# SUMMABILITY IN MIXED-NORM HARDY SPACES

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**Abstract.** The mixed norm Hardy spaces $H_{\vec{p}}(\mathbb{R}^d)$ is investigated, where $\vec{p} = (p_1, \ldots, p_d) \in (0, \infty]^d$. A general summability method, the so called $\theta$-summability is considered for multi-dimensional Fourier transforms. Under some conditions on $\theta$, it is proved that the maximal operator of the $\theta$-means is bounded from $H_{\vec{p}}(\mathbb{R}^d)$ to $L_{\vec{p}}(\mathbb{R}^d)$. This implies some norm and almost everywhere convergence results for the $\theta$-means, amongst others the generalization of the well known Lebesgue's theorem.

## 1. Introduction

It is due to Lebesgue [16] that the Fejér means [6] of the trigonometric Fourier transforms of a function $f \in L_p(\mathbb{R})$ $(1 \leq p < \infty)$ converge almost everywhere to the function. In this paper we generalize this result to mixed norm Lebesgue spaces and other summability methods as well. A general method of summation, the so called $\theta$-summation method, which is generated by a single function $\theta$ and which includes all well known summations, is studied intensively in the literature (see e.g. Butzer and Nessel [2], Trigub and Belinsky [24], Gát [7, 8, 9], Goginava [10, 11, 12], Persson, Tephnadze and Wall [18], Simon [19, 20] and Feichtinger and Weisz [4, 5, 27, 28, 29]). The means generated

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by the $\theta$-summation are defined for multi-dimensional functions by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\left(\frac{|u|}{T}\right) \widehat{f}(u) e^{2\pi\imath x\cdot u}\, du,$$

where $|\cdot|$ denotes the Euclidean norm and $\widehat{f}$ the Fourier transform of $f$. The choice $\theta(u) = \max(1-|u|, 0)$ yields the Fejér summation.

Stein, Taibleson and Weiss [22] proved for the Bochner-Riesz summability that the maximal operator $\sigma_*^\theta$ of the $\theta$-means is bounded from the Hardy space $H_p(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ if $p > p_0$ (see also Grafakos [13] and Lu [17]). Later we generalized this result to other summability methods in [4, 5, 27, 29].

In this paper, we generalize these results to mixed norm Lebesgue and Hardy spaces, $L_{\vec{p}}(\mathbb{R}^d)$ and $H_{\vec{p}}(\mathbb{R}^d)$, where $\vec{p} = (p_1, \dots, p_d) \in (0,\infty]^d$. We give the atomic decomposition of this Hardy space. If $\vec{p}$ is the vector $(p, \dots, p)$, then we get back the classical Lebesgue and Hardy spaces. Under some conditions on $\theta$, we will prove that the maximal operator $\sigma_*^\theta$ is bounded from $H_{\vec{p}}(\mathbb{R}^d)$ to $L_{\vec{p}}(\mathbb{R}^d)$ when each $p_i > p_0$. As a consequence, we prove some norm and almost everywhere convergence results for the $\theta$-means. In this way, the well known Lebesgue's theorem is generalized. As special cases of the $\theta$-summation, we consider the Riesz, Bochner-Riesz, Weierstrass, Picard and Bessel summations.

## 2. Mixed norm Lebesgue spaces

The $L_p(\mathbb{R}^d)$ space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p\, dx\right)^{1/p} \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$. Here we integrate with respect to the Lebesgue measure $\lambda$. The Lebesgue measure of a set $H$ will be denoted also by $|H|$. Benedek and Panzone [1] generalized this definition as follows. Let $\vec{p} = (p_1, \dots, p_d) \in (0,\infty]^d$. The *mixed-norm Lebesgue space* $L_{\vec{p}}(\mathbb{R}^d)$ is defined to be the set of all measurable functions $f$ such that

$$\|f\|_{L_{cp}(\mathbb{R}^d)} :=$$

$$:= \left(\int_{\mathbb{R}} \dots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, \dots, x_d)|^{p_1}\, dx_1\right)^{p_2/p_1} dx_2\right)^{p_3/p_2} \dots dx_d\right)^{1/p_d} < \infty,$$

with the usual modifications if $p_i = \infty$ for some $i = 1, \dots, d$. If $\vec{p} = (p, \dots, p)$, then we get back the space $L_p(\mathbb{R}^d)$. Let

$$p_- := \min\{p_1, \cdots, p_d\} \qquad \text{and} \qquad \underline{p} = \min\{p_-, 1\}.$$

It is known that

$$\| |f|^s \|_{L_{\vec{p}}(\mathbb{R}^d)} = \|f\|^s_{L_{s\vec{p}}(\mathbb{R}^d)}. \tag{2.1}$$

Given a locally integrable function $f$, the *Hardy-Littlewood maximal operator* $M$ is defined by

$$Mf(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy \qquad (x \in \mathbb{R}^d),$$

where the supremum is taken over all balls $B$ of $\mathbb{R}^d$ containing $x$. It is known that $M$ is bounded on $L_p(\mathbb{R}^d)$ if $1 < p < \infty$. This is extended to the mixed norm spaces in Huang at al. [14].

**Lemma 1.** *If $p_- > 1$ and $f \in L_{\vec{p}}(\mathbb{R}^d)$, then*

$$\|Mf\|_{L_{\vec{p}}(\mathbb{R}^d)} \le C \|f\|_{L_{\vec{p}}(\mathbb{R}^d)}. \tag{2.2}$$

The vector-valued extension of inequality (2.2) holds also. In the classical case see Fefferman–Stein [3], for the mixed norm spaces see Huang at al. [14].

**Lemma 2.** *If $p_- > 1$ and $1 < r < \infty$, then*

$$\left\| \left( \sum_{j=1}^\infty (Mf_j)^r \right)^{1/r} \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \le C \left\| \left( \sum_{j=1}^\infty |f_j|^r \right)^{1/r} \right\|_{L_{\vec{p}}(\mathbb{R}^d)}.$$

We will write $A \lesssim B$ if there exists a constant $C$ such that $A \le CB$.

## 3. Mixed norm Hardy spaces

Now we introduce the *mixed norm Hardy spaces* and give the atomic decompositions. Denote by $S(\mathbb{R}^d)$ the space of all Schwartz functions and by $S'(\mathbb{R}^d)$ the space of all tempered distributions. For $N \in \mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^d) := \left\{ \psi \in S(\mathbb{R}^d) : \sup_{\|\alpha\|_1 \le N} \sup_{x \in \mathbb{R}^d} (1 + |x|)^N |\partial^\alpha \psi(x)| \le 1 \right\},$$

where $\|\alpha\|_1 = |\alpha_1| + \cdots + |\alpha_d|$. For $t \in (0, \infty)$ and $\xi \in \mathbb{R}^d$, let

$$\psi_t(\xi) := t^{-d}\psi(\xi/t).$$

For any $d(1/p_- - 1) + 1 < N < \infty$, the *non-tangential grand maximal function* of $f \in S'(\mathbb{R}^d)$ is defined by

$$f_\Box(x) := \sup_{\psi \in \mathcal{F}_N(\mathbb{R}^d)} \sup_{0<t<\infty, |y-x|<t} |f * \psi_t(y)|.$$

Let $d(1/p_- - 1) + 1 < N < \infty$ be a positive integer. The mixed norm Hardy spaces $H_{\vec{p}}(\mathbb{R}^d)$ are consisting of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that

$$\|f\|_{H_{\vec{p}}(\mathbb{R}^d)} := \|f_\Box\|_{L_{\vec{p}}(\mathbb{R}^d)} < \infty.$$

It is known that different integers $N$ give the same space with equivalent norms. Moreover, all $f \in H_{\vec{p}}(\mathbb{R}^d)$ are bounded distributions, i.e. $f * \phi \in L_\infty(\mathbb{R}^d)$ for all $\phi \in S(\mathbb{R}^d)$. Similarly to the classical case, one can show (see Huang at al. [14]) that

$$H_{\vec{p}}(\mathbb{R}^d) \sim L_{\vec{p}}(\mathbb{R}^d)$$

whenever $p_- > 1$. If each $p_i = p$, then we get back the classical Hardy spaces $H_p(\mathbb{R}^d)$ investigated in Fefferman, Stein and Weiss [3, 23, 21], Lu [17], Uchiyama [25].

The atomic decomposition is a useful characterization of the Hardy spaces by the help of which some boundedness results, duality theorems, inequalities and interpolation results can be proved. A measurable function $a$ is called a $\vec{p}$-atom if there exists a ball $B$ such that

(a) supp $a \subset B$,

(b) $\|a\|_{L_\infty(\mathbb{R}^d)} \leq \dfrac{1}{\|\chi_B\|_{L_{\vec{p}}(\mathbb{R}^d)}}$,

(c) $\int_{\mathbb{R}^d} a(x)x^\alpha dx = 0$ for all multi-indices $\alpha$ with $|\alpha| \leq s$,

where $d(1/p_- - 1) < s < \infty$ is an integer. In the classical case, i.e., if each $p_i = p$, the atomic decomposition theorem can be formulated as follows (see e.g. Latter [15], Lu [17]). Assume that $0 < p \leq 1$. A tempered distribution $f$ is in $H_p(\mathbb{R}^d)$ if and only if there exist a sequence $\{a_i\}_{i\in\mathbb{N}}$ of $p$-atoms with support $\{B_i\}_{i\in\mathbb{N}}$ and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of positive numbers such that $f = \sum_{i\in\mathbb{N}} \lambda_i a_i$ in $S'(\mathbb{R}^d)$. Moreover,

$$\|f\|_{H_p(\mathbb{R}^d)} \sim \inf \left( \sum_{i\in\mathbb{N}} \lambda_i^p \right)^{1/p}.$$

It is easy to see that the right hand sides of the previous and next equations are the same. Thus

$$\|f\|_{H_p(\mathbb{R}^d)} \sim \inf \left\| \left( \sum_{i\in\mathbb{N}} \left( \frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_p(\mathbb{R}^d)}} \right)^p \right)^{1/p} \right\|_{L_p(\mathbb{R}^d)}.$$

And this form can be generalized to all $0 < p < \infty$ as follows. The next theorem is due to Huang at al. [14].

**Theorem 1.** *A tempered distribution $f \in S'(\mathbb{R}^d)$ is in $H_{\vec{p}}(\mathbb{R}^d)$ if and only if there exist a sequence $\{a_i\}_{i\in\mathbb{N}}$ of $\vec{p}$-atoms with support $\{B_i\}_{i\in\mathbb{N}}$ and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of positive numbers such that*

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \qquad \text{in} \qquad S'(\mathbb{R}^d).$$

*Moreover,*

$$\|f\|_{H_{\vec{p}}(\mathbb{R}^d)} \sim \inf \left\| \left( \sum_{i\in\mathbb{N}} \left( \frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}} \right)^{\underline{p}} \right)^{1/\underline{p}} \right\|_{L_{\vec{p}}(\mathbb{R}^d)},$$

*where the infimum is taken over all decompositions of $f$ as above.*

## 4. $\theta$-summability of Fourier transforms

The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\widehat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$ and $x \cdot t := \sum_{k=1}^d x_k t_k$. Suppose first that $f \in L_p(\mathbb{R}^d)$ for some $1 \le p \le 2$. The Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(t) e^{2\pi\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d)$$

holds if $\widehat{f} \in L^1(\mathbb{R}^d)$. This motivates the following definition of $\theta$-summability, which is a general summation generated by a single function $\theta : [0, \infty) \to \mathbb{R}$.

This summation was considered in a great number of papers and books, see e.g. Butzer and Nessel [2], Grafakos [13], Trigub and Belinsky [24] and Feichtinger and Weisz [5, 27, 28, 29] and the references therein. Let $\theta_0(x) := \theta(|x|)$ and suppose that

$$\theta \in C_0[0,\infty), \qquad \theta(0) = 1, \qquad \theta_0 \in L_1(\mathbb{R}^d), \qquad \widehat{\theta_0} \in L^1(\mathbb{R}^d), \tag{4.1}$$

where $C_0[0,\infty)$ denotes the spaces of continuous functions vanishing at infinity and $|\cdot|$ denotes the Euclidean norm. For $T>0$, the $T$th $\theta$-*mean* of the function $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ is given by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\left(\frac{|u|}{T}\right) \widehat{f}(u) e^{2\pi \imath x \cdot u}\, du \qquad (x \in \mathbb{R}^d, T>0).$$

This integral is well defined because $\theta_0 \in L^p(\mathbb{R}^d)$ and $\widehat{f} \in L_{p'}(\mathbb{R}^d)$, where $1/p + 1/p' = 1$.

For an integrable function $f$, it is known that we can rewrite $\sigma_T^\theta f$ as

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t)\, dt = f * K_T^\theta(x) \qquad (x \in \mathbb{R}^d, T>0),$$

where the $T$th $\theta$-kernel is given by

$$K_T^\theta(x) := \int_{\mathbb{R}^d} \theta\left(\frac{|t|}{T}\right) e^{2\pi \imath x \cdot t}\, dt = T^d \widehat{\theta}_0(Tx) \qquad (x \in \mathbb{R}^d, T>0).$$

We can extend the $\theta$-means to all $f \in L_{\vec{p}}(\mathbb{R}^d)$ with $p_- \ge 1$ and to all $f \in H_{\vec{p}}(\mathbb{R}^d)$ with $p_- > 0$ by

$$\sigma_T^\theta f := f * K_T^\theta \qquad (T>0).$$

The maximal $\theta$-operator is introduced by

$$\sigma_*^\theta f := \sup_{T>0} \left|\sigma_T^\theta f\right|.$$

For a ball $B$ with center $c$ and radius $\rho$, let $\tau B$ denotes the ball with the same center and with radius $\tau\rho$ $(\tau > 0)$. The following theorem can be proved as in [26].

**Theorem 2.** *Suppose that* (4.1) *is satisfied,* $\widehat{\theta_0}$ *is* $(N+1)$*-times differentiable for some* $N \in \mathbb{N}$ *and there exists* $d+N < \beta \le d+N+1$ *such that*

$$\left|\partial_1^{i_1} \dots \partial_d^{i_d} \widehat{\theta}_0(x)\right| \le C|x|^{-\beta} \qquad (x \ne 0) \tag{4.2}$$

*whenever* $i_1+\ldots+i_d=N$ *or* $i_1+\ldots+i_d=N+1$. *Then*

$$\left|\sigma_*^\theta a(x)\right| \le C\,\|\chi_B\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-1}\,|M\chi_B(x)|^{\beta/d} \tag{4.3}$$

*for all* $\vec{p}$*-atoms* $a$ *and all* $x\notin 2B$, *where the ball* $B$ *is the support of the atom. If* $\beta=d+N+1$, *then it is enough to suppose that* (4.2) *holds whenever* $i_1+\ldots+i_d=N+1$.

## 5. Boundedness in $H_{\vec{p}}(\mathbb{R}^d)$

In the proof of the boundedness of $\sigma_*^\theta$, we will use the next lemma.

**Lemma 3.** *Let* (4.1) *be satisfied. If* $\lim_{k\to\infty} f_k=f$ *in the* $H_{\vec{p}}(\mathbb{R}^d)$*-norm, then* $\lim_{k\to\infty}\sigma_t^\theta f_k=\sigma_t^\theta f$ *in* $S'(\mathbb{R}^d)$ *for all* $t>0$.

**Proof.** The proof is similar to that of Theorem 7 in [26], so we outline the differences, only. We have to show that $\sigma_t^\theta f$ is a tempered distribution for each $f\in H_{\vec{p}}(\mathbb{R}^d)$ and $t>0$. To this end, the man point is to show that $f*\breve{h}_k$ is uniformly bounded in $k$ if $\lim_{k\to\infty}h_k=h$ in $S(\mathbb{R}^d)$, where $\breve{h}(x):=h(-x)$. We may suppose that $h\in\mathcal{F}_N(\mathbb{R}^d)$, and hence that $h_k\in\mathcal{F}_N(\mathbb{R}^d)$ for large $k$'s. Then for such a $k$,

$$\left|(f*\breve{h}_k)(x)\right|\le f_\square(y)\qquad\text{for every } y \text{ with } |x-y|\le 1.$$

Thus, with the same $x$ and $y$,

$$\begin{aligned}&\left|f*\breve{h}_k(x)\right|\le\\ &\le\left(\int_{-1/2}^{1/2}\cdots\left(\int_{-1/2}^{1/2}\left(\int_{-1/2}^{1/2}|f_\square(y_1,\ldots,y_d)|^{p_1}\,dy_1\right)^{p_2/p_1}dy_2\right)^{p_3/p_2}\ldots dy_d\right)^{1/p_d}\le\\ &\le\|f\|_{H_{\vec{p}}(\mathbb{R}^d)},\end{aligned}$$

which shows the uniform boundedness of $f*\breve{h}_k$. The proof can be finished as Theorem 7 in [26]. ■

**Theorem 3.** *If* (4.1) *and* (4.2) *are satisfied and* $p_->d/\beta$, *then*

$$\left\|\sigma_*^\theta f\right\|_{L_{\vec{p}}(\mathbb{R}^d)}\lesssim\|f\|_{H_{\vec{p}}(\mathbb{R}^d)}\qquad\left(f\in H_{\vec{p}}(\mathbb{R}^d)\right).$$

**Proof.** By the atomic decomposition theorem, $f \in H_{\vec{p}}(\mathbb{R}^d)$ can be written as

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i,$$

where $\lambda_i$ is positive and $a_i$ is a $\vec{p}$-atom with support $B_i$. It is known (see e.g. Weisz [28]) that the series converge in the $H_1(\mathbb{R}^d)$-norm as well as in the $L_1(\mathbb{R}^d)$-norm if $f \in H_{\vec{p}}(\mathbb{R}^d) \cap H_1(\mathbb{R}^d)$. It is easy to see that $\sigma_t^\theta$ is bounded on the $L_1(\mathbb{R}^d)$ space, hence

$$\sigma_t^\theta(f) = \sum_{i \in \mathbb{N}} \lambda_i \sigma_t^\theta(a_i) \qquad (t > 0)$$

and so

$$\sigma_*^\theta(f) \le \sum_{i \in \mathbb{N}} \lambda_i \sigma_*^\theta(a_i).$$

Then

$$\begin{aligned} \|\sigma_*^\theta f\|_{L_{\vec{p}}(\mathbb{R}^d)} \lesssim \Big\| \sum_{i \in \mathbb{N}} \lambda_i \sigma_*^\theta(a_i) \chi_{2B_i} \Big\|_{L_{\vec{p}}(\mathbb{R}^d)} + \Big\| \sum_{i \in \mathbb{N}} \lambda_i \sigma_*^\theta(a_i) \chi_{(2B_i)^c} \Big\|_{L_{\vec{p}}(\mathbb{R}^d)} = \\ =: A_1 + A_2. \end{aligned}$$

Using (2.1) and the fact that $\underline{p} \le 1$, we can see

$$A_1 \le \Big\| \sum_{i \in \mathbb{N}} \lambda_i^{\underline{p}} \sigma_*^\theta(a_i)^{\underline{p}} \chi_{2B_i} \Big\|_{L_{\vec{p}/\underline{p}}(\mathbb{R}^d)}^{1/\underline{p}}.$$

Let $(\vec{p})' = (p_1', \ldots, p_d')$ denote the conjugate index vector, where $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ for every $i = 1, \ldots, d$. By Theorem 1 of Benedek and Panzone [1], there exists $g \in L_{(\vec{p}/\underline{p})'}(\mathbb{R}^d)$ with $\|g\|_{L_{(\vec{p}/\underline{p})'}(\mathbb{R}^d)} \le 1$ such that

$$\Big\| \sum_{i \in \mathbb{N}} \lambda_i^{\underline{p}} \sigma_*^\theta(a_i)^{\underline{p}} \chi_{2B_i} \Big\|_{L_{\vec{p}/\underline{p}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_{i \in \mathbb{N}} \lambda_i^{\underline{p}} \sigma_*^\theta(a_i)^{\underline{p}} \chi_{2B_i} g.$$

Choosing a real number $r$ such that $p_i/\underline{p} < r < \infty$ for all $i = 1, \ldots, d$ and applying Hölder's inequality, we deduce

$$\begin{aligned} A_1^{\underline{p}} &\le \int_{\mathbb{R}^d} \sum_{i \in \mathbb{N}} \lambda_i^{\underline{p}} \sigma_*^\theta(a_i)^{\underline{p}} \chi_{2B_i} g \le \\ &\le \sum_{i \in \mathbb{N}} \lambda_i^{\underline{p}} \|\sigma_*^\theta(a_i)^{\underline{p}} \chi_{2B_i}\|_{L_r(\mathbb{R}^d)} \|\chi_{2B_i} g\|_{L_{r'}(\mathbb{R}^d)} \lesssim \\ &\lesssim \sum_{i \in \mathbb{N}} \lambda_i^{\underline{p}} \|\sigma_*^\theta(a_i)^{\underline{p}}\|_{L_\infty(\mathbb{R}^d)} \lambda(2B_i)^{1/r} \|\chi_{2B_i} g\|_{L_{r'}(\mathbb{R}^d)}. \end{aligned}$$

Observe that $\sigma_*^\theta$ is bounded from $L_\infty(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$. By the definition of the $\vec{p}$-atom,

$$A_1^{\underline{p}} \lesssim \sum_{i\in\mathbb{N}} \lambda_i^{\underline{p}} \|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-\underline{p}} \lambda(2B_i) \left( \frac{1}{\lambda(2B_i)} \int_{2B_i} g^{r'} \right)^{1/r'} \leq$$

$$\leq \int_{\mathbb{R}^d} \sum_{i\in\mathbb{N}} \lambda_i^{\underline{p}} \|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-\underline{p}} \chi_{2B_i} \left( M(g^{r'}) \right)^{1/r'} d\lambda.$$

Again by Hölder's inequality,

$$A_1^{\underline{p}} \lesssim \left\| \sum_{i\in\mathbb{N}} \lambda_i^{\underline{p}} \|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-\underline{p}} \chi_{2B_i} \right\|_{L_{\vec{p}/\underline{p}}(\mathbb{R}^d)} \left\| \left( M(g^{r'}) \right)^{1/r'} \right\|_{L_{(\vec{p}/\underline{p})'}(\mathbb{R}^d)}.$$

Since $p_i/\underline{p} < r < \infty$ imply $(p_i/\underline{p})' > r'$ $(i = 1, \ldots, d)$, we get by (2.1) and (2.2) that

$$A_1 \lesssim \left\| \sum_{i\in\mathbb{N}} \lambda_i^{\underline{p}} \|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-\underline{p}} \chi_{2B_i} \right\|_{L_{\vec{p}/\underline{p}}(\mathbb{R}^d)}^{1/\underline{p}} \left\| M(g^{r'}) \right\|_{L_{((\vec{p}/\underline{p})')/r'}(\mathbb{R}^d)}^{1/\underline{p}r'} \lesssim$$

$$\lesssim \left\| \sum_{i\in\mathbb{N}} \lambda_i^{\underline{p}} \|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-\underline{p}} \chi_{2B_i} \right\|_{L_{\vec{p}/\underline{p}}(\mathbb{R}^d)}^{1/\underline{p}} \|g\|_{L_{(\vec{p}/\underline{p})'}(\mathbb{R}^d)}^{1/\underline{p}} \lesssim$$

$$\lesssim \left\| \left( \sum_{i\in\mathbb{N}} \left( \frac{\lambda_i \chi_{2B_i}}{\|\chi_{2B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}} \right)^{\underline{p}} \right)^{1/\underline{p}} \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \lesssim \|f\|_{H_{\vec{p}}(\mathbb{R}^d)}.$$

On the other hand, using (4.3) and Lemma 2, we establish that

$$A_2 \lesssim \left\| \sum_{i\in\mathbb{N}} \lambda_i \, \|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-1} \, |M\chi_{B_i}|^{\beta/d} \chi_{(2B_i)^c} \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq$$

$$\leq \left\| \left( \sum_{i\in\mathbb{N}} \left( \lambda_i^{d/\beta} \, \|\chi_{B_i}\|_{\vec{p}}^{-d/\beta} \, |M\chi_{B_i}| \right)^{\beta/d} \right)^{d/\beta} \right\|_{L_{\beta\vec{p}/d}(\mathbb{R}^d)}^{\beta/d} \leq$$

$$\leq \left\| \left( \sum_{i\in\mathbb{N}} \lambda_i \, \|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}^{-1} \, \chi_{B_i} \right)^{d/\beta} \right\|_{L_{\beta\vec{p}/d}(\mathbb{R}^d)}^{\beta/d} \lesssim$$

$$\lesssim \left\| \sum_{i\in\mathbb{N}} \frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_{\vec{p}}(\mathbb{R}^d)}} \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \lesssim \|f\|_{H_{\vec{p}}(\mathbb{R}^d)},$$

which proves the theorem for $f \in H_{\vec{p}}(\mathbb{R}^d) \cap H_1(\mathbb{R}^d)$. Note that $\beta/d > 1$ and $p_- > d/\beta$. Using lemma 3, the proof can be finished by a standard density argument as in [26]. ■

Note that if each $p_i = p$, then we get back the classical result (see Weisz [27, 28]). The classical result was proved in a special case, for the Bochner-Riesz means in Stein, Taibleson and Weiss [22], Grafakos [13] and Lu [17]. For the same case [22] contains a counterexample which shows that the theorem is not true for $p \le n/\beta$.

Using Theorem 3 and a usual density argument, we obtain the next convergence results. We do not give the details here, because they can be found in similar cases in [26].

**Corollary 1.** *Suppose that* (4.1) *and* (4.2) *are satisfied and* $p_- > d/\beta$. *If* $f \in H_{\vec{p}}(\mathbb{R}^d)$, *then* $\sigma_T^\theta f$ *converges almost everywhere as well as in the* $L_{\vec{p}}(\mathbb{R}^d)$-*norm as* $T \to \infty$.

For functions from the Hardy spaces, the limit of $\sigma_T^\theta f$ will be exactly the function.

**Corollary 2.** *Suppose that* (4.1) *and* (4.2) *are satisfied and* $p_- > d/\beta$. *If* $f \in H_{\vec{p}}(\mathbb{R}^d)$ *and there exists an interval* $I \subset \mathbb{R}^d$ *such that the restriction* $f|_I \in L_{\vec{r}}(I)$ *with* $r_- \ge 1$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \qquad \text{for a.e. } x \in I \text{ as well as in the } L_{\vec{p}}(I)\text{-norm.}$$

The next consequence follows from the fact that $L_{\vec{p}}(\mathbb{R}^d)$ is equivalent to $H_{\vec{p}}(\mathbb{R}^d)$ if $p_- > 1$.

**Corollary 3.** *Suppose that* (4.1) *and* (4.2) *are satisfied and* $p_- > d/\beta$. *If* $p_- > 1$ *and* $f \in L_{\vec{p}}(\mathbb{R}^d)$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d \text{ as well as in the } L_{\vec{p}}(\mathbb{R}^d)\text{-norm.}$$

## 6. Some summability methods

As special cases, we consider some summability methods. The details of the necessary computations are left to the reader.

### 6.1. Riesz summation

The function

$$\theta_0(t) = \begin{cases} (1-|t|^\gamma)^\alpha, & \text{if } |t| > 1; \\ 0, & \text{if } |t| \le 1 \end{cases} \qquad (t \in \mathbb{R}^d)$$

defines the *Riesz summation* if $0 < \alpha < \infty$ and $\gamma$ is a positive integer. It is called *Bochner-Riesz summation* if $\gamma = 2$. The next lemma can be found in Stein and Weiss [23] (see also Lu [17, p. 132] and Weisz [29]).

**Lemma 4.** *Condition* (4.1) *is satisfied if* $\alpha > \frac{d-1}{2}$ *and*

$$\left|\partial_1^{i_1} \dots \partial_d^{i_d} \widehat{\theta}_0(x)\right| \le C|x|^{-d/2-\alpha-1/2} \qquad (x \neq 0)$$

*for all* $i_1, \dots, i_d \in \mathbb{N}$.

The following result follows from Theorem 3.

**Corollary 4.** *If*

$$\alpha > \frac{d-1}{2}, \qquad \frac{d}{d/2+\alpha+1/2} < p_- < \infty,$$

*then*

$$\left\|\sigma_*^\theta f\right\|_{L_{\vec{p}}(\mathbb{R}^d)} \lesssim \|f\|_{H_{\vec{p}}(\mathbb{R}^d)} \qquad (f \in H_{\vec{p}}(\mathbb{R}^d)).$$

*Moreover, the corresponding Corollaries* 1–3 *hold as well.*

### 6.2. Weierstrass summation

The *Weierstrass summation* is defined by

$$\theta_0(t) = e^{-|t|^2/2} \qquad (t \in \mathbb{R}^d) \tag{6.1}$$

or by

$$\theta_0(t) = e^{-|t|} \qquad (t \in \mathbb{R}^d), \tag{6.2}$$

or, in the one-dimensional case, by

$$\theta_0(t) = e^{-|t|^\gamma} \qquad (t \in \mathbb{R}, 1 \le \gamma < \infty). \tag{6.3}$$

It is called **Abel summation** if $\gamma = 1$. It is known that in the first case $\widehat{\theta}_0(x) = e^{-|x|^2/2}$ and in the second one $\widehat{\theta}_0(x) = c_d/(1+|x|^2)^{(d+1)/2}$ for some $c_d \in \mathbb{R}$ (see Stein and Weiss [23, p. 6.]). The following lemma is easy to verify.

**Lemma 5.** *Let* $\theta_0$ *be as in* (6.1) *or in* (6.2) *or in* (6.3). *Then condition* (4.1) *is satisfied and for any* $N \in \mathbb{N}$,

$$\left|\partial_1^{i_1} \dots \partial_d^{i_d} \widehat{\theta}_0(x)\right| \le C|x|^{-d-N-1} \qquad (x \neq 0),$$

*where* $i_1 + \dots + i_d = N+1$.

The following result is an easy consequence of Theorem 3.

**Corollary 5.** *Let $\theta_0$ be as in* (6.1) *or in* (6.2) *or in* (6.3). *Then*

$$\left\|\sigma_*^\theta f\right\|_{L_{\vec{p}}(\mathbb{R}^d)} \lesssim \|f\|_{H_{\vec{p}}(\mathbb{R}^d)} \qquad (f \in H_{\vec{p}}(\mathbb{R}^d)).$$

*Moreover, the corresponding Corollaries* 1–3 *and hold as well.*

### 6.3. Picard–Bessel summation

Now let

$$\theta_0(t) = \frac{1}{(1+|t|^2)^{(d+1)/2}} \qquad (t \in \mathbb{R}^d). \tag{6.4}$$

Here $\widehat{\theta_0}(x) = c_d e^{-|x|}$ for some $c_d \in \mathbb{R}^d$.

**Corollary 6.** *Let $\theta_0$ be as in* (6.4). *Then Lemma* 5 *and Corollary* 5 *hold.*

## References


[1] **Benedek, A. and R. Panzone,** The spaces $L^p$, with mixed norm, *Duke Math. J.*, **28** (1961), 301–324.

[2] **Butzer, P.L. and R.J. Nessel,** *Fourier Analysis and Approximation*, Birkhäuser Verlag, Basel, 1971.

[3] **Fefferman, C. and E.M. Stein,** $H^p$ spaces of several variables, *Acta Math.*, **129** (1972), 137–194.

[4] **Feichtinger, H.G. and F. Weisz,** The Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ and norm summability of Fourier series and Fourier transforms, *Monatsh. Math.*, **148** (2006), 333–349.

[5] **Feichtinger, H.G. and F. Weisz,** Wiener amalgams and pointwise summability of Fourier transforms and Fourier series,. *Math. Proc. Cambridge Philos. Soc.*, **140** (2006), 509–536.

[6] **Fejér, L.,** Untersuchungen über Fouriersche Reihen, *Math. Ann.*, **58** (1904), 51–69.

[7] **Gát, G.,** Pointwise convergence of cone-like restricted two-dimensional $(C,1)$ means of trigonometric Fourier series, *J. Approx. Theory.*, **149** (2007), 74–102.

[8] **Gát, G.,** Almost everywhere convergence of sequences of Cesàro and Riesz means of integrable functions with respect to the multidimensional Walsh system, *Acta Math. Sin., Engl. Ser.*, **30(2)** (2014), 311–322.

[9] **Gát, G., U. Goginava and K. Nagy,** On the Marcinkiewicz-Fejér means of double Fourier series with respect to Walsh-Kaczmarz system, *Studia Sci. Math. Hungar.*, **46** (2009), 399–421.

[10] **Goginava, U.,** Marcinkiewicz–Fejér means of $d$-dimensional Walsh-Fourier series, *J. Math. Anal. Appl.*, **307** (2005), 206–218.

[11] **Goginava, U.,** Almost everywhere convergence of (C, a)-means of cubical partial sums of d-dimensional Walsh-Fourier series, *J. Approx. Theory*, **141** (2006), 8–28.

[12] **Goginava, U.,** The maximal operator of the Marcinkiewicz-Fejér means of $d$-dimensional Walsh-Fourier series, *East J. Approx.*, **12** (2006), 295–302.

[13] **Grafakos, L.,** *Classical and Modern Fourier Analysis.* Pearson Education, New Jersey, 2004.

[14] **Huang, L., J. Liu, D. Yang and W. Yuan,** Atomic and Littlewood–Paley characterizations of anisotropic mixed-norm Hardy spaces and their applications, `http://arxiv.org/pdf/1801.06251.pdf`, 2018.

[15] **Latter, R.H.,** A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms, *Studia Math.*, **62** (1978), 92–101.

[16] **Lebesgue, H.,**. Recherches sur la convergence des séries de Fourier, *Math. Ann.*, **61** (1905), 251–280.

[17] **Lu, S.,** *Four Lectures on Real $H^p$ Spaces*, World Scientific, Singapore, 1995.

[18] **Persson, L.E., G. Tephnadze and P. Wall,** Maximal operators of Vilenkin-Nörlund means, *J. Fourier Anal. Appl.*, **21(1)** (2015), 76–94.

[19] **Simon, P.,** Cesàro summability with respect to two-parameter Walsh systems, *Monatsh. Math.*, **131** (2000), 321–334.

[20] **Simon, P.,** $(C, \alpha)$ summability of Walsh-Kaczmarz-Fourier series, *J. Approx. Theory*, **127** (2004), 39–60.

[21] **Stein, E.M.,** *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N.J., 1993.

[22] **Stein, E.M., M.H. Taibleson and G. Weiss,** Weak type estimates for maximal operators on certain $H^p$ classes, *Rend. Circ. Mat. Palermo, Suppl.*, **1** (1981), 81–97.

[23] **Stein, E.M. and G. Weiss,** *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

[24] **Trigub, R.M. and E.S. Belinsky,** *Fourier Analysis and Approximation of Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2004.

[25] **Uchiyama, A.,** *Hardy Spaces on the Euclidean Space*, Springer Monographs in Mathematics. Springer, Berlin, 2001.

[26] **Weisz, F.,** Summability of Fourier transforms in variable Hardy and Hardy-Lorentz spaces, (preprint).

[27] **Weisz, F.,** *Summability of Multi-dimensional Fourier Series and Hardy Spaces*, Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[28] **Weisz, F.,** Summability of multi-dimensional trigonometric Fourier series, *Surv. Approx. Theory*, 7:1–179, 2012.

[29] **Weisz, F.,** *Convergence and Summability of Fourier Transforms and Hardy Spaces*, Applied and Numerical Harmonic Analysis. Springer, Birkhäuser, Basel, 2017.

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