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ON THE UNIFORM DISTRIBUTION MODULO 1 OF SEQUENCES $f(m/n)$

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Abstract. The series of sequences of real numbers $\langle f(r)\rangle_{r\in\mathbb{Q}_x}$ are considered, where $\mathbb{Q}_x, x \geqslant 1$, are some finite sets of rational numbers. Using the Weyl's criterion the conditions for the uniform distribution modulo 1 are derived.

1. Introduction

The theory of uniform distribution modulo 1 is rich in results of different kind: examples of specific sequences, general conditions implying this phenomenon, quantitative evaluations of discrepancies, metric theorems, etc.

The Weyl's criterion is the main tool in the theory; originally it was proved in [4], [5]; nowadays the criterion can be found in many works dedicated to the subject; see, for example, [2].

We are interested in series of sequences of real numbers, generating the uniform distribution modulo 1. The sequences of these series will be interpreted as sequences of values of functions, defined on some finite sets of rational numbers.

The sets of integers, positive integers, positive rational numbers, real numbers will be denoted by $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$. The fractional part of the real number t is denoted as usual by $\{t\}$; $\#A$ means the number of elements in the finite set A. We shall use also the conventional notation $\ll$.

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Let $f : \mathbb{Q} \to \mathbb{R}$ be some function and $\mathbb{Q}_x, x = 1, 2 \ldots,$ a sequence of finite subsets of rational numbers $\mathbb{Q}_x \subset \mathbb{Q}$, such that $\#\mathbb{Q}_x \to \infty$ as $x \to \infty$. We denote by $\langle f(r) \rangle_{r \in \mathbb{Q}_x}$ the finite sequence of values of f, obtained as r runs over $\mathbb{Q}_x$ in ascending order and write $\langle f, \mathbb{Q}_x \rangle$ for the whole series of sequences corresponding to $x = 1, 2, \ldots$.

Definition 1.1. We say, that the series of sequences $\langle f, \mathbb{Q}_x \rangle$ generate the uniform distribution modulo 1, if for an arbitrary $u \in (0; 1)$

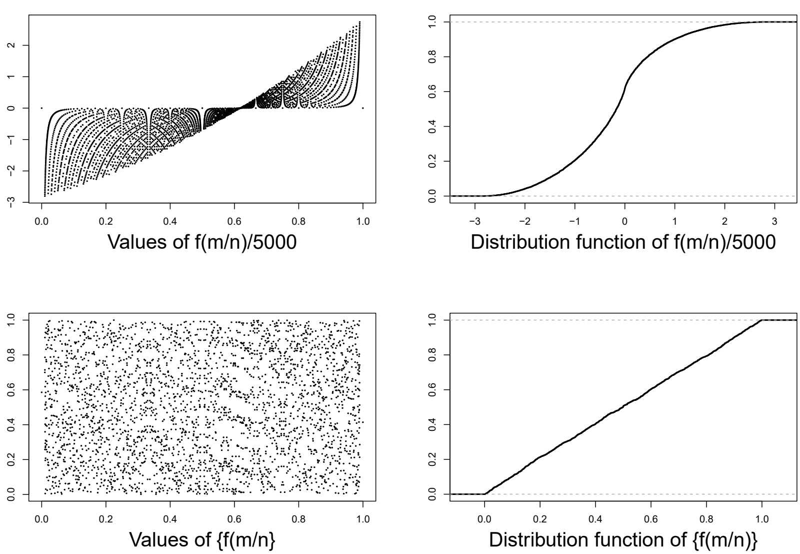
$$\#\{r \in \mathbb{Q}_x : \{f(r)\} < u\} \sim u \cdot \#\mathbb{Q}_x, \quad x \to \infty.$$

The Weyl's criterion is usually applied for an infinite sequence of real numbers, but its proof is valid for the series of finite, but growing sequences, too.

Proposition 1.1. (Weyl's criterion.) *The series of sequences $\langle f, \mathbb{Q}_x \rangle$ generate the uniform distribution modulo* 1 *if and only if for an arbitrary integer $h \neq 0$*

$$\#\mathbb{Q}_x^{-1} \sum_{r \in \mathbb{Q}_x} \exp\{2\pi i h f(r)\} \to 0, \quad x \to \infty.$$

We specify the sets $\mathbb{Q}_x$ for the use in the sequel. Let $I_x = (\alpha_x; \beta_x)$, where $0 < \alpha_x < \beta_x$, be some system of intervals. We shall usually omit the index x and write $I = (\alpha_x; \beta_x)$ having the dependence on x in mind.



The plots of normalized values of $f(m/n) = \sqrt{2}(m^2 - n^2 + mn)$ and corresponding distribution functions

Let us define

$$\mathbb{Q}_x^I = \left\{\frac{m}{n} : n \leqslant x, (m,n) = 1\right\} \cap I.$$

It is known, that if $(\beta_x - \alpha_x) \cdot x \to \infty$, then

$$\#\mathbb{Q}_x^I \sim \frac{3}{\pi^2}(\beta_x - \alpha_x)x^2, \quad x \to \infty; \tag{1.1}$$

see, for example [3]. With this restriction we shall consider the conditions which imply that $\langle f, \mathbb{Q}_x^I \rangle$ generate the uniform distribution modulo 1.

Just for illustration we present the scatterplots of values of $f(r)$ and $\{f(r)\}$ for $f(m/n) = \sqrt{2}(m^2 - n^2 + mn)$ as r runs over $\mathbb{Q}_x^I, x = 100, I = (0;1)$, and graphs of corresponding distribution functions.

2. A metrical theorem

Consider the sequences of real numbers $\langle f(m|\sigma)\rangle_{m\geqslant 1}$, depending on the parameter $\sigma \in \mathbb{R}$. The fact, that for every irrational number σ the values of $f(m|\sigma) = m\sigma$ are uniformly distributed modulo 1, was established independently by several authors (see, exhaustive references in [2]). The Weyl's proof in [5] is based on the evaluation of exponential sums.

For $f(m|\sigma) = g(m)\sigma$ to be uniformly distributed modulo 1 for almost all σ it is sufficient that for a real-valued function $g(m)$ $\liminf_{m\neq n} |g(m) - g(n)| > 0$, (follows from the general theorem of Koksma in [1]; see, also [2]). This is true, for example, if $g : \mathbb{N} \to \mathbb{Z}$ is injective.

We shall prove a metrical result for the series $\langle f(\cdot|\sigma), \mathbb{Q}_x^I \rangle$.

Theorem 2.1. *Let $f(m/n|\sigma) = P(m/n)\sigma + Q(m/n)$, where $P(m/n) \in \mathbb{Z}$, and $Q(m/n) \in \mathbb{R}$. Let $I = (\alpha_x; \beta_x), x \geqslant 1$, be some system of intervals and let $C(x)$ be a function such that*

$$\max_a \#\left\{\frac{m}{n} \in \mathbb{Q}_x^I : P\left(\frac{m}{n}\right) = a\right\} \leqslant C(x).$$

If the series

$$\sum_{x=1}^{\infty} \frac{C(x)}{(\beta_x - \alpha_x)x^2} \tag{2.1}$$

converges, then for almost all σ the series of sequences $\langle f(\cdot|\sigma), \mathbb{Q}_x^I \rangle$ generate the uniform distribution modulo 1.

Given f the condition (2.1) determines the lengths of intervals I. Obviously, this condition is satisfied if $C(x)(\log x)^{1+\delta}/x \ll \beta_x - \alpha_x$ with an arbitrary $\delta > 0$.

Let us consider some examples. If a, b are coprime positive integers and $P(m/n) = a^m b^n$, then $C(x) = 1$ and we can take $\beta_x - \alpha_x = \rho(x)/x$, supposed that the series $\sum_{x=1}^{\infty} 1/(\rho(x)x)$ converges.

If $P(m/n) = p_1(m)p_2(n)$, where $p_1, p_2 \in \mathbb{Z}[u]$ are polynomials of positive degrees k_1, k_2, then

$$\#\Big\{\frac{m}{n} \in \mathbb{Q}_x^I : P\big(\frac{m}{n}\big) = a\Big\} \leqslant \sum_{d|a} \#\{m : p_1(m) = d\} \cdot \#\{n : p_2(n) = a/d\}.$$

Because $\#\{m : p_1(m) = d\}, \#\{n : p_2(n) = a/d\}$ do not exceed the degrees of polynomials, then

$$\#\Big\{\frac{m}{n} \in \mathbb{Q}_x^I : P\big(\frac{m}{n}\big) = a\Big\} \ll \tau(a),$$

where $\tau(a)$ is the number of distinct divisors of a. Because of $|a| \ll x^{k_2}(\beta_x x)^{k_1}$, and $\tau(a) \ll a^{\epsilon}$ for an arbitrary $\epsilon > 0$, we get with $\delta > 0$

$$\#\Big\{\frac{m}{n} \in \mathbb{Q}_x^I : P\big(\frac{m}{n}\big) = a\Big\} \ll \big(x^{k_1+k_2}\beta_x^{k_1}\big)^{\delta/(k_1+k_2)} \ll (x\beta_x)^{\delta}.$$

If $\beta_x \ll x^k$ for some $k > 0$, we can take $C(x) = x^{\delta}$ with an arbitrary $\delta > 0$.

If $P(m/n) = p(m, n)$, where $p \in \mathbb{Z}[u, v]$ is a polynomial of positive degrees both in u and v, then for fixed m (or n) the number of solutions to $p(m, n) = a$ does not exeed the degree of p and this way the bound $C(x) \ll \max(x, \beta_x x)$ follows.

Proof of the Theorem 2.1. It is sufficient to show that

$$S(x|h, \sigma) = \frac{1}{\#\mathbb{Q}_x^I} \sum_{m/n \in \mathbb{Q}_x^I} \exp\{2\pi i h(P(m/n)\sigma + Q(m/n))\} \to 0, \quad x \to \infty,$$

for all $h \in \mathbb{Z}, h \neq 0$ and almost all $\sigma \in (0; 1)$.

Consider

$$|S(x|h, \sigma)|^2 = \frac{1}{(\#\mathbb{Q}_x^I)^2} \sum_{\substack{m_1/n_1 \in \mathbb{Q}_x^I \\ m_2/n_2 \in \mathbb{Q}_x^I}} \exp\Big\{2\pi i h \Delta_P\big(\frac{m_1}{n_1}, \frac{m_2}{n_2}\big)\sigma + 2\pi i h \Delta_Q\big(\frac{m_1}{n_1}, \frac{m_2}{n_2}\big)\Big\},$$

where $\Delta_P\big(\frac{m_1}{n1}, \frac{m_2}{n_2}\big) = P\big(\frac{m_1}{n_1}\big) - P\big(\frac{m_2}{n_2}\big), \Delta_Q\big(\frac{m_1}{n_1}, \frac{m_2}{n_2}\big) = Q\big(\frac{m_1}{n_1}\big) - Q\big(\frac{m_2}{n_2}\big)$.

Integrating over σ and taking notice of $\Delta_P\left(\frac{m_1}{n_1}, \frac{m_2}{n_2}\right) \in \mathbb{Z}$ we have

$$\begin{aligned} I(x,h) &= \int_0^1 |S(x|h,\sigma)|^2 \mathrm{d}\sigma = \\ &= \frac{1}{(\#\mathbb{Q}_x^I)^2} \sum_{m_1/n_1 \in \mathbb{Q}_x^I} \sum_{\substack{m_2/n_2 \in \mathbb{Q}_x^I \\ \Delta_P(m_1/n_1, m_2/n_2)=0}} \exp\left\{2\pi i h \Delta_Q\left(\frac{m_1}{n_1}, \frac{m_2}{n_2}\right)\right\}. \end{aligned}$$

Now

$$I(x,h) \leqslant \frac{1}{(\#\mathbb{Q}_x^I)^2} \sum_{m_1/n_1 \in \mathbb{Q}_x^I} \#\left\{\frac{m_2}{n_2} \in \mathbb{Q}_x^I : \Delta_P\left(\frac{m_1}{n_1}, \frac{m_2}{n_2}\right) = 0\right\} < \frac{C(x)}{\#\mathbb{Q}_x^I}.$$

Because $C(x) \geqslant 1$, (2.1) implies $x(\beta_x - \alpha_x) \to \infty$, and the asymptotics (1.1) is valid. We have then

$$I(x,h) \ll \frac{C(x)}{(\beta_x - \alpha_x)x^2}.$$

Fix now an arbitrary $\epsilon > 0$ and consider the set

$$A_{\epsilon,x} = \{\sigma \in (0;1) : |S(x|h,\sigma)| \geqslant \epsilon\}.$$

Let μ stand for the Lebesgue measure. Then

$$\epsilon^2 \mu(A_{\epsilon,x}) \leqslant I(x,h) \ll \frac{C(x)}{(\beta_x - \alpha_x)x^2},$$

and due to (2.1)

$$\sum_{x=1}^{\infty} \mu(A_{\epsilon,x}) \ll \epsilon^{-2} \sum_{x=1}^{\infty} \frac{C(x)}{(\beta_x - \alpha_x)x^2} < +\infty.$$

Hence, $\mu(\limsup_x A_{\epsilon,x}) = 0$, i.e., for almost all $\sigma \in (0;1)$ $|S(x|h,\sigma)| < \epsilon$ as $x \geqslant x_0(\sigma)$. Because $\epsilon > 0$ is arbitrary, we conclude that for almost all $\sigma \in (0;1)$ and $h \in \mathbb{Z}, h \neq 0$, $S(x|h,\sigma) \to 0$ as $x \to \infty$.

The proof of the Theorem 2.1 is complete. ■

3. The uniform distribution of $P(n)m\sigma + Q(n)$ modulo 1

In this section we prove an analogue of the theorem on the uniformity of distribution of $m\sigma$ modulo 1 as σ is an irrational number. For a real number ξ we denote the distance from ξ to the nearest integer by $\|\xi\|$.

Theorem 3.1. *Let* $f(m/n|\sigma) = P(n)m\sigma + Q(n)$, *where* $P(n) \in \mathbb{Z}, Q(n) \in \mathbb{R}$. *Let* σ *be an irrational number and* $I = (\alpha_x; \beta_x)$ *be some system of intervals. Suppose* $\#\{n \leqslant x : P(n) = 0\}/((\beta_x - \alpha_x)x) \to 0, x \to \infty$, *and let* G, ψ *be two non-decreasing functions such that*

$$\max\{|P(n)| : n \leqslant q\} \ll G(q), \quad 1 \ll \|q\sigma\| \cdot q\psi(q), \quad q \to \infty.$$

If the lengths of intervals $I = (\alpha_x; \beta_x)$ *satisfy the condition*

$$\min\{G(x), (G(x)x)^\epsilon, x\} \cdot \log^2 x \cdot \frac{G(x)\psi(G(x)\lambda(x)x)}{(\beta_x - \alpha_x)x} \to 0, \quad x \to \infty, \tag{3.1}$$

where $\epsilon \in (0;1)$, *and* $\lambda(x)$ *is an arbitrary non-decreasing function growing unboundedly as* $x \to \infty$, *then the series of sequences* $\langle f(\cdot|\sigma), \mathbb{Q}_x^I \rangle$ *generate the uniform distribution modulo* 1.

Let $\sigma \in (0;1)$ be an irrational number, consider the continued fraction expansion $\sigma = [0; a_1, a_2, \ldots]$ with $a_n \in \mathbb{Z}$ positive integers. Let $\langle q_n \rangle_{n \geqslant 0}$ be the increasing sequence of denominators of the convergents of σ; we have then $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geqslant 2$. It is known, that the convergents are the best approximations to σ, and for $q_{n-1} \leqslant q < q_n$ we have $\|q\sigma\| > \|q_{n-1}\sigma\| = q_n^{-1}$. Then

$$q\|q\sigma\| > \frac{q}{q_n} \geqslant \frac{q_{n-1}}{q_n} = \frac{1}{a_n + q_{n-2}/q_{n-1}} > \frac{1}{a_n + 2}.$$

Hence, if in the range $q_{n-1} \leqslant q < q_n$ we define $\psi(q) = \max\{a_m + 2 : 1 \leqslant m \leqslant n\}$ then $q\psi(q)\|q\sigma\| \gg 1$.

For badly approximable irrationals, i.e., for having the elements a_n in the continued fraction expansion bounded, we can take $\psi(q) = 1$. Because the quadratic irrationals have the periodic expansions, they are badly approximable. For the algebraic irrationals of larger degree it follows from the Roth's theorem, that $q^{1+\delta}\|q\sigma\| \gg 1$ with an arbitrary $\delta > 0$; hence we can take $\psi(q) = q^\delta$.

Corollary 3.1. *Let* $f(m/n|\sigma) = P(n)m\sigma + Q(n)$, *where* $P(n) \in \mathbb{Z}, Q(n) \in \mathbb{R}$, *and let* $\#\{n : P(n) = 0\} \ll 1$. *For an arbitrary irrational* σ *there exists a function* $\rho(x)$, *such that if*

$$\frac{\beta_x - \alpha_x}{\rho(x)} \to \infty,$$

then $\langle f(\cdot|\sigma), \mathbb{Q}_x^I \rangle$ *generate the uniform distribution modulo* 1.

If σ is badly approximable and $P(n) = \lfloor \log^k n \rfloor$, we can take $\rho(x) = (\log x)^{2k+2}/x$; if $P(n)$ is a polynomial of degree k, we can take $\rho(x) = x^{k+\delta}$ with an arbitrary $\delta \in (0;1)$.

Proof of the Theorem 3.1. We have to show, that for all integers $h \neq 0$

$$S(x|h,\sigma) = \frac{1}{\#\mathbb{Q}_x^I} \sum_{m/n \in \mathbb{Q}_x^I} \exp\{2\pi i h(P(n)m\sigma + Q(n))\} \to 0, \quad x \to \infty,$$

supposed that the conditions for $I = (\alpha_x; \beta_x)$ are satisfied. We have

$$S(x|h,\sigma) = \frac{1}{\#\mathbb{Q}_x^I} \sum_{n \leqslant x} \sum_{\substack{\alpha_x n < m < \beta_x n \\ (m,n)=1}} \exp\{2\pi i h(P(n)m\sigma + Q(n))\}.$$

The total contribution to the sum of summands corresponding to the values $n, P(n) = 0$, is $o((\beta_x - \alpha_x)x^2)$ as $x \to \infty$, and it can be neglected because of $\#\mathbb{Q}_x^I \sim (3/\pi^2)(\beta_x - \alpha_x)x^2, x \to \infty$. Hence, we suppose now, that $P(n) \neq 0$ for all n. Rewrite the sum as

$$S(x|h,\sigma) =$$
$$= \frac{1}{\#\mathbb{Q}_x^I} \sum_{d \leqslant x} \mu(d) \sum_{nd \leqslant x} \exp\{2\pi i h Q(nd)\} \sum_{\alpha_x n < m < \beta_x n} \exp\{2\pi i h P(nd) m d\sigma\}.$$

The inner sum consists of the terms of geometrical progression. If we define $\underline{m} = \lceil \alpha_x n \rceil, \overline{m} = \lfloor \beta_x n \rfloor$, then

$$\sum_{\alpha_x n < m < \beta_x n} \exp\{2\pi i h P(nd) m d\sigma\} =$$
$$= \frac{\exp\{2\pi i h P(nd)\underline{m} d\sigma\} - \exp\{2\pi i h P(nd)\overline{m} d\sigma\}}{1 - \exp\{2\pi i h P(nd) d\sigma\}}.$$

Using the elementary inequality

$$|\exp\{2\pi i u\} - 1| = |\sin(\pi u)| = \sin(\pi\|u\|) \geqslant 2\|u\|,$$

we get

$$S(x|h,\sigma) \ll \frac{1}{\#\mathbb{Q}_x^I} \sum_{d \leqslant x} \mu^2(d) \sum_{nd \leqslant x} \frac{1}{\|hP(nd)d\sigma\|}. \tag{3.2}$$

Let $\mathbb{G}_x$ be the set of different values of $P(n), n \leqslant x$. We have, obviously, $\#\mathbb{G}_x \ll \min(G(x), x)$. Then

$$\frac{1}{\|hP(nd)d\sigma\|} \leqslant \sum_{g \in \mathbb{G}_x} \frac{1}{\|ghd\sigma\|}.$$

Now we estimate (3.2) as

(3.3)

$$S(x|h,\sigma) \ll \frac{1}{\#\mathbb{Q}_x^I} \sum_{d \leqslant x} \mu^2(d) \sum_{nd \leqslant x} \sum_{g \in \mathbb{G}_x} \frac{1}{\|ghd\sigma\|} \ll \frac{x}{\#\mathbb{Q}_x^I} \sum_{g \in \mathbb{G}_x} \sum_{d \leqslant x} \frac{\mu^2(d)}{d\|ghd\sigma\|}.$$

Consider the sum

$$s(u) = s(u|g,h,\sigma) = \sum_{d\leqslant u} \frac{\mu^2(d)}{\|ghd\sigma\|}, \quad u \geqslant 1. \tag{3.4}$$

Note, that $\|ghd\sigma\| \gg (|g|hd\psi(|g|hd))^{-1} \gg (Ghu\psi(Ghu))^{-1}$, here and in what follows we set $G = G(x)$. We shall estimate $s(u)$ using the simple identity

$$|\|u\| - \|v\|| = \min\{\|u-v\|, \|u+v\|\}, \tag{3.5}$$

which can easily verified for $u, v \in (0; 1/2)$ and proved for arbitrary u, v using the obvious relations $\|-u\| = \|u\| = \|1-u\|$.

Using this identity for the different integers $d_1, d_2 \leqslant u, u \leqslant x$, we get

$$|\|ghd_1\sigma\| - \|ghd_2\sigma\|| = \min\{\|gh(d_1-d_2)\sigma\|, \|gh(d_1+d_2)\sigma\|\} > (2Ghu\psi(2Ghu))^{-1}.$$

It follows from this inequality, that in the interval

$$\Big[\frac{i}{2Ghu\psi(2Ghu)}; \frac{i+1}{2Ghu\psi(2Ghu)}\Big], \quad i \geqslant 1,$$

can lay at most one value $\|ghd\sigma\|, d \leqslant u$. Therefore

$$s(u) \ll \sum_{1\leqslant i\leqslant u} \frac{2Ghu\psi(2Ghu)}{i} \ll Ghu\psi(2Ghu)\log u.$$

Integrating by parts we have

$$\sum_{d\leqslant x} \frac{\mu^2(d)}{d\|ghd\sigma\|} \ll \int_1^x \frac{\mathrm{d}s(u)}{u} \ll Gh\psi(2Ghx)\log^2 x.$$

Using this bound in (3.3) we arrive to

$$S(x|h,\sigma) \ll \frac{\#\mathbb{G}_x}{\#\mathbb{Q}_x^I} G\psi(2Ghx)x\log^2 x \ll \frac{1}{\#\mathbb{Q}_x^I} G^2\psi(2Ghx)x\log^2 x.$$

Due to $\#\mathbb{Q}_x^I \sim (3/\pi^2)x^2(\beta_x - \alpha_x), x \to \infty$, we get

$$S(x|h,\sigma) \ll G\log^2 x \cdot \frac{G\psi(2Ghx)}{(\beta_x-\alpha_x)x} \ll G\log^2 x \cdot \frac{G\psi(Gx\lambda(x))}{(\beta_x-\alpha_x)x}$$

as $x \geqslant x_0(h)$. For to complete the proof we have to show that for an arbitrary ϵ the bound

$$S(x|h,\sigma) \ll \min(x, (Gx)^\epsilon) \cdot \log^2 x \cdot \frac{G\psi(Gx\lambda(x))}{(\beta_x-\alpha_x)x} \tag{3.6}$$

is valid, too. Then the condition (3.1) will lead to $S(x|h,\sigma)\to 0$ as $x\to\infty$. For this purpose we derive a different estimate for the sum in (3.2). For $d\leqslant x$ denote by n_d the natural number satisfying the condition:

$$\|hP(n_d d)d\sigma\| = \min\{\|hP(nd)d\sigma\| : nd\leqslant x\}.$$

Then

$$S(x|h,\sigma)\ll \frac{x}{\#\mathbb{Q}_x^I}\sum_{d\leqslant x}\frac{\mu^2(d)}{d\|hP(n_d d)d\sigma\|}.$$

However, $P(n_d d)d,\ d\leqslant x$, are not necessarily different. Let V be the set of all different values of $|P(n_d d)d|,\ d\leqslant x$. Then

$$S(x|h,\sigma)\ll \frac{x}{\#\mathbb{Q}_x^I}\sum_{v\in V}\frac{1}{\|hv\sigma\|}\sum_{\substack{d\leqslant x\\ |P(n_d d)d|=v}}\frac{\mu^2(d)}{d}.$$

The number of occurences N_v of the value $v=|P(n_d d)d|$ as $d\leqslant x$ is bounded by $N_v\ll\min(x,\tau(v))$, here τ is the divisor function. Because of

$$\tau(v)=\tau(P(n_d d)d)\leqslant\tau(P(n_d d))\tau(d)\ll (Gx)^\delta,$$

we have a uniform bound

$$N_v\ll\min(x,(Gx)^\delta)$$

with $\delta>0$ arbitrary. For each v find a smallest integer $d\leqslant x$ with the property $|P(n_d d)d|=v$. Let D be the set of such d. Then

$$S(x|h,\sigma)\ll\frac{x\min(x,(Gx)^\delta)}{\#\mathbb{Q}_x^I}\sum_{\substack{d\leqslant x\\ d\in D}}\frac{\mu^2(d)}{d\|hP(n_d d)d\sigma\|}. \tag{3.7}$$

Because $|P(n_d d)d|$ in the last sum are different, we can use the identity (3.5) to derive the bound

$$|\|hP(n_{d_1}d_1)d_1\sigma\| - \|hP(n_{d_2}d_2)d_2\sigma\||\geq (2hG\psi(2Gu)u)^{-1}$$

for $d_1,d_2\in D, d_1<d_2<u$. Consider the sum

$$s^*(u)=s^*(u|h,\sigma)=\sum_{\substack{d\leqslant x\\ d\in D}}\frac{\mu^2(d)}{\|hP(n_d d)d\sigma\|}$$

and estimate this function as $s(u)$ defined in (3.4). We shall get

$$s^*(u)\ll\sum_{1\leqslant i\leqslant u}\frac{2Ghu\psi(2Ghu)}{i}\ll Ghu\psi(2Ghu)\log u,$$

and integrating by parts as before

$$\sum_{\substack{d \leqslant x \\ d \in D}} \frac{\mu^2(d)}{d\|hP(n_d d)d\sigma\|} \ll Gh\psi(2Ghx)\log^2 x.$$

Using this bound in (3.7) we derive (3.6). ■

References

[1] **Koksma, J.F.,** Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins, *Compositio Math.*, **2** (1935), 250–258.

[2] **Kuipers, L. and H. Niederreiter,** *Uniform Distribution of Sequences*, John Wiley & Sons, New York, London, Sydney and Toronto, 1974.

[3] **Stakėnas, V.,** A sieve result for Farey fractions, *Lith. Math. J.*, **39** (1999), 86–101.

[4] **Weyl, H.,** Über ein Problem aus dem Gebiete der diophantischen Approximationen, *Nachr. Ges. Wiss. Göttingen, Math.-phys. Klasse*, (1914), 234–244.

[5] **Weyl H.,** Über die Gleichverteilung von Zahlen mod. Eins, *Math. Ann.*, **77** (1916), 313–352.

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