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SPHERICAL MONOMIALS ON AFFINE GROUPS

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Abstract. Spherical monomials are the building blocks of spherical spectral synthesis. In this work we exhibit the basic ideas how to describe spherical monomials on some types of affine groups using invariant differential operators. In particular, we show that if the algebra of invariant differential operators is generated by a single operator then the classes of spherical monomials and of spherical moment functions coincide.

1. Introduction

Let V be finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose that K is a subgroup of the general linear group $GL(\mathbb{K}^n)$ which we realize as a matrix group with respect to the standard basis of $\mathbb{K}^n$. We define G as the *affine group of K over $\mathbb{K}^n$*, or simply the *affine group of K* as the semidirect product of K and $\mathbb{K}^n$:

$$G = \mathrm{Aff}\, K = K \ltimes \mathbb{K}^n.$$

Recall that G as a set is identified with $K \times \mathbb{K}^n$, and the group operation is given as

$$(k, u) \cdot (l, v) = (k \circ l, kv + u)$$

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for each k, l in K and u, v in $\mathbb{K}^n$, where $k \circ l$ is the composition in K and kv is the image of the vector u under the linear map k. Equipped G with the ordinary Euclidean topology it is a locally compact topological group, in fact, it is a Lie group. The identity of G is $(id, 0)$ where id is the identity map in K, and the inverse of (k, u) is $(k^{-1}, -k^{-1}u)$. Typical examples for affine groups are the following:

i) Let $K = \{id\}$, the trivial subgroup of $GL(\mathbb{K}^n$. In this case Aff K identifies with $\{id\} \times \mathbb{K}^n = \mathbb{K}^n$ itself, the *group of all translations* on $\mathbb{K}^n$.

ii) Let $\mathbb{K} = \mathbb{R}$ and $K = O(n)$, the *orthogonal group*, then

$$\mathrm{Aff}\, O(n) = O(n) \ltimes \mathbb{R}^n$$

is the group of *Euclidean motions.* The affine group of $O(n)$ acts on $\mathbb{R}^n$ in the natural way: for (k, u) in Aff $O(n)$ and x in $\mathbb{R}^n$ we let

$$(k, u) \cdot x = kx + u,$$

which represents the affine map which is composition of a rotation corresponding to k, and a translation corresponding to u. The "product" of (k, u) and (l, v) corresponds to the composition of the respective affine mappings.

iii) Replacing $O(n)$ by $SO(n)$, the *special orthogonal group* in the previous example, we obtain the affine group

$$\mathrm{Aff}\, SO(n) = SO(n) \ltimes \mathbb{R}^n,$$

the group of *proper Euclidean motions.*

iv) Let $\mathbb{K} = \mathbb{C}$ and $K = U(n)$, the *unitary group*, then

$$\mathrm{Aff}\, U(n) = U(n) \ltimes \mathbb{C}^n$$

is the group of *unitary motions.* The action of $U(n)$ on $\mathbb{C}^n$ is similar to that of $O(n)$ over $\mathbb{R}^n$. Also, we can replace $U(n)$ by $SU(n)$, the *special unitary group* to obtain the affine group

$$\mathrm{Aff}\, SU(n) = SU(n) \ltimes \mathbb{C}^n$$

of *proper unitary motions* in $\mathbb{C}^n$.

v) Let $\mathbb{K} = \mathbb{R}$ and let $K = O(1, 3)$ denote the *Lorentz group* on the Minkowski spacetime $\mathbb{R}^{1,3} = \mathbb{R} \oplus \mathbb{R}^3$, that is the group of linear maps on $\mathbb{R}^4$ which leave the indefinite scalar product

$$\langle x, y \rangle = x_0 y_0 - \sum_{j=1}^{3} x_j y_j$$

invariant. Here $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ are in $\mathbb{R}^4$. The corresponding affine group is

$$\mathrm{Aff}\, O(1,3) = O(1,3) \ltimes \mathbb{R}^{1,3}$$

is the *Poincaré group.*

The affine group $G = \mathrm{Aff}\, K$ has a closed subgroup $\{(k, 0) : k \in K\}$ which is topologically isomorphic to K, therefore we shall identify it with K. Also, the closed commutative subgroup $\{(id, u) : u \in \mathbb{K}^n\}$ is topologically isomorphic to $\mathbb{K}^n$, therefore we shall identify it with $\mathbb{K}^n$. In fact, the additive group of $\mathbb{K}^n$ is embedded as a closed normal subgroup into $\mathrm{Aff}\, K$, and the factor group $\mathrm{Aff}\, K/\mathbb{K}^n$ is topologically isomorphic to K.

2. Invariant functions and measures

Given the affine group $G = \mathrm{Aff}\, K$ we call the function $f : G \to \mathbb{C}$ K-*invariant* if f is invariant with respect to left and right multiplications with elements of K: more exactly, we have for each k, l', l'' in K and u in $\mathbb{K}^n$

$$f\big((l', 0) \cdot (k, u) \cdot (l'', 0)\big) = f(k, u).$$

We have the following simple statement:

Proposition 2.1. *The function $f : G \to \mathbb{C}$ is K-invariant if and only if*

$$f(k, u) = f(id, lu)$$

holds for each k, l in K and u in $\mathbb{K}^n$.

Proof. If f satisfies the above condition, then we have

$$f\big((l', 0) \cdot (k, u) \cdot (l'', 0)\big) = f(l' \circ k \circ l'', l'u) = f(id, lu)$$

for each k, l, l', l'' in K and u in $\mathbb{K}^n$. Also, $f(k, u) = f(id, lu)$ for each k, l in K and u in $\mathbb{K}^n$, hence f is K-invariant.

Conversely, if f is K-invariant, then given k, l in K and u in $\mathbb{K}^n$ we choose $l' = l$ and $l'' = k^{-1}l^{-1}$, then we get

$$f(k, u) = f\big((l', 0) \cdot (k, u) \cdot (l'', 0)\big) = f(id, lu)$$

and the statement is proved.

By this statement, K-invariant functions on the affine group $\mathrm{Aff}\, K$ can be identified with those functions on $\mathbb{K}^n$ which are invariant under the action of K on $\mathbb{K}^n$:

$$f(k,u) = \varphi(u), \text{ with } \varphi(lu) = \varphi(u)$$

for each k, l in K and u in $\mathbb{K}^n$, where $\varphi : \mathbb{K}^n \to \mathbb{C}$ is a function. In fact, $\varphi(u) = f(id, u)$ holds for each u in $\mathbb{K}^n$.

We use the standard notation $\mathcal{C}(X)$ for the set of all continuous complex valued functions on the locally compact Hausdorff topological space X. Equipped with the topology of uniform convergence on compact sets and with the pointwise linear operations $\mathcal{C}(X)$ is a locally convex topological vector space. The dual of this space can be identified with the space $\mathcal{M}_c(X)$ of all compactly supported complex Borel measures on X. The pairing between $\mathcal{M}_c(X)$ and $\mathcal{C}(X)$ is given by

$$\langle \mu, f\rangle = \int f\, d\mu.$$

Given the affine group G as above the space of all continuous K-invariant functions is identified with the closed subspace $\mathcal{C}_K(\mathbb{K}^n)$ of all K-invariant continuous functions on $\mathbb{K}^n$. The dual of this subspace is identified with a weak*-closed subspace $\mathcal{M}_K(\mathbb{K}^n)$ of $\mathcal{M}_c(\mathbb{K}^n)$ whose elements μ are characterized by the property that $\mu * f$ is K-invariant whenever f is K-invariant. In other words, $\mathcal{C}_K(\mathbb{K}^n)$ is a topological module over the ring $\mathcal{M}_K(\mathbb{K}^n)$, where the ring operation in $\mathcal{M}_K(\mathbb{K}^n)$ is the convolution of measures:

$$(\mu * \nu)(f) = \int\int f(x+y)\, d\mu(x)\, d\nu(y)$$

for each f in $\mathcal{C}_K(\mathbb{K}^n)$.

3. Projections

Now we suppose that K is a compact subgroup of $GL(\mathbb{K}^n)$ and ω is the normalized Haar measure on K. We introduce the map $f \mapsto f^\#$ from $\mathcal{C}(G)$ to $\mathcal{C}(G)$ defined by

$$f^\#(k,u) = \int_K \int_K f(l'kl'', l'u)\, d\omega(l')\, d\omega(l'')$$

for each k in K and u in $\mathbb{K}^n$. Clearly, the right hand side is independent of k, by the left and right invariance of the Haar measure ω. Hence we can write

$$f^\#(k,u) = \int_K \int_K f(l'', l'u)\, d\omega(l')\, d\omega(l'') = f^\#(id, u)$$

for each k in K and u in $\mathbb{K}^n$. On the other hand, we have for each l in K:

$$f^{\#}(k, lu) = \int_K \int_K f(l'', l'lu)\, d\omega(l')\, d\omega(l'') = f^{\#}(k, u) = f^{\#}(id, lu),$$

hence $f^{\#}$ is K-invariant. Consequently, $f \mapsto f^{\#}$ is a continuous linear mapping from $\mathcal{C}(G)$ to $\mathcal{C}_K(G)$. This mapping is also surjective, as it is the identity on $\mathcal{C}_K(G)$. In fact, the continuous function f on G is K-invariant if and only if $f = f^{\#}$. The mapping $f \mapsto f^{\#}$ is called *K-projection.*

The projection of the measure μ in $\mathcal{M}_c(G)$ is defined by the equation

$$\langle \mu^{\#}, f\rangle = \langle \mu, f^{\#}\rangle$$

whenever f is in $\mathcal{C}(G)$. Then $\mu^{\#}$ is a K-invariant measure, and the map $\mu \mapsto \mu^{\#}$ is a weak*-continuous linear map from $\mathcal{M}_c(G)$ onto $\mathcal{M}_K(\mathbb{K}^n)$. In particular, μ in $\mathcal{M}_c(G)$ is K-invariant if and only if $\mu = \mu^{\#}$.

Let for each $\delta_{(k,u)}$ denote the point mass at (k, u) – then $\delta_{(id,0)}$ is the identity in $\mathcal{M}_c(G)$. Its projection $\delta^{\#}_{(id,0)}$ is the measure on $\mathcal{C}(G)$ given by

$$\begin{aligned}\langle \delta^{\#}_{(id,0)}, f\rangle = \langle \delta_{(id,0)}, f^{\#}\rangle &= \int_{\mathbb{K}^n} \int_K \int_K f(l'', l'u)\, d\omega(l')\, d\omega(l'')\, d\delta_{(id,0)}(k, u) = \\ &= \int_K f(l, 0)\, d\omega(l),\end{aligned}$$

which is the average of the function $l \mapsto f(l, 0)$ over K. The projection of $\delta_{(l,v)}$ is

$$\begin{aligned}\langle \delta^{\#}_{(l,v)}, f\rangle = \langle \delta_{(l,v)}, f^{\#}\rangle &= \int_{\mathbb{K}^n} \int_K \int_K f(l'', l'u)\, d\omega(l')\, d\omega(l'')\, d\delta_{(l,v)}(k, u) = \\ &= \int_K \int_K f(l'', l'v)\, d\omega(l')\, d\omega(l'').\end{aligned}$$

If f is K-invariant, then we have $\langle \delta^{\#}_{(l,v)}, f\rangle = f(v) = \langle \delta_v, f\rangle$, where δ_v is the point mass on $\mathbb{K}^n$ with support at v. In fact, $\delta^{\#}_{(l,v)}$ is independent of l, hence we can simply write $\delta^{\#}_v$, instead.

As convolution in $\mathcal{M}_K(\mathbb{K}^n)$ reduces to ordinary convolution of measures on $\mathbb{K}^n$, $\mathcal{M}_K(\mathbb{K}^n)$ is a commutative algebra, and (G, K) is a *Gelfand pair.*

4. Spherical functions

We introduce K*-translation operators* on the space $\mathcal{C}_K(\mathbb{K}^n)$: let for each y in $\mathbb{K}^n$ and f in $\mathcal{C}_K(\mathbb{K}^n)$

$$\tau_y f(x) = \delta_y^{\#} * f(x) = \int_K f(x+ky)\,d\omega(k).$$

As the algebra $\mathcal{M}_K(\mathbb{K}^n)$ is commutative, all K-translation operators form a commutative family of linear operators. Common normalized eigenfunctions of all K-translations are called K*-spherical functions*. The following proposition is easy to prove.

Proposition 4.1. *The continuous K-invariant function $f : \mathbb{K}^n \to \mathbb{C}$ is a K-spherical function if and only if $f(0) = 1$, and*

$$\int_K f(x+ky)\,d\omega(k) = f(x)f(y)$$

holds for each x, y in $\mathbb{K}^n$.

Proof. If f satisfies the given conditions, then it is clearly a K-spherical function.

Conversely, suppose that f is a common normalized eigenfunction of all K-translation operators, then there exists a function $\lambda : \mathbb{K}^n \to \mathbb{C}$ such that

$$\int_K f(x+ky)\,d\omega(k) = \lambda(y)f(x)$$

holds for each x, y in $\mathbb{K}^n$. As $f(0) = 1$ we have

$$\int_K f(ky)\,d\omega(k) = \lambda(y),$$

and, by the K-invariance of f, the left hand side is $f(y)$. ■

The following theorem characterizes K-spherical functions on affine groups. Recall that linear operators of the form $f \mapsto \mu * f$ with f in $\mathcal{C}_K(\mathbb{K}^n)$ and μ in $\mathcal{M}_K(\mathbb{K}^n)$ are called *convolution operators.*

Theorem 4.2. *Given the affine group $G = K \ltimes \mathbb{K}^n$ with K compact K-spherical functions are exactly the common normalized eigenfunctions of all convolution operators in $\mathcal{C}_K(\mathbb{K}^n)$.*

Proof. As all K-translations are convolution operators, the first part of the statement is obvious: every common normalized eigenfunction of all convolution operators is a K-spherical function.

For the converse we recall that linear combinations of point masses form a weak*-dense subset in $\mathcal{M}_c(G)$, which implies that the linear combinations of their projections form a weak*-dense subset in $\mathcal{M}_K(\mathbb{K}^n)$. ∎

As a simple example we consider the case $K = \{id\}$ which leads to the group of translations $G = \mathbb{K}^n$. In this case every function is K-invariant on $\mathbb{K}^n$ and the equation characterizing K-spherical functions reduces to

$$f(x + y) = f(x)f(y)$$

with $f(0) = 1$. Clearly, the continuous solutions are exactly the exponential functions $f(x) = \exp \lambda \cdot x$ with λ in $\mathbb{C}^n$, where $\lambda \cdot x$ is the inner product in $\mathbb{K}^n$.

The example is the case of the group G of Euclidean motions: $G = \operatorname{Aff} O(n)$.

Theorem 4.3. *The continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is an $O(n)$-spherical function if and only if $f(0) = 1$, f is C^∞, and f is an eigenfunction of the Laplacian in $\mathbb{R}^n$.*

The proof can be found in [1]. As $O(n)$-invariant functions are *radial*, that is, depend on the norm only, the continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is an $O(n)$-spherical function if and only if it has the form

$$f(x) = \varphi(\|x\|)$$

for x in $\mathbb{R}^n$, where $\varphi : \mathbb{R} \to \mathbb{C}$ is a regular solution of

$$\frac{d^2\varphi}{dr^2}(r) + \frac{n-1}{r}\frac{d\varphi}{dr}(r) = \lambda\varphi(r)$$

for some complex number λ with $\varphi(0) = 1$. In fact, there is a unique solution for every complex number λ. If the unique $O(n)$-spherical function corresponding to λ is denoted by s_λ then s_λ is the following:

$$s_\lambda(x) = J_{\lambda,n}(\|x\|) = \Gamma\Big(\frac{n}{2}\Big) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!\Gamma\Big(k + \frac{n}{2}\Big)} \Big(\frac{\|x\|}{2}\Big)^k$$

for x in $\mathbb{R}^n$ (see [1]). Here J_λ is the Bessel function with parameters λ, n.

As $SO(n)$ acts on the spheres in $\mathbb{R}^n$ transitively (see [2]), it follows that the space of $SO(n)$-invariant functions coincides with the set of radial functions. Hence $SO(n)$-spherical functions are the same as the $O(n)$-spherical functions on $\mathbb{R}^n$.

In our fourth example above we consider the affine groups of the unitary motions and of the proper unitary motions. In [3], $SU(n)$-spherical functions have been described. We note that a similar description is possible for $U(n)$-spherical functions.

In all these examples K is a compact subgroup of $GL(\mathbb{K}^n)$ and clearly, compactness of K plays a basic role in the definition of K-spherical functions. In the last example we have $K = O(1,3)$, the Lorentz group, and its affine group, the Poincaré group over the Minkowski spacetime. In this case K is non-compact. Still, the subgroup $SO(1,3)$ of orientation preserving operators in $O(1,3)$ do not form a compact group. In fact, its maximal compact subgroup is $S(O(1) \times O(3))$.

On the other hand, the identity component of $O(1,3)$ identifies with $SO^+(1,3)$, which refers to preserving orientation in the first (temporal) dimension. It is still not compact, but its maximal compact subgroup is $K = SO(1) \times SO(3)$ which identifies with $SO(3)$. The corresponding affine group is

$$G = \mathrm{Aff}\,[SO(1) \times SO(3)] = [SO(1) \times SO(3)] \ltimes \mathbb{R}^{1,3} = SO(3) \ltimes \mathbb{R}^{1,3},$$

where (k,t,v) in $\mathrm{Aff}\,SO(3)$ acts on $\mathbb{R} \oplus \mathbb{R}^3$ for (t,v) and (u,x) in $\mathbb{R}^{1,3}$ as

$$(k,t,v) \cdot (s,x) = (t+u, kx+v).$$

K-invariant functions on G are those independent of the first variable and $SO(3)$-invariant in the second in the sense:

$$f(k,t,x) = f(id,t,lx)$$

for each k,l in $SO(3)$, t in $\mathbb{R}$ and x in $\mathbb{R}^3$. Hence the space of K-invariant functions is identified with the set of those continuous functions f on $\mathbb{R} \oplus \mathbb{R}^3$ satisfying $f(t,kx) = f(t,x)$ for each t in $\mathbb{R}$, x in $\mathbb{R}^3$ and k in $SO(3)$. The functional equation of K-spherical functions is:

$$\int_{SO(3)} s(t+u, x+ky)\, d\omega(k) = s(t,x)s(u,y), \quad s(0,0) = 1.$$

Putting $t = 0, y = 0$ we have

$$s(u,x) = s(0,x)s(u,0).$$

On the other hand, putting $x = y = 0$ it follows

$$s(t+u,0) = s(t,0)s(u,0),$$

hence $s(t,0)=e^{\lambda t}$ with λ in $\mathbb{C}$. Finally, putting $t=u=0$ we have

$$\int_{SO(3)} s(0,x+ky)\,d\omega(k)=s(0,x)s(0,y),\ \ s(0,0)=1,$$

hence

$$s(0,x)=J_{\xi,3}(\|x\|),$$

where ξ is in $\mathbb{C}$ and J_ξ is the Bessel function with parameters $\xi,3$. The explicit form is

$$J_{\xi,3}(r)=\Gamma\Big(\frac{3}{2}\Big)\sum_{k=0}^{\infty}\frac{\xi^k}{k!\,\Gamma\big(k+\frac{3}{2}\big)}\Big(\frac{r}{2}\Big)^{2k},$$

which reduces to

$$J_{\xi,3}(r)=\frac{\sinh\xi r}{\xi r}.$$

Finally, we have for the K-spherical functions s:

$$s(t,x)=s_{\lambda,\xi}(t,x)=\frac{\sinh\xi\|x\|}{\xi\|x\|}\cdot e^{\lambda t}$$

with λ,ξ in $\mathbb{C}$.

5. Spherical monomials and moment functions

For each K-spherical function s and y in $\mathbb{K}^n$ we define the *modified K-difference* as the measure:

$$\Delta_{s;y}=\delta^{\#}_{-y}-s(y)\delta_0.$$

For the higher order modified K-differences we use the following notation:

$$\Delta_{s;y_1,y_2,\dots,y_{j+1}}=\Pi_{i=1}^{j+1}\Delta_{s;y_i}$$

whenever j is a natural number and $y_1,y_2,\dots,y_{j+1}$ are in $\mathbb{K}^n$. In particular, if $y=y_1=y_2=\dots=y_{j+1}$, then we write

$$\Delta^{n+1}_{s;y}=\Delta_{s;y_1,y_2,\dots,y_{j+1}}.$$

The explicit form of $\Delta_{s;y_1,y_2,\dots,y_{j+1}}*f(x)$ can be given as

$$\sum_{i=0}^{j+1}\sum_{1\le j_1<j_2<\dots<j_l\le j+1}(-1)^{j_1+j_2+\dots+j_l}s(y_{j_1})s(y_{j_2})\cdots s(y_{j_l})\cdot$$
$$\cdot\int_K\cdots\int_K f(x+\epsilon_1k_1\cdot y_1+\dots+\epsilon_{j+1}k_{j+1}\cdot y_{j+1})\,d\omega(k_1)\dots d\omega(k_{j+1}),$$

where in each summand ϵ_i is 0 or 1 depending on if i is in the set $\{j_1, j_2, \dots, j_l\}$ or it is not. For instance, for any y, z in $\mathbb{K}^n$ we have

$$\Delta_{s;y,z} * f(x) = \int_K \int_K f(x + k \cdot y + l \cdot z)\, d\omega(k)\, d\omega(l) -$$
$$- s(y) \int_K f(x + l \cdot z)\, d\omega(l) - s(z) \int_K f(x + k \cdot y)\, d\omega(k) + s(y)s(z)f(x).$$

If $s \equiv 1$, then we write Δ_y for $\Delta_{1;y}$, and $\Delta_{y_1,y_2,\dots,y_{j+1}}$ for $\Delta_{1;y_1,y_2,\dots,y_{j+1}}$.

We call the continuous K-invariant function f a *K-spherical monomial*, if it generates a finite dimensional submodule, and if there is a K-spherical function s and a natural number d such that

$$\Delta_{s;y_1,y_2,\dots,y_{d+1}} * f = 0$$

holds for each $y_1, y_2, \dots, y_{d+1}$ in $\mathbb{K}^n$. It can be shown that if $f \neq 0$, then s is uniquely determined by f. Then we call f an *s-monomial* and the smallest d which satisfies the above equation is called its *degree*. The 1-monomials are called K-polynomials. The K-polynomials of degree 0 are the constants, and the K-polynomials of degree 1 have the form $A + c$, where c is a complex number, and A satisfies

$$\int_K A(x + ky)\, d\omega(k) = A(x) + A(y)$$

for each x, y in $\mathbb{K}^n$. For obvious reasons such functions are called *K-additive functions*.

A special and important subclass of spherical monomials is formed by the spherical moment functions. Let N be a positive integer. The sequence $f_0, f_1, \dots, f_N$ of continuous K-invariant functions is called a *generalized K-spherical moment sequence*, or shortly *K-moment sequence*, if

$$\int_K f_j(x + ky)\, d\omega(k) = \sum_{l=0}^{j} \binom{j}{l} f_l(x) f_{j-l}(y) \tag{5.1}$$

holds for $j = 0, 1, \dots, N$ and for all x, y in $\mathbb{R}^n$ and $f_0 \neq 0$. We call the functions in a generalized K-spherical moment sequence *generalized K-spherical moment functions*, or shortly *K-moment functions*, and f_j is called of *j-th order*. Clearly, f_0 is a K-spherical function, hence the elements of the K-moment function sequence starting with $f_0 = s$ are called *s-moment functions*. By definition, K-moment functions generate finite dimensional submodules. Now we show that K-moment functions are K-monomials (see also [3, Theorem 3]).

Theorem 5.1. *If f is a K-moment function, then f is a K-monomial. In particular, if the order of f is at most j, then the degree of f is at most j.*

Proof. We prove the statement by induction, and it is clearly true for $j = 0$. Let $f_0 = s$ and suppose we have proved the statement for j and now we prove it for $j + 1$. Let $y_1, y_2, \ldots, y_{j+2}$ be arbitrary in $\mathbb{R}^n$, then we have for each x in $\mathbb{R}^n$:

$$\Delta_{s;y_1,y_2,\ldots,y_{j+2}} * f_{j+1}(x) = \Delta_{s;y_1,y_2,\ldots,y_{j+1}} * [\Delta_{s;y_{j+2}} * f](x) =$$

$$\Delta_{s;y_1,y_2,\ldots,y_{j+1}} * \Big[\int_K f_{j+1}(x + ky)\, d\omega(k) - s(y_{j+2})f(x)\Big] =$$

$$\Delta_{s;y_1,y_2,\ldots,y_{j+1}} * \Big[\sum_{l=0}^{j} \binom{j+1}{l} f_l(x) f_{j+1-l}(y_{j+2})\Big] =$$

$$\sum_{l=0}^{j} \binom{j+1}{l} \Delta_{s;y_1,y_2,\ldots,y_{j+1}} * f_l(x) \cdot f_{j+1-l}(y_{j+2}) = 0,$$

by assumption. Our statement is proved. ■

6. Invariant differential operators

In the previous example of $O(n)$-spherical functions we observed that those are exactly the common normalized eigenfunctions of the Laplacian on $\mathbb{R}^n$. This property is related to the concept of invariant differential operators. Let $\mathcal{E}(\mathbb{R}^n)$ denote the Schwartz's space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ with the usual topology: the net (f_j) converges to f in $\mathcal{E}(\mathbb{R}^n)$ if the net $\big(P(\partial)f_i\big)$ is uniformly converges to $P(\partial)f$ on every compact set for every differential operator $P(\partial)$, where P is a complex polynomial of the form

$$P(\xi) = \sum_\alpha c_\alpha \xi^\alpha, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n).$$

We call a continuous linear mapping $D : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ a *differential operator*, if it is support-decreasing: for each f in $\mathcal{E}(\mathbb{R}^n)$ we have

$$\operatorname{supp} Df \subseteq \operatorname{supp} f.$$

The differential operator D is said to be *K-invariant*, if

$$D(f \circ k) = Df \circ k$$

holds for each f in $\mathcal{E}(\mathbb{R}^n)$ and k in K (see [4]). All K-invariant differential operators form commutative unital algebra, as it is easy to see.

Let $\mathbb{C}[z_1, z_2, \dots, z_k]$ denote the ring of complex polynomials in k variables. Then we have the following result (see [4]):

Theorem 6.1. *Let $K = O(n)$ or $SO(n)$. The K-spherical functions are exactly the those normalized $\mathcal{C}^\infty$-functions f on $\mathbb{R}^n$ for which there exists a ring-homomorphism $\lambda : \mathbb{C}[z] \to \mathbb{C}$ such that for each P in $\mathbb{C}[z]$ we have*

$$P(\Delta)f = \lambda(P)f.$$

Proof. By the previous results we have that the C^∞-functions f on $\mathbb{R}^n$ is a K-spherical function if and only if there exists a function $\lambda : \mathbb{C}[z] \to \mathbb{C}$ such that for each P in $\mathbb{C}[z]$ we have

$$P(\Delta)f = \lambda(P)f.$$

It is enough to show that λ is a ring-homomorphism. We have for P, Q in $\mathbb{C}[z]$ and for an arbitrary $O(n)$-spherical function f:

$$\lambda(P \cdot Q)f = (P \cdot Q)(\Delta)f = [P(\Delta) \circ Q(\Delta)]f = P(\Delta)\big(Q(\Delta)f\big) =$$

$$P(\Delta)\big(\lambda(Q)f\big) = \lambda(Q)P(\Delta)f = \lambda(Q)\lambda(P)f,$$

which implies the statement after evaluating both sides at $x = 0$. ■

It is known that on the polynomial ring $\mathbb{C}[z]$ every complex ring homomorphism is the evaluation functional at some point. In fact, if f is the spherical function s_{λ_0} corresponding to the eigenvalue λ_0, then we have $\lambda(P) = P(\lambda_0)$ for each polynomial P.

A fundamental result about spherical functions is the following (see [4]):

Theorem 6.2. *The K-spherical functions are exactly the common normalized $\mathcal{C}^\infty$-eigenfunctions of all K-invariant differential operators.*

In other words, K-spherical functions are the solutions of a system of partial differential equations which is determined by the algebra of K-invariant differential operators. Now we prove a similar result for spherical monomials. Let $\mathcal{D}_K$ denote the algebra of K-invariant differential operators and I denotes the identity operator..

Theorem 6.3. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a K-monomial of degree at most N. The f is $\mathcal{C}^\infty$ and there is a complex ring homomorphism $\lambda : \mathcal{D}_K \to \mathbb{C}$ such that*

$$(D - \lambda_D I)^{n+1} f = 0$$

holds for each K-invariant differential operator D.

Proof. In [3], we proved that every K-monomial is infinitely differentiable. Suppose that f is an s-monomial of degree at most N with the K-spherical function $s : \mathbb{R}^n \to \mathbb{C}$. Then there is a complex ring homomorphism $\lambda : \mathcal{D}_K \to \mathbb{C}$ such that

$$(D - \lambda_D I)s = 0.$$

First we show that

$$[(D - \lambda_D I)_y(\Delta_{s;y} * f)](x) = [(D - \lambda_D I)f](x) \tag{6.1}$$

holds for $y = 0$ and for each x in $\mathbb{R}^n$, where $(D - \lambda_D I)_y$ denote the differential operator acting on functions depending on y. Indeed, this is a K-invariant differential operator, further we can differentiate under the integral to get

$$(D-\lambda_D I)_y\Big[\int_K f(x+ky)\,d\omega(k)-s(y)f(x)\Big] = \int_K (D-\lambda_D I)_y[f(x+ky)]\,d\omega(k) =$$

$$\int_K [(D - \lambda_D I)f](x + ky)]\,d\omega(k),$$

and substituting $y = 0$ we get the statement.

By assumption, we have

$$\Delta_{s;y_1} * \Delta_{s;y_2} * \cdots * \Delta_{s;y_{N+1}} * f(x) = 0$$

for each $x, y_1, \ldots, y_{N+1}$ in $\mathbb{R}^n$. If we apply (6.1) with $y = y_j$ $j = 1, 2, \ldots, N + 1$ repeatedly, then we get the statement. ∎

In some cases we can prove the converse statement, too. First we prove the following lemma.

Lemma 6.1. *Suppose that the algebra $\mathcal{D}_K$ is generated by a single operator D, i.e. it is isomorphic to the polynomial ring $\mathbb{C}[z]$. Let $j \le N$ be natural numbers, λ a complex number and denote s_λ the unique K-spherical function corresponding to the eigenvalue λ of D. Then the functions $s_\lambda^{(l)}$ for $l = 0, 1, 2, \ldots, j$ form a K-moment sequence.*

Proof. We know that s_λ satisfies

$$(D - \lambda I)s_\lambda = 0, \text{ or } Ds_\lambda = \lambda s_\lambda.$$

It follows that $\lambda \mapsto s_\lambda(x)$ is a $\mathcal{C}^\infty$ function for each x in $\mathbb{R}^n$. For each complex λ and for every x, y in $\mathbb{R}^n$ we have

$$\int_K s_\lambda(x + ky)\,d\omega(k) = s_\lambda(x)s_\lambda(y).$$

Differentiating both sides j times with respect λ for $j = 1, 2, \ldots, N$ we get

$$\int_K s_\lambda^{(j)}(x + ky)\, d\omega(k) = \sum_{l=0}^{j} \binom{j}{l} s_\lambda^{(l)}(x) s_\lambda^{(j-l)}(y),$$

which proves that the functions $s_\lambda^{(l)}$ for $l = 0, 1, 2, \ldots, j$ form a K-moment sequence. ■

Theorem 6.4. *Suppose that the algebra $\mathcal{D}_K$ is generated by a single operator D, i.e. it is isomorphic to the polynomial ring $\mathbb{C}[z]$. Let N be a natural number and λ a complex number. Then every $\mathcal{C}^\infty$ K-invariant solution $f : \mathbb{R}^n \to \mathbb{C}$ of the partial differential equation*

$$(D - \lambda I)^{N+1} f = 0$$

is a K-monomial of degree at most N.

Proof. By differentiating the given equation l times with respect to λ we have

$$(D - \lambda I) \frac{d^l s_\lambda}{d\lambda^l} = l \frac{d^{l-1} s_\lambda}{d\lambda^{l-1}}.$$

For the sake of simplicity, we denote $\frac{d^l s_\lambda}{d\lambda^l}$ by $s_\lambda^{(l)}$. Hence we have

$$(D - \lambda I) s_\lambda^{(l)} = l s_\lambda^{(l-1)}$$

for $l = 1, 2, \ldots$.

Now suppose that $f : \mathbb{R}^n \to \mathbb{C}$ is a K-invariant $\mathcal{C}^\infty$ function satisfying

$$(D - \lambda I)^{N+1} f = 0.$$

We prove by induction on N that f is a linear combination of the functions $s_\lambda^{(l)}$ with $l = 0, 1, \ldots, N$. This is clearly true for $N = 0$, by Theorem 6.2. Suppose that the statement holds for $l = 0, 1, \ldots, N - 1$, and we prove it for $N \geq 1$. By assumption, we have

$$(D - \lambda I) f = \sum_{j=0}^{N-1} c_j s_\lambda^{(j)}.$$

We let

$$\varphi = \sum_{j=0}^{N-1} c_j \frac{1}{j+1} s_\lambda^{(j)},$$

then

$$(D - \lambda I)\varphi = \sum_{j=0}^{N-1} c_j \frac{1}{j+1} (D - \lambda I) s_\lambda^{(j+1)}$$

$$= \sum_{j=0}^{N-1} c_j (D - \lambda I) \frac{1}{j+1} s_\lambda^{(j+1)} = \sum_{j=0}^{N-1} c_j s_\lambda^{(j)},$$

hence

$$(D - \lambda I)(f - \varphi) = 0.$$

It follows that $f - \varphi = ds_\lambda$, where $d = f(0) - \varphi(0)$. Finally, we have

$$f = ds_\lambda + \sum_{j=0}^{N-1} c_j \frac{1}{j+1} s_\lambda^{(j)},$$

as it was to be proved. ■

We can summarize our results about spherical monomials and spherical moment functions as follows.

Corollary 6.1. *Suppose that the algebra $\mathcal{D}_K$ is generated by a single operator D, i.e. it is isomorphic to the polynomial ring $\mathbb{C}[z]$. For each natural number N and complex number λ let s_λ denote the unique K-spherical function corresponding the eigenvalue λ of D. Then the set of all s_λ-monomials of degree at most N and the the set of all s_λ-moment functions of order at most N coincides with the linear span of the functions $\frac{d^l s_\lambda}{d\lambda^l}$ with $l = 0, 1, \ldots, l$.*

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