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ON THE LOCAL CONVERGENCE AND COMPLEX GEOMETRY OF EIGHTH ORDER ITERATION FUNCTION

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Abstract. The convergence order of numerous iterative methods is obtained using derivatives of higher order, although these derivatives are not involved in the methods. Therefore, these methods cannot be used to solve equations with functions that do not have such high order derivatives, since their convergence is not guaranteed. In this paper, we study local convergence of an efficient eighth order method under hypotheses only on the first derivative. That is how we expand the applicability of some popular methods. Moreover, the study of local convergence of iteration functions is important because it provides the degree of difficulty for choosing initial points. We also verify the theoretical results on some numerical problems. Finally, stability of the method is checked through complex geometry shown by drawing basins of attraction of the solutions.

1. Introduction

Let B_1, B_2 be Banach spaces and Ω be a convex subset of B_1. Further, suppose that $\mathfrak{L}(B_1, B_2)$ is the set of bounded linear operators from B_1 into B_2.

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In applied mathematics many problems can be formulated in the form

$$F(x) = 0, \tag{1.1}$$

wherein $F : \Omega \subset B_1 \to B_2$ is a differentiable operator in the sense of Fréchet. Most of the methods for finding a solution x_* of (1.1) are iterative, because closed form solutions can be found only in some special cases (see [4, 10]).

Cordero et al. in [8] considered an eighth order iterative method for approximating a solution of the nonlinear equation $F(x) = 0$, where $F : \Omega \subset \subset \mathbb{R}^m \to \mathbb{R}^m$ is a differentiable nonlinear function. In this paper, we study the method developed in [8] using hypotheses only on first derivative of the function. Precisely, we present the local convergence analysis of the method defined for $n = 0, 1, 2....$, by

$$\begin{aligned} y_n =&\ x_n - F'(x_n)^{-1}F(x_n),\\ z_n =&\ y_n - \Big(\frac{5}{4}I - \frac{1}{2}F'(y_n)^{-1}F'(x_n) + \frac{1}{4}(F'(y_n)^{-1}F'(x_n))^2\Big)\times\\ &\times F'(y_n)^{-1}F(y_n),\\ x_{n+1} =&\ z_n - \Big(\frac{3}{2}I - F'(y_n)^{-1}F'(x_n) + \frac{1}{2}(F'(y_n)^{-1}F'(x_n))^2\Big)\times\\ &\times F'(y_n)^{-1}F(z_n), \end{aligned} \tag{1.2}$$

where $x_0 \in \Omega$ is an initial point.

The convergence of the method (1.2) was shown using Taylor expansion and conditions reaching up to the eighth derivative of F. Such assumptions restrict the applicability of method, especially since only first derivative is used in the method. As a motivational example, consider a function F on $\Omega = [-\frac{5}{2}, 2]$ and $B_1 = B_2 = \mathbb{R}$, by

$$F(x) = \begin{cases} x^3\log(\pi^2x^2) + x^5\sin\frac{1}{x}, & x \neq 0,\\ 0, & x = 0. \end{cases}$$

We have that

$$\begin{aligned} F'(x) =&\, 2\,x^2 - x^3\cos\Big(\frac{1}{x}\Big) + 3\,x^2\log(\pi^2x^2) + 5\,x^4\sin\Big(\frac{1}{x}\Big),\\ F''(x) =& - 8\,x^2\cos\Big(\frac{1}{x}\Big) + 2\,x(5 + 3\log(\pi^2x^2)) + x(20\,x^2 - 1)\sin\Big(\frac{1}{x}\Big) \end{aligned}$$

and

$$F'''(x) = \frac{1}{x}\Big[(1 - 36\,x^2)\cos\Big(\frac{1}{x}\Big) + x\Big(22 + 6\log(\pi^2x^2) + (60\,x^2 - 9)\sin\Big(\frac{1}{x}\Big)\Big)\Big].$$

It is clear that $F'''(x)$ is not bounded on Ω, so earlier results cannot be applied. In our convergence analysis, we only used hypotheses on the first derivative. Therefore, our results can be applied. Hence, we extend the applicability of method (1.2). Many authors have studied convergence analysis of iterative methods, see, for example [1, 2, 3, 4, 5, 6, 7, 11, 13, 15].

We summarize the contents of the paper. In section 2, the local convergence analysis of method (1.2) is presented. The numerical examples are performed in section 3. Section 4 is devoted to check the convergence domain of the iterative technique geometrically by means of drawing basin of attractors. Concluding remarks are given in section 5.

2. Convergence analysis

We shall first introduce some scaler functions and parameters that appear in the local convergence analysis of method (1.2) that follows in Theorem 2.1. Let $w_0 : [0, \infty) \to [0, \infty)$ be a continuous and increasing function with $w_0(0) = 0$. Suppose that equation

$$w_0(t) = 1, \tag{2.1}$$

has at least one positive solution. Denote by δ_0 the smallest such solution. Let also $w : [0, \delta_0) \to [0, +\infty)$, $w_1 : [0, \delta_0) \to [0, +\infty)$ be continuous and increasing functions such that $w(0) = 0$. Define functions φ_1 and $\bar{\varphi}_1$ on the interval $[0, \delta_0)$ by

$$\varphi_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)}$$

and

$$\bar{\varphi}_1(t) = \varphi_1(t) - 1.$$

We have $\bar{\varphi}_1(0) = -1$ and $\bar{\varphi}_1(t) \to \infty$ as $t \to \delta_0^-$. Then, intermediate value theorem guarantees that the equation $\bar{\varphi}_1(t) = 0$ has at least one solution in $(0, \delta_0)$. Denote by r_1 the smallest such solution. Suppose that equation

$$w_0(\varphi_1(t)t) = 1, \tag{2.2}$$

has at least one positive solution. Denote by δ_1 the smallest such solution. Define functions φ_2 and $\bar{\varphi}_2$ on $[0, \delta_1)$ by

$$\varphi_2(t) = \Big[\frac{\int_0^1 w((1-\theta)\varphi_1(t)t)d\theta}{1 - w_0(\varphi_1(t)t)} + \frac{1}{4}\frac{(w_0(\varphi_1(t)t) + w_0(t))^2 \int_0^1 w_1(\theta\varphi_1(t)t)d\theta}{(1 - w_0(\varphi_1(t)t))^3}\Big]\varphi_1(t)$$

and

$$\bar{\varphi}_2(t) = \varphi_2(t) - 1.$$

We get $\bar{\varphi}_2(0) = -1$ and $\bar{\varphi}_2(t) \to \infty$ as $t \to \delta_1^-$. Denote by r_2 the smallest solution of equation $\bar{\varphi}_2(t) = 0$ in $(0, \delta_1)$. Suppose that equation

$$w_0(\varphi_2(t)t) = 1, \tag{2.3}$$

has at least one positive solution. Denote by δ_2 the smallest such solution. Set $\delta = \min\{\delta_1, \delta_2\}$. Define functions φ_3 and $\bar{\varphi}_3$ on $[0, \delta)$ by

$$\varphi_3(t) = \Big[\frac{\int_0^1 w((1-\theta)\varphi_2(t)t)d\theta}{1 - w_0(\varphi_2(t)t)} + \frac{(w_0(\varphi_2(t)t) + w_0(\varphi_1(t)t))\int_0^1 w_1(\theta\varphi_2(t)t)d\theta}{(1 - w_0(\varphi_2(t)t))(1 - w_0(\varphi_1(t)t))} + \\ + \frac{1}{2}\frac{(w_0(\varphi_1(t)t) + w_0(t))^2\int_0^1 w_1(\theta\varphi_2(t)t)d\theta}{(1 - w_0(\varphi_1(t)t))^3}\Big]\varphi_2(t)$$

and

$$\bar{\varphi}_3(t) = \varphi_3(t) - 1.$$

We obtain $\bar{\varphi}_3(0) = -1$ and $\bar{\varphi}_3(t) \to \infty$ as $t \to \delta^-$. Denote by r_3 the smallest solution of equation $\bar{\varphi}_3(t) = 0$ in $(0, \delta)$. Define a radius of convergence r by

$$r = \min\{r_j\} \qquad j = 1, 2, 3. \tag{2.4}$$

It follows from (2.4) that for each $t \in [0, r)$

$$0 \le w_0(t) \le 1, \tag{2.5}$$

$$0 \le w_0(\varphi_1(t)t) \le 1, \tag{2.6}$$

$$0 \le w_0(\varphi_2(t)t) \le 1 \tag{2.7}$$

and

$$0 \le \varphi_j(t) \le 1. \tag{2.8}$$

Let $S(\mu, \lambda)$ and $\bar{S}(\mu, \lambda)$ stand for the open and closed balls in B_1, respectively with center $\mu \in B_1$ and of radius $\lambda > 0$.
The following conditions shall be used in the local convergence analysis (A):

(a_1) $F : \Omega \to B_2$ is continuously differentiable in the sense of Frèchet and there exists $x_* \in \Omega$ such that $F(x_*) = 0$ and $F'(x_*)^{-1} \in \mathfrak{L}(B_2, B_1)$.

(a_2) There exists function $w_0 : [0,\infty) \to [0,\infty)$ with $w_0(0) = 0$ continuous and increasing such that for each $x \in \Omega$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap S(x_*, \delta_0)$, where δ_0 is given in (2.1).

(a_3) There exist functions $w : [0, \delta_0) \to [0,\infty)$, $w_1 : [0, \delta_0) \to [0, +\infty)$, continuous and increasing with $w(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leq w(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \leq w_1(\|x - x_*\|).$$

(a_4) $\bar{S}(x_*, r) \subset \Omega$, where r is defined in (2.4).

(a_5) There exists $R \geq r$ such that

$$\int_0^1 w_0(\theta R)d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{S}(x_*, R)$.

Next, we present the local convergence analysis of method (1.2) using the preceding notation and the conditions (A).

Theorem 2.1. *Suppose that the conditions* (A) *hold. Then, for* $x_0 \in S(x_*, r) - -\{x_*\}$ *sequence* $\{x_n\}$ *generated by method* (1.2) *is well defined, remains in* $S(x_*, r)$ *for each* $n = 0, 1, 2......$ *and converges to* x_*. *Moreover, the following error bounds hold*

$$\|y_n - x_*\| \leq \varphi_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \tag{2.9}$$

$$\|z_n - x_*\| \leq \varphi_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \tag{2.10}$$

and

$$\|x_{n+1} - x_*\| \leq \varphi_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{2.11}$$

where the functions φ_j *are given previously and* r *is defined in* (2.4). *Furthermore, the limit point* x_* *is only solution of equation* $F(x) = 0$ *in* Ω_1 *given in* (a_5).

Proof. We base our proof on mathematical induction. Choose $x \in S(x_*, r) - -\{x_*\}$. Then, by (2.4), (a_1) and (a_2), we obtain in turn that

$$\|F'(x_*)^{-1}\big(F'(x) - F'(x_*)\big)\| \leq w_0(\|x - x_*\|) < w_0(r) \leq 1. \tag{2.12}$$

It follows by (2.12) and the Banach lemma on invertible operators [4] that $F'(x)^{-1} \in \mathfrak{L}(B_2, B_1)$ and

$$\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - w_0(\|x - x_*\|)}. \tag{2.13}$$

We also have that y_0 is well defined by (2.13). Using (2.4), (2.8) (for $j = 1$), (a_1)–(a_3), (2.13) and the first substep of method (1.2) for $n = 0$, we get in turn that

$$\begin{aligned} \|y_0 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| = \\ &= \|F'(x_0)^{-1}F'(x_*)\| \Big\| \int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - \\ &\qquad - F'(x_0))d\theta(x_0 - x_*)\Big\| \leq \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x_*\|)d\theta \|x_0 - x_*\|}{1 - w_0(\|x_0 - x_*\|)} = \\ &= \varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{2.14}$$

which shows (2.9) for $n = 0$ and $y_0 \in S(x_*, r)$. We can write by (a_1) that

$$F(x) = F(x) - F(x_*) = \int_0^1 F'(x_* + \theta(x - x_*))d\theta(x - x_*). \tag{2.15}$$

Then, by the second condition in (a_3) and (2.15), we get that

$$\|F'(x_*)^{-1}F(x)\| \leq \int_0^1 w_1(\theta\|x - x_*\|)d\theta \cdot \|x - x_*\|. \tag{2.16}$$

The iterates z_0 and x_1 are well defined by (2.13) for $x = y_0$. We have by the second substep of method (1.2) for $n = 0$

$$
\begin{aligned}
z_0 - x_* = &\ y_0 - x_* - F'(y_0)^{-1}F(y_0) - \\
-\Big(\frac{1}{4}I - &\frac{1}{2}F'(y_0)^{-1}F(x_0) + \frac{1}{4}(F'(y_0)^{-1}F'(x_0))^2\Big)F'(y_0)^{-1}F(y_0) = \\
&= y_0 - x_* - F'(y_0)^{-1}F(y_0) + \\
+\frac{1}{4}\Big(F'(y_0)^{-1}&\big((F'(y_0) - F'(x_*)) + (F'(x_*) - F'(x_0))\big)\Big)^2 \times \\
&\times F'(y_0)^{-1}F(y_0),
\end{aligned} \tag{2.17}
$$

since

$$
\begin{aligned}
\frac{1}{4}I - \frac{1}{2}F'(y_0)^{-1}F'(x_0) + &\frac{1}{4}(F'(y_0)^{-1}F'(x_0))^2 = \\
&= \frac{1}{4}\big(I - 2F'(y_0)^{-1}F'(x_0) + (F'(y_0)^{-1}F'(x_0))^2\big) = \\
&= \frac{1}{4}(I - F'(y_0)^{-1}F'(x_0))^2 = \\
&= \frac{1}{4}\big(F'(y_0)^{-1}(F'(y_0) - F'(x_0))\big)^2.
\end{aligned}
$$

In view of (2.4), (2.8) (for $j = 2$), (2.13) (for $x = y_0$), (2.14), (2.16) (for $x = y_0$) and (2.17), we have that

$$
\begin{aligned}
\|z_0 - x_*\| \leq &\ \ \|y_0 - x_* - F'(y_0)^{-1}F(y_0)\| + \\
+\frac{1}{4}\|F'(y_0)^{-1}F'(x_*)\|^2&\Big(\|F'(x_*)^{-1}(F'(y_0) - F'(x_*))\| + \\
+\|F'(x_*)^{-1}&(F'(x_0) - F'(x_*))\|\Big)^2 \leq \\
\leq \Big(&\frac{\int_0^1 w((1-\theta)\|y_0 - x_*\|)d\theta}{1 - w_0(\|y_0 - x_*\|)} + \\
+\frac{1}{4}\frac{\big(w_0(\|y_0 - x_*\|) + w_0(\|x_0 - x_*\|)\big)^2 \int_0^1 w_1(\theta\|y_0 - x_*\|)d\theta}{\big(1 - w_0\big(\|y_0 - x_*\|\big)\big)^3}&\Big) \times \\
&\times \|y_0 - x_*\| \leq \\
\leq \varphi_2(\|x_0 - x_*\|)\|x_0 - x_*\| &\leq \|x_0 - x_*\| < r,
\end{aligned} \tag{2.18}
$$

so (2.10) holds for $n = 0$ and $z_0 \in S(x_*, r)$. Using the third substep of method

(1.2) for $n=0$, we get in turn that

$$x_1 - x_* = z_0 - x_* - F'(z_0)^{-1}F(z_0) + \big(F'(z_0)^{-1} - F'(y_0)^{-1}\big)F(z_0) - \\ - \frac{1}{2}\Big(I - 2F'(y_0)^{-1}F'(x_0) + \big(F'(y_0)^{-1}F'(x_0)\big)^2\Big)F'(y_0)^{-1}F(z_0) = \\ = z_0 - x_* - F'(z_0)^{-1}F(z_0) + F'(z_0)^{-1}\big(F'(y_0) - F'(z_0)\big)F'(y_0)^{-1}F(z_0) - \\ - \frac{1}{2}\Big(F'(y_0)^{-1}\big(F'(y_0) - F'(x_0)\big)\Big)^2 F'(y_0)^{-1}F(z_0). \tag{2.19}$$

Then, by (2.4), (2.8) (for $j=1$), (2.13) (for $x = y_0, z_0$), (2.18) and (2.19), we obtain in turn that

$$\|x_1 - x_*\| \le \|z_0 - x_* - F'(z_0)^{-1}F(z_0)\| + \|F'(z_0)^{-1}F'(x_*)\| \times \\ \times \Big(\|F'(x_*)^{-1}\big(F'(z_0) - F'(x_*)\big)\| + \|F'(x_*)^{-1}\big(F'(y_0) - F'(x_*)\big)\|\Big) \times \\ \times \|F'(x_*)^{-1}F(z_0)\| + \frac{1}{2}\|F'(y_0)^{-1}F'(x_*)\|^2 \\ \Big(\|F'(x_*)^{-1}\big(F'(y_0) - F'(x_*)\big)\| + \|F'(x_*)^{-1}\big(F'(x_0) - F'(x_*)\big)\|\Big)^2 \times \\ \times \|F'(y_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(z_0)\| \le \\ \le \Big(\frac{\int_0^1 w((1-\theta)\|z_0 - x_*\|)d\theta}{1 - w_0(\|z_0 - x_*\|)} + \\ + \frac{\big(w_0(\|z_0 - x_*\|) + w_0(\|z_0 - x_*\|)\big)\int_0^1 w_1(\theta\|z_0 - x_*\|)d\theta}{\big(1 - w_0(\|z_0 - x_*\|)\big)\big(1 - w_0(\|y_0 - x_*\|)\big)} + \\ + \frac{1}{2}\frac{\big(w_0(\|y_0 - x_*\|) + w_0(\|x_0 - x_*\|)\big)^2\int_0^1 w_1(\theta\|z_0 - x_*\|)d\theta}{\big(1 - w_0(\|y_0 - x_*\|)\big)^3}\Big)\|z_0 - x_*\| \le \\ \le \varphi_3(\|x_0 - x_*\|)\|x_0 - x_*\| \le \|x_0 - x_*\| < r, \tag{2.20}$$

so (2.11) holds for $n=0$ and $x_1 \in S(x_*, r)$. If we replace x_0, y_0, z_0, x_1 by x_i, y_i, z_i, x_{i+1} in the preceding estimates, we complete the induction (2.9)-(2.11). Then, it follows from the estimate

$$\|x_{i+1} - x_*\| \le \mu\|x_i - x_*\| < r, \tag{2.21}$$

where $\mu = \varphi_3(\|x_0 - x_*\|) \in [0,1)$, we get that $\lim_{i\to\infty} x_i = x_*$ and $x_{i+1} \in S(x_*, r)$.

The uniqueness part is shown by letting $T = \int_0^1 F'(x_* + \theta(y_* - x_*))d\theta$ for some $y_* \in \Omega_1$ with $F(y_*) = 0$. Then, using (a_5), we get that

$$\|F'(x_*)^{-1}(T - F'(x_*))\| \leq \int_0^1 w_0(\theta\|x_* - y_*\|)d\theta \leq$$
$$\leq \int_0^1 w_0(\theta R)d\theta < 1,$$

so $T^{-1} \in \mathfrak{L}(B_2, B_1)$. Finally, by

$$0 = F(y_*) - F(x_*) = T(y_* - x_*),$$

we deduce that $x_* = y_*$. ■

3. Numerical results

Example 1. Consider the motivational example as given in the introduction of the paper. Note that $x_* = 0$ is zero of this function. Then, we have $w_0(t) = Lt$, $w(t) = Lt$, $w_1(t) = 2$, where $L = \frac{2}{2\pi+1}(80 + 16\pi + (11 + 12\log 2)\pi^2)$. Using the definition, we obtain the parameter values are

$r_1 = 7.5648\times10^{-3}$, $r_2 = 6.2566\times10^{-3}$, $r_3 = 5.5908\times10^{-3}$ and $r = 5.5908\times10^{-3}$.

Example 2. Suppose that the motion of an object in three dimensions is governed by system of differential equations

$$f_1'(x) - f_1(x) - 1 = 0,$$
$$f_2'(y) - (e-1)y - 1 = 0,$$
$$f_3'(z) - 1 = 0,$$

with $x, y, z \in \Omega$ for $f_1(0) = f_2(0) = f_3(0) = 0$. Then, the solution of the system is given for $v = (x, y, z)^T$ by function $F := (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ defined by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have

$$w_0(t) = (e-1)t, \quad w(t) = et, \quad w_1(t) = \frac{e}{2}.$$

Using the definition, the parameter values are given as

$$r_1 = 0.324947,\ r_2 = 0.276393,\ r_3 = 0.248044 \text{ and } r = 0.248044.$$

Example 3. Let us consider the function $F := (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ defined by

$$F(x) = \ (10x_1 + \sin(x_1 + x_2) - 1, 8x_2 - \cos^2(x_3 - x_2) - 1, 12x_3 + \sin(x_3) - 1)^T,$$

where $x = (x_1, x_2, x_3)^T$.

Fréchet-derivative of $F(x)$ is given by

$$F'(x) = \begin{bmatrix} 10 + \cos(x_1 + x_2) & \cos(x_1 + x_2) & 0 \\ 0 & 8 + \sin 2(x_2 - x_3) & -\sin 2(x_2 - x_3) \\ 0 & 0 & 12 + \cos(x_3) \end{bmatrix}.$$

Then, we get that $w_0(t) = w(t) = 0.269812t$ and $w_1(t) = 1.08139$. The parameters values of r_1, r_2, r_3 and r for this example are given as

$$r_1 = 2.47086,\ r_2 = 2.13854,\ r_3 = 1.96885 \text{ and } r = 1.96885.$$

Example 4. Next, we consider Kepler's equation

$$F(x) = \ x - \beta\sin(x) - K = 0,$$

where $0 \leq \beta < 1$, $0 \leq K \leq \pi$. A numerical study, for different values of β and K, has been performed in [9]. In this example, we take values $K = 0.1$ and $\beta = 0.27$, so the solution $x_* \approx 0.13682853547099\ldots$ is obtained. Since

$$F'(x) = \ 1 - \beta\cos(x),$$

we have

$$\begin{aligned} |F'(x_*)^{-1}(F'(x) - F'(y))| &= \frac{|\beta(\cos(x) - \cos(y))|}{|1 - \beta\cos(x_*)|} = \\ &= \frac{2\beta|\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})|}{|1 - \beta\cos(x_*)|} \leq \\ &\leq \frac{\beta}{|1 - \beta\cos(x_*)|}|x - y| \end{aligned}$$

and

$$|F'(x_*)^{-1}F'(x)| = \frac{|1-\beta\cos(x)|}{|1-\beta\cos(x_*)|} \leq \frac{1+\beta}{|1-\beta\cos(x_*)|}.$$

Then, we have $w_0(t) = w(t) = 0.3685888t$ and $w_1(t) = 1.7337327$. The calculated values of parameters are given by

$$r_1 = 1.8087, \quad r_2 = 1.5784, \quad r_3 = 1.4992 \quad \text{and} \quad r = 1.4992.$$

Example 5. Let $B_1 = C[0,1]$, be the space of continuous functions defined on the interval $[0,1]$ and be equipped with max norm. Let $\Omega = \bar{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \phi(x) - 10\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 30\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \quad \text{for each } \xi \in \Omega.$$

Then for $x_* = 0$ we have that $w_0(t) = 15t$, $w(t) = 30t$ and $w_1(t) = \frac{15}{2}$. The parameters r_1, r_2, r_3 and r are given by

$$r_1 = 3.33333 \times 10^{-2}, \quad r_2 = 2.48384 \times 10^{-2},$$
$$r_3 = 2.02103 \times 10^{-2} \quad \text{and} \quad r = 2.02103 \times 10^{-2}.$$

4. Complex geometry

The study of complex geometry of the rational function associated to an iterative method gives important information about convergence and stability of the method, see for example [14, 12]. To start with, let us recall some basic dynamical concepts of rational function. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a rational function, the orbit of a point $x_0 \in \mathbb{R}$ is defined as the set

$$\{x_0, \phi(x_0), \phi^2(x_0), \ldots, \phi^m(x_0), \ldots\}$$

of successive images of x_0 by the rational function.

The dynamical behavior of the orbit of a point of $\mathbb{R}$ can be classified depending on its asymptotic behavior. In this way, a point $x_0 \in \mathbb{R}$ is a fixed point of $\phi(\alpha)$ if it satisfies $\phi(\alpha) = \alpha$. Moreover, x_0 is called a periodic point of period $p > 1$ if it is a point such that $\phi^p(x_0) = x_0$ but $\phi^k(x_0) \neq x_0$, for each $k < p$. Moreover, a point x_0 is called pre-periodic if it is not periodic but there exists a $k > 0$ such that $\phi^k(x_0)$ is periodic. There exist different types of fixed points depending on the associated multiplier $|\phi^k(x_0)|$. Taking the associated multiplier into account, a fixed point x_0 is called:

- attractor if $|\phi^k(x_0)| < 1$,
- superattractor if $|\phi^k(x_0)| = 0$,
- repulsor if $|\phi^k(x_0)| > 1$,
- parabolic if $|\phi^k(x_0)| = 1$.

If x_* an attracting fixed point of the rational function ϕ, its basin of attraction $\mathcal{A}(x_*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x_*) = \{x_0 \in \mathcal{R} : \phi^m(x_0) \to x_*, \, m \to \infty\}. \tag{4.1}$$

The set of points whose orbits tend to an attracting fixed point x_* defined as the Fatou set. Its complementary set, called Julia set, is the closure of the set consisting of repelling fixed points, and establishes the borders between the basins of attraction. That means the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

We take the initial point as $z_0 \in \Omega$, where Ω is a rectangular region in complex plane ($\mathbb{C}$) containing all the roots of $f(z) = 0$. The numerical methods starting at point z_0 in a rectangle can converge to the zero of the function $f(z)$ or eventually diverge. We consider the stopping criterion for convergence as 10^{-3} up to a maximum of 25 iterations. If we have not obtained the desired tolerance in 25 iterations, we do not continue and decide that the iterative method starting at z_0 does not converge to any root. The strategy taken into account is following: A color is assigned to each starting point z_0 in the basin of attraction of a zero. If the iteration starting from the initial point z_0 converges then it represents the basins of attraction with that particular color assigned to it and if it fails to converge in 25 iterations then it shows the black color. In this way, we distinguish the attraction basins by their colors for the method.

To view complex geometry, we analyze the basins of attraction of the method on following polynomials:

Test problem 1. Consider the polynomial $P_1(z) = z^2 + 1$ having two simple zeros $\{-i, i\}$. The basin of attractors for this polynomial are shown in Figure 1. From this figure, it can be observed that method (1.2) has very stable behavior. In addition, the method does not exhibits chaotic behavior on the boundary points.

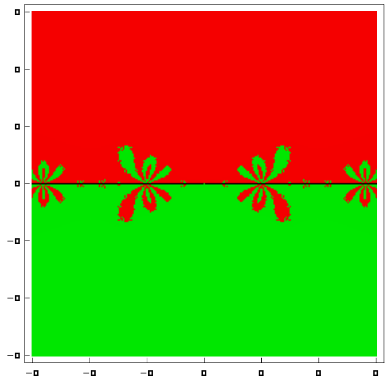

Figure 1. Basins of attraction of method (1.2) for test problem 1

Test problem 2. Let $P_2(z) = z^3 - z$ having three simple zeros $\{-1, 0, 1\}$. The basin of attractors for this polynomial are shown in Figure 2. From this figure, we observe the stable behavior of method (1.2).

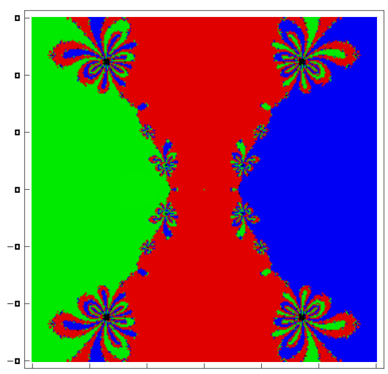

Figure 2. Basins of attraction of method (1.2) for test problem 2

Test problem 3. Let $P_3(z) = z^4 - 5z^2 + 4$ having four simple zeros $\{-1, 1, -\frac{5}{2}, \frac{5}{2}\}$. The basin of attractors for this polynomial are shown in Figure 3. From figure, it can be observed that method (1.2) has beautiful fractal geometry. Few black spots at the borders between the basins of attraction indicate the diverging points.

Test problem 4. Consider the polynomial $P_4(z) = z^5 - \frac{29}{4}z^3 + \frac{25}{4}z$ having five simple zeros $\{0, -1, 1, -\frac{5}{2}, \frac{5}{2}\}$. The basin of attractors are shown in Figure 4. In this case we also observe the beautiful shapes of the basins of attraction of

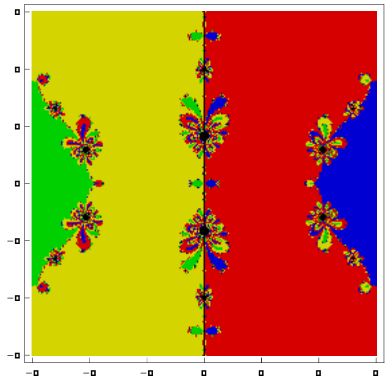

Figure 3. Basins of attraction of method (1.2) for test problem 3

different roots. However, few black spots at the borders between the basins of attraction indicate the diverging nature.

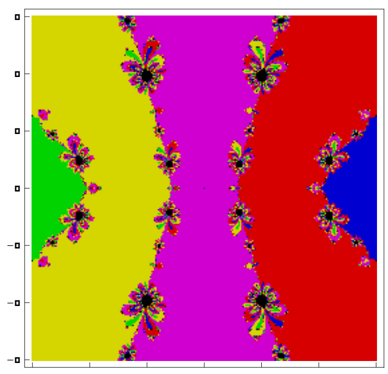

Figure 4. Basins of attraction of method (1.2) for test problem 4

5. Conclusions

In the forgoing study, we have studied the local convergence of an efficient eighth order method by assuming conditions only on the first derivative of the operator. The iterative scheme does not use second or higher-order derivative of the considered function. However, in earlier study of convergence the hypotheses used were based on Taylor series expansions reaching up to the eighth or higher-order derivatives of function although the iterative scheme uses first-order derivative. These conditions restrict the usage of the iterative scheme. We have extended the suitability of method by considering suppositions only on the first-order derivative. The local convergence we have studied is also important in the sense that it provides estimates on radius of convergence and error bounds of the solution. Such estimates are not provided in the procedures that use Taylor expansions of higher derivatives which may not exist or may be very expensive to compute. We have also verified the theoretical results so derived on some numerical problems. Finally, we have checked the stability of the method by means of using complex dynamics tool, namely, basin of attraction.

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