

AN ELEMENTARY METHOD OF SOLVING FUNCTIONAL EQUATIONS

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Abstract. In the present paper we prove in an elementary way that if for a fixed $\lambda \in (0, 1)$ the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = g(\lambda a + (1 - \lambda)b)$$

for all $b > a$ then f is a quadratic polynomial, and $g = f'$. Moreover, if $\lambda \neq \frac{1}{2}$, then f is a linear polynomial and $g = f'$. This result is obtained with no regularity assumptions on f or g and generalizes a theorem from [4].

1. Introduction

Roman Ger (cf. [5]) has drawn our attention to the paper [4] where the authors ask the following questions:

Question 1. Which differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy

$$(1.1) \quad \frac{f(b) - f(a)}{b - a} = f' \left(\frac{a + b}{2} \right)$$

for all $b > a$?

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Question 2. Let us fix a $\lambda \in (0, 1)$. Which differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy

$$(1.2) \quad \frac{f(b) - f(a)}{b - a} = f'(\lambda a + (1 - \lambda)b)$$

for all $b > a$?

The following theorem is proved in [4].

Theorem. *Let us fix a $\lambda \in (0, 1)$, and suppose that a (differentiable) function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the condition*

$$\frac{f(b) - f(a)}{b - a} = f'(\lambda a + (1 - \lambda)b)$$

for all $a < b$. Then f is a quadratic polynomial. If $\lambda \neq \frac{1}{2}$, then f is a linear polynomial.

In the proof the authors use **triple** differentiability of f . It has been known, at least since J. Aczél's result (cf. [1]) that **no regularity** is needed to solve even a more general equation than (1.1). Indeed, Aczél has shown that

$$(1.3) \quad \frac{f(b) - f(a)}{b - a} = g(a + b),$$

has a unique solution and this is of the form

- $f(x) = \alpha x^2 + \beta x + \gamma$
- $g(x) = \alpha x + \beta$

for some constants α, β and γ . Later J. Aczél and M. Kuczma published a paper [2] where they proved, among others, the above result for functions f and g defined in an interval $I \subseteq \mathbb{R}$.

Functional equations stemming from the Mean Value Theorems have been studied by several mathematicians, let me quote J. Aczél [1], J. Aczél and M. Kuczma [2], S. Haruki [6], B. Kocłęga-Kulpa and T. Szostok [7], [8], [9] and [10], B. Kocłęga-Kulpa, T. Szostok and Sz. Wąsowicz [11] and [12], A. Lisak and M. S. [13], I. Pawlikowska [15], T. Riedel and M. S. [16], M. S. [17] and [18], T. Szostok [23], and quite recent papers by Z. M. Balogh, O. O. Ibrogimov and B. S. Mityagin [3], also M. Schwarzenberger [22], and above all by P. K. Sahoo and T. Riedel in the book [20]. It is astonishing that the authors were unaware of these items, especially the last one.

Let us observe that (1.2) may be written in more general form

$$(1.4) \quad \frac{f(b) - f(a)}{b - a} = g(\lambda a + (1 - \lambda)b)$$

for all $a < b$. Here f and g are arbitrary functions mapping \mathbb{R} into itself. Since the left-hand side is symmetric in a, b we infer that

$$(1.5) \quad \frac{f(b) - f(a)}{b - a} = g((1 - \lambda)a + \lambda b)$$

for all $b < a$. Denoting

$$(1.6) \quad m_\lambda(a, b) = \begin{cases} \lambda a + (1 - \lambda)b & \text{if } a < b \\ (1 - \lambda)a + \lambda b & \text{if } a \geq b, \end{cases}$$

we can write (1.4) and (1.5) as

$$\frac{f(b) - f(a)}{b - a} = g(m_\lambda(a, b))$$

for all $a, b, a \neq b$, or, after multiplying both sides by $b - a$ as

$$(1.7) \quad f(b) - f(a) = g(m_\lambda(a, b))(b - a)$$

for all $a, b \in \mathbb{R}$.

Let us substitute $x := m_\lambda(a, b)$ and $y := b - a$. This substitution is an automorphism of \mathbb{R}^2 onto itself. Moreover, we have

$$(1.8) \quad \begin{cases} a = x - (1 - \lambda)y \\ b = x + \lambda y \end{cases}$$

if $y > 0$, and

$$(1.9) \quad \begin{cases} a = x - \lambda y \\ b = x + (1 - \lambda)y \end{cases}$$

if $y \leq 0$. Now, substituting (1.8) or (1.9) into (1.7) we obtain

$$(1.10) \quad g(x)y = \begin{cases} f(x + \lambda y) - f(x - (1 - \lambda)y) & \text{if } y > 0 \\ f(x + (1 - \lambda)y) - f(x - \lambda y) & \text{if } y \leq 0. \end{cases}$$

It is the above equation that we are going to solve.

1.1. Solution of (1.10)

Let us introduce the following operator acting in the space of functions of two variables. For every function $\varphi : (x, y) \mapsto \varphi(x, y)$, and every pair (u, v) we put

$$\Gamma_{(u,v)}\varphi(x, y) := \varphi(x + u, y + v) - \varphi(x, y),$$

for $(x, y) \in \text{Dom}\varphi$ fulfilling $(x + u, y + v) \in \text{Dom}\varphi$ as well. Observe that in particular we have for $\varphi(x, y) = g(x)y$ the following

$$(1.11) \quad \begin{aligned} \Gamma_{(u,v)}g(x)y &= g(x+u)(y+v) - g(x)y = \\ &= [g(x+u) - g(x)]y + g(x+u)v = \\ &= [\Delta_u g(x)]y + g(x+u)v, \end{aligned}$$

where Δ is the Fréchet difference operator. For arbitrary $(x, y) \in \mathbb{R}^2$ let us choose arbitrarily $u_1 \in \mathbb{R}$, and select a $v_1 \in \mathbb{R}$ so that $u_1 + \lambda v_1 = 0$ (resp. $u_1 + (1 - \lambda)v_1 = 0$) depending on whether $y > 0$ or $y \leq 0$. Applying $\Gamma_{(u_1, v_1)}$ to both sides of (1.10) we obtain

$$(1.12) \quad [\Delta_{u_1} g(x)]y + g(x + u_1)v_1 = \begin{cases} -\Gamma_{(u_1, v_1)}f(x - (1 - \lambda)y) & \text{if } y > 0 \\ -\Gamma_{(u_1, v_1)}f(x - \lambda y) & \text{if } y \leq 0. \end{cases}$$

Now, let choose arbitrarily $u_2 \in \mathbb{R}$ and select a $v_2 \in \mathbb{R}$ so that $u_2 - (1 - \lambda)v_2 = 0$ (resp. $u_2 - \lambda v_2 = 0$) depending on whether $y > 0$ or $y \leq 0$. Applying $\Gamma_{(u_2, v_2)}$ to both sides of (1.12) we obtain

$$(1.13) \quad [\Delta_{u_2} \Delta_{u_1} g(x)]y + \psi(x, u_1, u_2) = 0,$$

where $\psi(x, u_1, u_2) = \Delta_{u_1} g(x + u_2)v_2 + \Delta_{u_2} g(x + u_1)v_1$ does not depend on y . The left-hand side of (1.13) is a polynomial in y , so we have

$$\Delta_{u_2} \Delta_{u_1} g(x) = 0,$$

for every $x, u_1, u_2 \in \mathbb{R}$. But this means (cf. eg. [21]) that there exist a constant β and an additive function A_1 such that

$$(1.14) \quad g(x) = \beta + A_1(x)$$

for every $x \in \mathbb{R}$. From (1.7), putting $a = 0$ and using equalities (1.6) we obtain

$$(1.15) \quad f(b) = f(0) + \begin{cases} g((1 - \lambda)b)b & \text{if } 0 < b \\ g(\lambda b)b & \text{if } 0 \geq b. \end{cases}$$

We insert the general forms (1.14) and (1.15) into (1.10) to obtain among others that

$$(1.16) \quad \begin{aligned} A_1(x)y + \beta(y) &= [\beta + A_1((1 - \lambda)(x + \lambda y))](x + \lambda y) - \\ &- [\beta + A_1((1 - \lambda)(x - (1 - \lambda)y))](x - (1 - \lambda)y) \end{aligned}$$

for all $x > 0, y \in \left(0, \frac{x}{1 - \lambda}\right)$. Rearranging the terms in (1.16), and comparing those containing x and y we obtain in particular the equality

$$A_1(\lambda x)y = A_1((1 - \lambda)y)x$$

for all $x > 0$ and $y \in \left(0, \frac{x}{1-\lambda}\right)$. Dividing both sides by xy we get

$$\frac{A_1(\lambda x)}{x} = \frac{A_1((1-\lambda)y)}{y}$$

or

$$(1.17) \quad \lambda \frac{A_1(\lambda x)}{\lambda x} = (1-\lambda) \frac{A_1((1-\lambda)y)}{(1-\lambda)y}$$

for all $x > 0$ and $y \in \left(0, \frac{x}{1-\lambda}\right)$. Fix $x > 0$ and define $\alpha := \frac{A_1(\lambda x)}{\lambda x}$. Putting $z := (1-\lambda)y$ we get from (1.17)

$$(1.18) \quad (1-\lambda)A_1(z) = \lambda\alpha z$$

for all $z \in (0, x)$. Thus A_1 is continuous on an interval $(0, x)$, and being additive it has to be so on the whole \mathbb{R} . From (1.17) we get $\lambda\alpha = (1-\lambda)\alpha$ or

$$(2\lambda - 1)\alpha = 0.$$

Thus either $\lambda = \frac{1}{2}$ or $\alpha = 0$. In the case $\lambda = \frac{1}{2}$ we get from (1.17) that $\frac{A_1(x)}{x} = \frac{A_1(y)}{y}$ for all $y \in (0, 2x)$ and the linearity of A_1 easily follows.

In other words we proved that if (f, g) yields a solution to (1.10) then

$$(1.19) \quad g(x) = \begin{cases} \beta & \text{if } \lambda \neq \frac{1}{2} \\ \beta + \alpha x & \text{if } \lambda = \frac{1}{2}. \end{cases}$$

Now, going back to formula (1.15) we see that (with $\gamma = f(0)$)

$$(1.20) \quad f(x) = \begin{cases} \gamma + \beta x & \text{if } \lambda \neq \frac{1}{2} \\ \gamma + \beta x + \frac{1}{2}\alpha x^2 & \text{if } \lambda = \frac{1}{2}. \end{cases}$$

We summarize the results of the present section in

Theorem 1.1. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be functions and let $\lambda \in (0, 1)$. The pair (f, g) satisfies the equation*

$$\frac{f(b) - f(a)}{b - a} = g(\lambda a + (1-\lambda)b)$$

for all $a < b$ if, and only if, f and g are given by (1.20) and (1.19), respectively. In particular, $g = f'$.

2. Remarks

Remark 2.1. We do not assume any regularity of f or g . However, some regularity is "tacitly" assumed. Namely, if we considered (1.7) with a, b belonging to a linear space X over \mathbb{R} then clearly $g(\lambda a + (1 - \lambda)b)$ would belong to the space of additive mappings defined on X . Usually such a mapping needs not to be linear or continuous (if the space X is endowed with a topology). In our present case ($X = \mathbb{R}$), following the authors of [4], we assume that acting on the difference $b - a$ is actually multiplying the coefficient by the argument. This means that the additive mapping $g(\lambda a + (1 - \lambda)b)$ is highly regular, as highly as possible.

Remark 2.2. The main problem was to get (1.7) so that instead of investigating the problem for $a < b$, we have obtained an equation holding for **all** arguments from \mathbb{R} . This has an important meaning for the method of solving. However, in (1.7) we have now an operation m_λ appearing in the argument of g . This prevents us from directly applying the general theory developed for some general equations for functions with arguments and values in Abelian groups (cf. [21], [18], [15], [13] and especially [19]). In all the mentioned papers the operations appearing in arguments of unknown functions were of the form $(x, y) \longrightarrow \mu(x) + \nu(y)$ where μ and ν are homomorphisms.

Remark 2.3. We are going to develop further the general theory with operations not necessarily being sums of homomorphisms. The results will be published elsewhere.

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