infty$. Assume that one of the following two sets of conditions is valid:*

(A) the set of the numbers r is bounded, $\dfrac{n}{N^2} \to 0$, $K\dfrac{1}{r!}\left(\dfrac{n}{N}\right)^r e^{-\frac{n}{N}} \to \beta$;

(B) $$\frac{r}{N} \to 0, \quad \frac{r^2}{n} \to 0, \quad \frac{n}{N} \to 0, \quad K\frac{1}{r!}\left(\frac{n}{N}\right)^r \to \beta.$$

Then (2.5) is satisfied.

We can apply Theorem 2.4 to the random variable

$$\eta_{(K1)} = \min_{1 \le i \le K} \eta_i$$

that is the minimal content of a box from the first K boxes.

Corollary 2.2. *Consider the equidistributed allocation scheme (2.12) of n distinct balls into N different boxes. Let the set of r values be bounded. Suppose that $n, K \to \infty$ such that $\frac{n}{N} \to \infty$ and $K\frac{1}{r!}\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}} \to \beta$. Then we have*

$$\mathbb{P}\{\eta_{(K1)} \le r - 1\} = \mathrm{o}(1), \quad \mathbb{P}\{\eta_{(K1)} = r\} = 1 - e^{-\beta} + \mathrm{o}(1). \tag{2.13}$$

If $K = N$ then (2.13) was proved in [11] under another conditions: $\alpha > r$, $\frac{\alpha}{\ln N} \to 1$, $Np_r \to \lambda$, where $\alpha = \frac{n}{N}$. The conditions $Np_r \to \lambda$, $\alpha > r$, $\frac{\alpha}{\ln N} \to 1$ imply $\frac{n}{rN} \to \infty$. The condition $\frac{\alpha}{\ln N} \to 1$ implies $\frac{n}{N^2} \to 0$. Therefore our above corollary is an extension of the result of [11] to the case $K < N$ and for $K = N$ we apply weaker conditions. In [11] the proof is based on a local limit theorem.

Next we consider allocations of indistinguishable balls. Let n, N be integer numbers. The homogeneous allocation scheme of n indistinguishable balls into N boxes is represented by the random variables $\eta_1, \dots, \eta_N$ with the distribution defined by formula

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = 1 \Big/ \binom{n+N-1}{N-1}, \tag{2.14}$$

where $k_1, k_2, \dots, k_N$ are non-negative integer numbers such that $k_1 + k_2 + \cdots + k_N = n$ (see Kolchin [7], Example 1.2.2). In other words, this scheme is the subdivision of n into N parts as $n = k_1 + k_2 + \cdots + k_N$, where $k_1, k_2, \dots, k_N$ are non-negative integers.

Let K be an integer number such that $0 < K \leq N$. Let r be a non-negative integer number.

Theorem 2.5. *In model (2.14), suppose that $n, K \to \infty$ such that*

$$\frac{r^2}{n} \to 0, \quad K \frac{N}{N+n} \left(\frac{n}{N+n} \right)^r \to \beta. \tag{2.15}$$

Then (2.5) is satisfied.

Now we turn to random forests. More precisely, in what follows a labelled graph containing rooted trees will be called a forest. The roots are labelled by $1, \dots, N$ and the non-root vertices are labelled by $1, \dots, n$. On the set of all such kind of forests the uniform distribution is considered. This model is called random forest. A random forest with N root vertices and n non-root vertices can be considered as a generalized allocation scheme with B-function

$$B_f(\alpha) = \sum\nolimits_{k=1}^{\infty} \frac{k^{k-1}}{k!} \alpha^k,$$

(see V. F. Kolchin [7], see also Yu. L. Pavlov [14], A. N. Timashev [17] for the theory of random forests). In this case the meaning of (2.4) is the following. In the random forest there are N rooted trees and the ith tree has η_i non-root vertices. So $\mu_r(N, K, n)$ is the number of those trees in the set of the first K trees which have r non-root vertices. We mention that the function $B_f(\alpha)$ satisfies condition $A1(2)$. Therefore we can apply Theorem 2.2. However, in Theorem 2.2 we have condition $\frac{n}{N} = m(\alpha)$. In the following theorem we shall avoid this condition, so we do not need to solve equation $\frac{n}{N} = m(\alpha)$.

Theorem 2.6. *Consider a random forest with N root vertices and n non-root vertices. Suppose that $r \geq 1$, and $n, N \to \infty$ such that*

$$\frac{r^2}{n} \to 0, \qquad K \frac{1}{(r+1)!} \left(\frac{(r+1)n}{n+N} \right)^r e^{-\frac{(r+1)n}{n+N}} \to \beta. \tag{2.16}$$

Then (2.5) is satisfied.

Now we turn to multi-colour urn schemes. In a box there are mN balls. The balls are coloured with N different colours. For each colour there are m balls in the urn with this colour. We choose n balls from the urn without replacement. Let η_i denote the number of balls chosen from the ith colour. So the random variables η_i, $1 \leq i \leq N$, represent the multi-colour urn scheme if their joint distribution is given by the formula

$$\mathbb{P}\{\eta_1 = n_1, \ldots, \eta_N = n_N\} = \frac{\binom{m}{n_1}\binom{m}{n_2}\cdots\binom{m}{n_N}}{\binom{Nm}{n}}, \tag{2.17}$$

where $n_1 + n_2 + \cdots + n_N = n$, $n_i < m$, $1 \leq i \leq N$, $n \leq Nm$. The multi-colour urn scheme is a general allocation scheme with binomial random variable ξ_i having parameters m and α (see V.F. Kolchin [7]). Therefore the limit in (2.5) depends also on m. We need the following technical lemma.

Lemma 2.5. *Consider the multi-colour urn scheme (2.17). Suppose that $n, N \to \infty$ such that*

$$r \leq m, \quad \frac{n}{Nm} < C \text{ for some } C < 1, \quad \frac{m}{N} \to 0, \quad \frac{r^2}{n} \to 0. \tag{2.18}$$

Let k be a fixed non-negative integer number. Then we have

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = \left(\binom{m}{r} \left(\frac{n}{Nm} \right)^r \left(1 - \frac{n}{Nm} \right)^{m-r} \right)^k (1 + \mathrm{o}(1)). \tag{2.19}$$

Using Le Cam's Poisson approximation theorem, we obtain the following result.

Theorem 2.7. *Consider the multi-colour urn scheme (2.17). Suppose that $n, N \to \infty$ such that (2.18) is satisfied. Let $K \to \infty$ such that*

$$K \frac{1}{r!} \left(\frac{n}{N} \right)^r e^{-\frac{n}{N}} \to \beta, \qquad Km \left(\frac{n}{mN} \right)^2 \to 0. \tag{2.20}$$

Then (2.5) is satisfied.

In order to prove our next theorem, we will use the following version of the Moivre–Laplace theorem (see [15], see also, p. 310 in [16])

$$\binom{m}{r} p^r q^{m-r} = \frac{1}{\sqrt{2\pi mpq}} e^{-x^2/2}\left[1+\frac{(q-p)(x^3-3x)}{6\sqrt{mpq}}\right]+\Delta, \tag{2.21}$$

where $0<p<1$, $q=1-p$, $x=\frac{(r-mp)}{\sqrt{mpq}}$, and

$$|\Delta| < \frac{0,15+0,25|p-q|}{(mpq)^{3/2}} + e^{-\frac{3}{2}\sqrt{mpq}}, \quad mpq \geq 25. \tag{2.22}$$

Theorem 2.8. *Consider the multi-colour urn scheme* (2.17). *Suppose* $n, N \to \to \infty$ *such that* (2.18) *is satisfied,* $K \to \infty$ *such that*

$$\frac{\left(r-\frac{n}{N}\right)^2}{2\frac{n}{N}\left(1-\frac{n}{Nm}\right)} < C \tag{2.23}$$

for some $0<C<\infty$ *and*

$$\frac{K}{\sqrt{2\pi\frac{n}{N}\left(1-\frac{n}{Nm}\right)}} e^{-\frac{\left(r-\frac{n}{N}\right)^2}{2\frac{n}{N}\left(1-\frac{n}{Nm}\right)}} \to \beta. \tag{2.24}$$

Then (2.5) *is satisfied.*

One can see that in Theorems 2.7 and 2.8 $\alpha=\frac{n}{N}$ can converge to infinity.

Let $D \subset \{1,2,\ldots,N\}$, such that $|D|=K$. Denote by $\mu_r(D,n)$ the number of boxes from D which contain r balls. Then the distribution of $\mu_r(D,n)$ coincides with the distribution of $\mu_r(N,K,n)$. Therefore the results of this paper can be considered as Poisson limit theorems for $\mu_r(D,n)$. In [5] limit theorems were obtained for $\mu_0(D,n)$ in some schemes of allocations of distinct balls. In several papers limit theorems for $\mu_r(N,N,n)$ were proved (see, for example, the monographs [7], [11], [14], [17]).

3. Proofs

Proof of Proposition 2.1. For the sake of completeness we present a proof which is a simplified version of the proof in [4]. Let $X_i=\mathbb{I}_{A_i}$, $S_K=\sum_{i=1}^K X_i$, $m_K^l=\mathbb{E}(S_K)^l$, $l=1,2,\ldots$. By Theorem 1.1.4 of [7] we obtain

$$S_K(S_K-1)\cdots(S_K-l+1)=\sum_{\{i_1,\ldots,i_l\}} X_{i_1}X_{i_2}\cdots X_{i_l},$$

where we summarize for all ordered selections without replacement $i_1, \dots, i_l$ of l elements from the set $\{1, 2, \dots, K\}$. The proof of this equation is simple. In the set $\{X_1, \dots, X_K\}$ let j be the number of ones (and $K - j$ be the number of zeros). If $j = 0, 1, \dots, (l-1)$, then both sides of the above equation are 0, if $j = l, (l+1), \dots, K$, then both sides are equal to $j(j-1)\cdots(j-(l-1))$. Now the above equation implies that

$$S_K(S_K-1)\cdots(S_K-l+1) = \sum\nolimits_{\{i_1,\dots,i_l\}} \mathbb{I}_{A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_l}}$$

and therefore

$$\mathbb{E}S_K(S_K-1)\cdots(S_K-l+1) = K(K-1)\cdots(K-l+1)\mathbb{P}(A_1\cap A_2\cap\cdots\cap A_l) =$$
$$= (1+\mathrm{o}(1))K^l\mathbb{P}(A_1\cap A_2\cap\cdots\cap A_l) \to \beta^l$$

as $K \to \infty$, because our condition (2.2) is

$$K^l\mathbb{P}(A_1\cap A_2\cap\cdots\cap A_l) \to \beta^l.$$

Recall that $\pi(\beta)$ is a Poisson random variable with parameter β. So we have

$$\mathbb{E}\pi(\beta)(\pi(\beta)-1)\cdots(\pi(\beta)-l+1) = e^{-\beta}\sum_{k=0}^{\infty} k(k-1)\cdots(k-l+1)\frac{\beta^k}{k!} = \beta^l.$$

Therefore

$$\mathbb{E}S_K(S_K-1)\cdots(S_K-l+1) \to \mathbb{E}\pi(\beta)(\pi(\beta)-1)\cdots(\pi(\beta)-l+1)$$

that is

$$m_K^l \to \mathbb{E}(\pi(\beta))^l$$

for $l = 1, 2, \dots$. Therefore we can apply the moment convergence theorem (see Theorem C in Section 11.4 of [13], see also Theorem 1.1.3 of [7]). The proof is complete. ■

Proof of Lemma 2.2. Consider independent random variables $\xi_i' = \xi_i'(\alpha)$, $i \in \mathbb{N}$, having power law distribution with function $B'(\alpha) = \frac{B(\alpha)}{\alpha^l}$. Let ϕ_α' be the characteristic function of ξ_i'. Then, by (2.9), we have

$$\phi_\alpha'(t) = \frac{B(\alpha e^{it})/(\alpha e^{it})^l}{B(\alpha)/\alpha^l} = e^{-itl}\frac{B(\alpha e^{it})}{B(\alpha)} = e^{-itl}\phi_\alpha(t).$$

Therefore $\xi_i' = \xi_i(\alpha) - l$, $i = 1, 2, \dots$, and $\zeta_N' = \xi_1' + \cdots + \xi_N' = \zeta_N - lN$. Then ξ_i' satisfies condition (A1), it has expectation $m' = m - l$ and variance σ^2. Therefore, using Lemma 2.1, we obtain

$$\sigma\sqrt{N}\mathbb{P}\{\zeta_N = n\} = \sigma\sqrt{N}\mathbb{P}\{\zeta'_N = n - Nl\} =$$
$$= \frac{1}{\sqrt{2\pi}} e^{\frac{(n-Nl-Nm')^2}{2\sigma^2 N}} + \mathrm{o}(1) = \frac{1}{\sqrt{2\pi}} e^{\frac{(n-Nm)^2}{2\sigma^2 N}} + \mathrm{o}(1).$$

The proof is complete. ∎

Proof of Lemma 2.4. The proof is a consequence of the following inequalities.

$$|\mathbb{P}(\zeta_N \in B) - \mathbb{P}(l\pi(Np_l(\alpha)) \in B)| \leq \left|\mathbb{P}(\zeta_N \in B) - \mathbb{P}\left(l\sum_{i=1}^{N} \mathbb{I}_{\{\xi_i = l\}} \in B\right)\right| +$$
$$+\left|\mathbb{P}\left(l\sum_{i=1}^{N} \mathbb{I}_{\{\xi_i = l\}} \in B\right) - \mathbb{P}(l\pi(Np_l(\alpha)) \in B)\right| \leq$$
$$\leq \sum_{i=1}^{N} \mathbb{P}(\xi_i \neq l\mathbb{I}_{\{\xi_i = l\}}) + \left|\mathbb{P}\left(\sum_{i=1}^{N} \mathbb{I}_{\{\xi_i(\alpha) = l\}} \in B/l\right) - \mathbb{P}(\pi(Np_l(\alpha)) \in B/l)\right| \leq$$
$$\leq N\mathbb{P}\left(\xi_i(\alpha) \geq l+1\right) + N(p_l(\alpha))^2.$$

In the last step we applied Le Cam's inequality. The proof is complete. ∎

Proof of Corollary 2.1. Using (2.11), we obtain

$$|\mathbb{P}(\zeta_N = ln) - \mathbb{P}(l\pi(Np_l(\alpha)) = n)| \leq$$
$$\leq N\left(\left(\frac{b_l\alpha^l}{l!B(\alpha)}\right)^2 + \frac{\alpha^{l+1}}{B(\alpha)}\sum_{k=l+1}^{\infty}\frac{b_k\alpha^{k-l-1}}{k!}\right) \leq$$
$$\leq N\alpha^{l+1}\left(\left(\frac{b_l}{l!B(\alpha)}\right)^2 \alpha^{l-1} + \frac{1}{B(\alpha)}\sum_{k=l+1}^{\infty}\frac{b_k\alpha^{k-l-1}}{k!}\right).$$

The proof is complete. ∎

Before proving Theorems 2.2 and 2.3, we observe that $Kp_r \to \beta$ implies that

$$Kp_r < C \quad \text{for some} \quad 0 < C < \infty.$$

Proof of Theorem 2.2. The radius of convergence of $\sum_{k=0}^{\infty} \frac{b_k}{k!}\alpha^k$ coincides with the radius of convergence of $\sum_{k=2}^{\infty} \frac{b_k}{k!}\alpha^{k-1}$. Therefore the radius of convergence of $\sum_{k=2}^{\infty} \frac{b_k}{k!}\alpha^{k-1}$ is equal to R and $\frac{b_r}{r!}\alpha^{r-1} \to 0$ as $\alpha \to 0$ uniformly for $r \geq l+2$. Now we apply the condition $Kp_r \to \beta$, so we obtain

$$N\alpha\frac{b_r\alpha^{r-1}}{r!(b_l\alpha^l + \mathrm{o}(1))} \geq K\frac{b_r\alpha^r}{r!B(\alpha)} = Kp_r = \beta + \mathrm{o}(1) > 0.$$

Consequently, using $r \geq l+2$, we get $N\alpha \to \infty$. Now, we use Lemma 2.2 to approximate the fractional in (2.6), so we obtain

$$K^k \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_k) =$$

$$= K^k (p_r)^k \frac{e^{-\frac{\left(n-kr-(N-k)\frac{n}{N}\right)^2}{2(N-k)\sigma^2}} + \mathrm{o}(1)}{\sqrt{2\pi(N-k)}\sigma} \Bigg/ \frac{e^{-\frac{\left(n-N\frac{n}{N}\right)^2}{2N\sigma^2}} + \mathrm{o}(1)}{\sqrt{2\pi N}\sigma} =$$

$$= K^k (p_r)^k \frac{e^{-\frac{\left(k\left(\frac{n}{N}-r\right)\right)^2}{2(N-k)\sigma^2}} + \mathrm{o}(1)}{1+\mathrm{o}(1)} \sqrt{\frac{N}{N-k}} = \beta^k (1+\mathrm{o}(1)). \tag{3.1}$$

So condition (2.2) of Proposition 2.1 is satisfied, therefore Theorem 2.2 follows from Proposition 2.1. The proof is complete. ∎

Proof of Theorem 2.3. Recall that $r = 1$. As $n^{2.5}/N \to 0$, so $\alpha = \frac{n}{N} \to 0$. Therefore, using Corollary 2.1 and Stirling's formula to approximate $n!$ and $(n-k)!$, we obtain from (2.6) that

$$K^k \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = (Kp_1(\alpha))^k \frac{\mathbb{P}(\zeta_{N-k} = n-k)}{\mathbb{P}(\zeta_N = n)} =$$

$$= (Kp_1(\alpha))^k \frac{\mathbb{P}(\pi((N-k)\alpha) = n-k) + \mathrm{O}((N-k)\alpha^2)}{\mathbb{P}(\pi(N\alpha) = n) + \mathrm{O}(N\alpha^2)} =$$

$$= (Kp_1(\alpha))^k \frac{e^{-(N-k)\frac{n}{N}} \left(\frac{(N-k)\frac{n}{N}e}{(n-k)}\right)^{n-k} \frac{1}{\sqrt{2\pi(n-k)}} + \mathrm{O}((N-k)\alpha^2)}{e^{-N\frac{n}{N}} \left(\frac{N\frac{n}{N}e}{n}\right)^n \frac{1}{\sqrt{2\pi n}} + \mathrm{O}(N\alpha^2)} (1+\mathrm{o}(1)) =$$

$$= (Kp_1(\alpha))^k \frac{e^{-n+\frac{kn}{N}+n-k} \frac{\left(1-\frac{k}{N}\right)^{N\frac{n-k}{N}}}{\left(1-\frac{k}{n}\right)^{n-k}} \sqrt{\frac{n}{n-k}} + \mathrm{O}\left(N\alpha^2\sqrt{n}\right)}{1+\mathrm{O}(N\alpha^2\sqrt{n})} (1+\mathrm{o}(1)) =$$

$$= (Kp_1(\alpha))^k \frac{e^{\frac{kn}{N}} \frac{\left(1-\frac{k}{N}\right)^{N\frac{n-k}{N}}}{e^k\left(1-\frac{k}{n}\right)^{n-k}} \sqrt{\frac{n}{n-k}} + \mathrm{O}\left(N\alpha^2\sqrt{n}\right)}{1+\mathrm{O}(N\alpha^2\sqrt{n})} (1+\mathrm{o}(1)). \tag{3.2}$$

Applying the well-known inequality $e^{-1} \geq \left(1-\frac{1}{y}\right)^y \geq e^{-1}\left(1-\frac{1}{y}\right)$ if $y > 1$, then using $\frac{n^{2,5}}{N} \to 0$ and $Kp_1(\alpha) \to \beta$, we obtain from (3.2) for fixed k that

$$K^k \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_k) =$$

$$= (Kp_1(\alpha))^k \frac{e^{\frac{kn}{N} - k \frac{e^{-k\frac{n-k}{N}\left(1+\mathrm{O}\left(\frac{k}{N}\right)\right)}}{e^{-k\frac{n-k}{n}\left(1+\mathrm{O}\left(\frac{k}{n}\right)\right)}}} \sqrt{\frac{n}{n-k}} + \mathrm{O}\left(\frac{n^2\sqrt{n}}{N}\right)}{1 + \mathrm{O}\left(\frac{n^2\sqrt{n}}{N}\right)} = \beta^k(1 + \mathrm{o}(1)).$$

Therefore condition (2.2) of Proposition 2.1 is valid. Consequently, Theorem 2.3 follows from Proposition 2.1. The proof is complete. ∎

Proof of Theorem 2.4. Let either (A) or (B) be valid. Since

$$\left(\frac{n}{N}\right)^{kr} > \frac{n}{N} \cdot \left(\frac{n}{N} - \frac{1}{N}\right) \cdots \left(\frac{n}{N} - \frac{kr-1}{N}\right) > \left(\frac{n}{N}\right)^{kr} \left(1 - \frac{kr}{n}\right)^{kr} =$$

$$= \left(\frac{n}{N}\right)^{kr} \left(1 - \frac{kr}{n}\right)^{\frac{n}{kr}\frac{(kr)^2}{n}},$$

so we have

$$\frac{n}{N} \cdot \left(\frac{n}{N} - \frac{1}{N}\right) \cdots \left(\frac{n}{N} - \frac{kr-1}{N}\right) = \left(\frac{n}{N}\right)^{kr} (1 + \mathrm{o}(1)).$$

Consequently, we obtain

$$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = \frac{n!}{(r!)^k (n-kr)!} \frac{1}{N^{kr}} \left(1 - \frac{k}{N}\right)^{n-kr} =$$

$$= \left(\frac{1}{r!}\right)^k \frac{n}{N} \cdot \frac{n-1}{N} \cdots \frac{n-kr+1}{N} \left(1 - \frac{k}{N}\right)^{n-kr} =$$

$$= \left(\frac{1}{r!}\right)^k \left(\frac{n}{N}\right)^{rk} \left(1 - \frac{k}{N}\right)^{n-kr} (1 + \mathrm{o}(1)). \tag{3.3}$$

Now we consider the case when condition (A) is valid. For a fixed k, it follows from (3.3) that

$$K^k \mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = \left(K\frac{1}{r!}\right)^k \left(\frac{n}{N}\right)^{kr} e^{n\ln\left(1 - \frac{k}{N}\right)} \left(1 - \frac{k}{N}\right)^{-Nk\frac{r}{N}} =$$

$$= \left(K\frac{1}{r!}\right)^k \left(\frac{n}{N}\right)^{kr} e^{-k\frac{n}{N} + \mathrm{O}\left(\frac{n}{N^2}\right)} (1 + \mathrm{o}(1)) = \left(K\frac{1}{r!}\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}}\right)^k (1 + \mathrm{o}(1)) =$$

$$= \beta^k (1 + \mathrm{o}(1)).$$

In the last two steps we used that $n/N^2 \to 0$ and $K\frac{1}{r!}\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}} \to \beta$. Therefore condition (2.2) is valid. So in the case of the assumption (A), our theorem follows from Proposition 2.1.

Now, assume that (B) is valid. Then for a fixed k, it follows from (3.3) that

$$K^k \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_k) =$$

$$= \left(K\frac{1}{r!}\right)^k \left(\frac{n}{N}\right)^{rk} \left(1 - \frac{k}{N}\right)^{\frac{N}{k}\frac{n-kr}{N}k} (1 + \mathrm{o}(1)) = \beta^k(1 + \mathrm{o}(1)).$$

Therefore condition (2.2) is valid. So in the case of the assumption (B), our theorem follows from Proposition 2.1, too. ∎

Proof of Corollary 2.2. By taking the logarithm of $K\frac{1}{r!}\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}} = \beta + \mathrm{o}(1)$, we obtain

$$\ln(N) - C_1\frac{n}{N} > \ln(K) - \ln(r!) + C\ln\left(\frac{n}{N}\right) - \frac{n}{N} > \ln(\beta + \mathrm{o}(1))$$

for some $C_1 > 0$ and $C > 1$. Therefore we have

$$1 - \frac{\ln(\beta + \mathrm{o}(1))}{\ln(N)} > C_1\frac{n}{N\ln(N)}.$$

Consequently, $\frac{n}{N\ln(N)}$ is bounded. Therefore we see that the condition $K\frac{1}{r!}\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}} \to \beta$ implies $\frac{n}{N^2} \to 0$. So we can apply Theorem 2.4 because its conditions are satisfied.

Now let $\alpha = \frac{n}{N}$ and let ξ be a Poisson random variable with parameter α. Then, since $\frac{r}{\alpha} \to 0$, we have

$$K\mathbb{P}\{\xi \le r-1\} = K\sum_{i=0}^{r-1} e^{-\alpha}\frac{\alpha^i}{i!} = \left(\frac{r}{\alpha} + \frac{r(r-1)}{\alpha^2} + \cdots + \frac{r!}{\alpha^r}\right) K e^{-\alpha}\frac{\alpha^r}{r!} =$$

$$= \mathrm{o}(1)(\beta + \mathrm{o}(1)) = \mathrm{o}(1).$$

Using this, by direct calculation we find

$$\mathbb{P}\{\eta_{(K1)} \le r-1\} = \mathbb{P}\{\min_{1\le i\le K} \eta_i \le r-1\} = \mathbb{P}\{\cup_{i=1}^K \{\eta_i \le r-1\}\} \le$$

$$\le K\mathbb{P}\{\eta_1 \le r-1\} = K\sum_{k=0}^{r-1} \binom{n}{k}\left(\frac{1}{N}\right)^k \left(1 - \frac{1}{N}\right)^{n-k} \le$$

$$\le K\sum_{k=0}^{r-1} \frac{1}{k!}\left(\frac{n}{N}\right)^k \left(1 - \frac{1}{N}\right)^{n-k} \le K\sum_{k=0}^{r-1} \frac{1}{k!}\left(\frac{\alpha}{1 - \frac{1}{N}}\right)^k e^{-\alpha} =$$

$$= KP\{\xi \le r-1\}(1 + \mathrm{o}(1)) = \mathrm{o}(1).$$

So the first equality in (2.13) is proved. Let $B_r = \{\mu(N,K,r) \geq 1\}$. Then

$$\{\eta_{(K1)} \leq r-1\} \cup B_r = \{\eta_{(K1)} \leq r\},$$

an so we have

$$\mathbb{P}\{B_r\} \leq \mathbb{P}\{\eta_{(K1)} \leq r\} \leq \mathbb{P}\{\eta_{(K1)} \leq r-1\} + \mathbb{P}\{B_r\}.$$

So the first equality in (2.13) and Theorem 2.4 imply the second equality in (2.13). The proof is complete. ∎

Proof of Theorem 2.5. Observe that

$$k \leq N \quad \text{and} \quad kr \leq n.$$

Recall that $A_i = \{\eta_i = r\}$. Let k be a fixed positive integer. Then we have

$$K^k \mathbb{P}\{A_1 \cap A_2 \cap \cdots \cap A_k\} =$$

$$= K^k \frac{\binom{n-kr+N-k-1}{N-k-1}}{\binom{n+N-1}{N-1}} = K^k \frac{(N-1)!}{(N-k-1)!} \frac{n!}{(n-kr)!} \frac{(n-kr+N-k-1)!}{(n+N-1)!}.$$

Using assumption $r^2/n \to 0$, we obtain that the above expression is equal to

$$\left(K \frac{Nn^r}{(N+n-1)^{r+1}}\right)^k (1+\mathrm{o}(1)) = \left(K \frac{N}{N+n} \left(\frac{n}{N+n}\right)^r\right)^k (1+\mathrm{o}(1)).$$

So condition (2.15) implies (2.2). Therefore our theorem follows from Proposition 2.1. The proof is complete. ∎

Proof of Theorem 2.6. First observe that $k \leq N$. Since

$$n^{kr} > n(n-1)\cdots(n-(kr-1)) > n^{kr}\left(1 - \frac{kr-1}{n}\right)^{kr} \tag{3.4}$$

and $\frac{r^2}{n} \to 0$, so for any fixed k we have

$$n(n-1)\cdots(n-(kr-1)) = n^{kr}(1+\mathrm{o}(1)). \tag{3.5}$$

Moreover, again using $\frac{r^2}{n} \to 0$, we obtain

$$\left(1 - \frac{k(r+1)}{n+N}\right)^{n-kr-1} = e^{(n-kr-1)\ln\left(1-\frac{k(r+1)}{n+N}\right)} =$$

$$= e^{(n-kr-1)\left(-\frac{k(r+1)}{n+N} + \mathrm{O}\left(\left(\frac{k(r+1)}{n+N}\right)^2\right)\right)} = e^{-\frac{k(r+1)n}{n+N}}(1+\mathrm{o}(1)). \tag{3.6}$$

It is known that the number of random forests with N roots and n non-root vertices is $N(n+N)^{n-1}$ (see Kolchin [7], Example 1.2.4). Therefore, using (3.5), (3.6) and condition $\frac{r^2}{n} \to 0$, we obtain that for any fixed k

$$K^k \mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = K^k \left(\frac{(r+1)^r}{(r+1)!}\right)^k \frac{(N-k)(n-kr+N-k)^{n-kr-1} n!}{N(n+N)^{n-1}(n-kr)!} =$$

$$= K^k \left(\frac{(r+1)^r}{(r+1)!}\right)^k \frac{(n-kr+N-k)^{n-kr-1} n(n-1)\dots(n-kr+1)}{(n+N)^{n-1}} \frac{N-k}{N} =$$

$$= \left(K \frac{(r+1)^r}{(r+1)!}\right)^k \frac{(n+N-kr-k)^{n-kr-1} n^{kr}}{(n+N)^{n-1}} (1+\mathrm{o}(1)) =$$

$$= \left(K \frac{(r+1)^r}{(r+1)!}\right)^k \frac{(n+N-k(r+1))^{n-kr-1}}{(n+N)^{n-kr-1}} \frac{n^{kr}}{(n+N)^{kr}} (1+\mathrm{o}(1)) =$$

$$= \left(K \frac{(r+1)^r}{(r+1)!} \left(1 - \frac{k(r+1)}{n+N}\right)^{(n-kr-1)/k} \left(\frac{n}{N+n}\right)^r\right)^k (1+\mathrm{o}(1)) =$$

$$= \left(K \frac{(r+1)^r}{(r+1)!} e^{-\frac{n(r+1)}{n+N}} \left(\frac{n}{N+n}\right)^r\right)^k (1+\mathrm{o}(1)) = \beta^k (1+\mathrm{o}(1)). \tag{3.7}$$

In the last step we used condition (2.16). Therefore the conditions of Proposition 2.1 are valid, so Theorem 2.6 follows from Proposition 2.1. The proof is complete. ■

Proof of Lemma 2.5. Observe that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_k) = \frac{\left(\binom{m}{r}\right)^k \binom{(N-k)m}{n-rk}}{\binom{mN}{n}} = \frac{\left(\binom{m}{r}\right)^k \frac{((N-k)m)!}{(n-rk)!(Nm-n-k(m-r))!}}{\frac{(mN)!}{n!(mN-n)!}} =$$

$$= \left(\binom{m}{r}\right)^k \frac{((N-k)m)!}{(mN)!} \frac{n!}{(n-kr)!} \frac{(Nm-n)!}{(Nm-n-k(m-r))!}. \tag{3.8}$$

As $r^2/n \to 0$, so (3.5) gives

$$\frac{n!}{(n-kr)!} = n^{kr}(1+\mathrm{o}(1)). \tag{3.9}$$

We have

$$(Nm)^{km} > \frac{(mN)!}{((N-k)m)!} > (Nm)^{km} \left(1 - \frac{km}{Nm}\right)^{km} = (Nm)^{km} \left(1 - \frac{k}{N}\right)^{Nk\frac{m}{N}},$$

$$(mN-n)^{k(m-r)} \geq$$
$$\geq \frac{(mN-n)!}{(Nm-n-k(m-r))!} > (mN-n)^{k(m-r)}\left(1-\frac{k(m-r)}{mN-n}\right)^{k(m-r)} =$$
$$= (mN-n)^{k(m-r)}\left(1-\frac{k(m-r)}{mN-n}\right)^{\frac{mN-n}{m-r}k\frac{(m-r)^2}{mN-n}}$$

and because $m \geq r$, $n/Nm < C$, so

$$\frac{m-r}{mN-n} \leq \frac{(m-r)^2}{mN-n} \leq \frac{m^2}{mN-n} = \frac{\frac{m}{N}}{1-\frac{n}{Nm}} \leq \frac{\frac{m}{N}}{1-C} \to 0.$$

So, using $m/N \to 0$, we obtain

$$\frac{(mN)!}{((N-k)m)!} = (Nm)^{km}(1+\mathrm{o}(1)), \tag{3.10}$$

$$\frac{(mN-n)!}{(Nm-n-k(m-r))!} = (mN-n)^{k(m-r)}(1+\mathrm{o}(1)). \tag{3.11}$$

Using relations (3.9), (3.10) and (3.11) in equation (3.8), we obtain (2.19). The proof is complete. ■

Proof of Theorem 2.7. Using Le Cam's inequality with $\lambda = m\frac{n}{Nm} = \frac{n}{N}$, we obtain

$$K\binom{m}{r}\left(\frac{n}{Nm}\right)^r\left(1-\frac{n}{Nm}\right)^{m-r} = K\frac{1}{r!}\left(\frac{n}{N}\right)^r e^{-\frac{n}{N}} + \mathrm{o}_1(1) = \beta + \mathrm{o}(1),$$

where

$$|\mathrm{o}_1(1)| \leq Km\left(\frac{n}{mN}\right)^2.$$

Therefore Theorem 2.7 follows from Lemma 2.5 and Proposition 2.1. The proof is complete. ■

Proof of Theorem 2.8. Let in (2.21), (2.22) be $p = \frac{n}{Nm}$, $q = 1-\frac{n}{Nm}$. From (2.23) and (2.24) it follows that

$$\frac{K}{\sqrt{2\pi\frac{n}{N}}} = \mathrm{O}(1).$$

Therefore, as $K \to \infty$, $\frac{n}{N} \to \infty$. Now

$$Ke^{-\sqrt{\frac{n}{N}\left(1-\frac{n}{Nm}\right)}} = \frac{K}{\sqrt{\frac{n}{N}\left(1-\frac{n}{Nm}\right)}}\sqrt{\frac{n}{N}\left(1-\frac{n}{Nm}\right)}e^{-\sqrt{\frac{n}{N}\left(1-\frac{n}{Nm}\right)}} = \mathrm{o}(1)$$

and

$$K \Big/ \sqrt{\frac{n}{N}}\frac{n}{N} = \mathrm{o}(1).$$

Therefore we have $K\Delta \to 0$. Consequently, using (2.23), (2.24), we obtain

$$K\binom{m}{r}\left(\frac{n}{Nm}\right)^r\left(1-\frac{n}{Nm}\right)^{m-r} = \frac{K}{\sqrt{2\pi\frac{n}{N}\left(1-\frac{n}{Nm}\right)}}e^{-\frac{\left(r-\frac{n}{N}\right)^2}{2\frac{n}{N}\left(1-\frac{n}{Nm}\right)}}(1+\mathrm{o}(1)) =$$
$$= \beta + \mathrm{o}(1).$$

Therefore Theorem 2.8 follows from Lemma 2.5 and Proposition 2.1. The proof is complete. ■

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