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ON A CHARACTERIZATION OF STARLIKE FUNCTIONS

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Abstract. The aim of this paper is to present several equivalent conditions for starlikeness of functions. Our results will generalize that of Losonczi by eliminating the differentiability assumptions. Instead of Fréchet differentiability we only assume directional upper semicontinuity and use the left hand sided upper and lower Dini derivatives. Thus, we will also obtain new characterization of convex functions.

1. Introduction

In the paper [4] by L. Losonczi various versions of conditional convexity of real valued functions were defined in the normed space setting. We recall one of those notions and call it starlikeness in this paper.

Let X be a real linear space throughout this paper. A subset $D \subset X$ is said to be a *starlike set* with respect to a point $w \in D$ if, for $x \in D$,

$$[x, w] := \{ \lambda x + (1-\lambda)w \ : \ \lambda \in [0,1] \} \subset D.$$

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A function $f : D \to \mathbb{R}$ is called *starlike with respect to the point* $w \in D$ if D is starlike with respect to w and

$$(1.1) \qquad f(\lambda x + (1-\lambda)w) \le \lambda f(x) + (1-\lambda)f(w) \qquad (x \in D,\ \lambda \in [0,1])$$

holds.

It is obvious, that if D is convex, then f is starlike with respect to all $w \in D$ if and only if it is convex on D. Therefore, starlikeness of functions can be considered as a particular version of conditional convexity. In [4] Losonczi obtained the following characterization of starlike functions.

Theorem A. *Let X be a real normed space and $D \subset X$ be an open starlike set with respect to $w \in D$. Let $f : D \to \mathbb{R}$ be a Fréchet differentiable function on D. Then f is starlike with respect to w if and only*

$$(1.2) \qquad f'(x)(w-x) \le f(w) - f(x) \qquad (x \in D)$$

is valid, where $f'(x)$ denotes the Fréchet derivative of f at the point x.

If D is convex, then Theorem A yields the known first-order characterization of convex functions as well (see [8]).

The main aim of this paper is to present several equivalent conditions for starlikeness of functions. We will also obtain a generalization of Theorem A, where no differentiability of f will be assumed; we shall use left hand sided upper and lower Dini derivatives instead of the Fréchet derivative. Thus, we will also obtain new characterization of convex functions.

2. Results

Let X be a real linear space, D be nonempty subset of X, $w \in D$ be fixed, and $f : D \to \mathbb{R}$ be an arbitrary function throughout this paper.

In order to formulate the first geometric characterization of starlike functions, we recall the notion of *epigraph of functions*:

$$\operatorname{epi}(f) := \{\ (x,t) \in D \times \mathbb{R} \mid f(x) \le t\ \}.$$

Our first result establishes a connection between starlikeness of a functions and starlikeness of its epigraph. Its proof is almost obvious, it is described only for reader's convenience.

Theorem 1. *The set $D \subseteq X$ and the function $f : D \to \mathbb{R}$ is starlike with respect to $w \in D$ if and only if* $\operatorname{epi}(f)$ *is a starlike set with respect the point $(w, f(w))$.*

Proof. Assume first that D and f are starlike with respect to w. Let $(x,t) \in \operatorname{epi}(f)$ and $\lambda \in [0,1]$. Then, due to inequality (1.1), we have

$$f(\lambda x + (1-\lambda)w) \le \lambda f(x) + (1-\lambda) f(w) \le \lambda t + (1-\lambda) f(w).$$

Therefore,

$$\lambda(x,t) + (1-\lambda)(w, f(w)) = \Big(\lambda x + (1-\lambda)w, \lambda t + (1-\lambda) f(w)\Big) \in \operatorname{epi}(f).$$

Thus

$$[(x,t),(w,f(w))] \subset \operatorname{epi}(f),$$

and hence, $\operatorname{epi}(f)$ is starlike with respect to $(w, f(w))$.

Assume now that $\operatorname{epi}(f)$ is starlike with respect to $(w, f(w))$. Let $x \in D$ and $\lambda \in [0,1]$ be arbitrary. Then, $(x, f(x))$ being in $\operatorname{epi}(f)$, we have that

$$\lambda(x, f(x)) + (1-\lambda)(w, f(w)) = \Big(\lambda x + (1-\lambda)w, \lambda f(x) + (1-\lambda) f(w)\Big) \in \operatorname{epi}(f).$$

Hence,

$$\lambda x + (1-\lambda)w \in D$$

and

$$f(\lambda x + (1-\lambda)w) \le \lambda f(x) + (1-\lambda) f(w).$$

Thus D and f are starlike with respect to w. ■

The second characterization of starlike functions can also be proved in an obvious way.

Theorem 2. *Let $D \subseteq X$ be a starlike set with respect to w. Then f is starlike with respect to w if and only if, for all fixed $v \in D$, the function*

$$\phi_v(t) = \frac{f(vx + (1-t)w) - f(w)}{t} \qquad (t \in]0,1]) \tag{2.1}$$

is increasing.

Proof. Assume that f is starlike with respect to w. Let $v \in D$ be fixed and let $0 < s \le t \le 1$. Then, applying (1.1) with $x := tv + (1-t)w$ and $\lambda := s/t$, we get

$$f\big((s/t)(tv + (1-t)w) + (1-s/t)w\big) \le (s/t) f(tv + (1-t)w) + (1-s/t) f(w),$$

that is,

$$tf(sv + (1-s)w) \le sf(tv + (1-t)w) + (t-s)f(w).$$

Thus, $\phi_v(s) \le \phi_v(t)$, and hence ϕ_v is increasing.

Conversely, if for all $v \in D$, the function ϕ_v is increasing, then, with $v := x$, we get $\phi_x(\lambda) \le \phi_x(1)$ for $\lambda \in]0,1]$, which is equivalent to (1.1). ■

We note that the monotonicity of the function ϕ_v defined in (2.1) was also discovered and used in the proof of Theorem A by Losonczi [4].

Now we are going to formulate our main result that will generalize Theorem A. In order to accomplish this goal, we have to recall the notion of upper and lower left and right hand sided Dini directional derivative.

Assume that $x \in D$ and $v \in X$ such that $x+tv \in D$ for small positive values of t. Then the *upper and lower right hand sided Dini directional derivatives of f at the point x* are defined by

$$d^+f(x;v) := \limsup_{t\to 0+} \frac{f(x+tv)-f(x)}{t}, \quad d_+f(x;v) := \liminf_{t\to 0+} \frac{f(x+tv)-f(x)}{t},$$

respectively. Similarly, if $x + tv \in D$ for small negative values of t, the upper and lower left hand sided Dini directional derivatives are defined by

$$d^-f(x;v) := \limsup_{t\to 0-} \frac{f(x+tv)-f(x)}{t}, \quad d_-f(x;v) := \liminf_{t\to 0-} \frac{f(x+tv)-f(x)}{t},$$

respectively.

Theorem 3. *Let $D \subseteq X$ be a starlike set with respect to an element $w \in D$ such that, for all $x \in D$ there exists $\varepsilon > 0$ satisfying $x+\varepsilon(x-w) \in D$. Let $f : D \to \mathbb{R}$ be a function such that, for all $x \in D$, the function $\lambda \mapsto f(\lambda x+(1-\lambda)w)$ is upper semicontinuous on $]0,1]$. Then the following conditions are pairwise equivalent*

(i) *f is starlike with respect to w;*

(ii) *$d^-f(x,w-x) \le f(w) - f(x)$ for all $x \in D$;*

(iii) *$d_-f(x,w-x) \le f(w) - f(x)$ for all $x \in D$.*

We note that if X is a topological linear space, then the regularity assumptions on D and on f are easily satisfied if D is open and f is upper semicontinuous on $D \setminus \{w\}$.

Proof. Assume first that f is starlike with respect to w. Let $x \in D$ be fixed. We may also assume that $x \ne w$, otherwise (ii) is trivial. Let $\varepsilon > 0$ such that $x + \varepsilon(x-w) \in D$. Then, for $-\varepsilon < t < 0$, we have that $x_t := (1-t)x + tw =$

$x + t(w - x) \in D$. Applying (1.1) with x_t instead of x and with $\lambda := \frac{-t}{1-t}$, we get

$$f(x) = f\Big(\frac{-t}{1-t}[(1-t)x + tw] + \frac{1}{1-t}w\Big)$$
$$\leq \frac{-t}{1-t}f((1-t)x + tw) + \frac{1}{1-t}f(w).$$

Therefore, for $-\varepsilon < t < 0$, we have

$$f(w) - f(x) \geq \frac{f(x + t(w - x)) - f(x)}{t}.$$

Taking the limsup $t \to 0-$ in the above inequality, we obtain that (ii) is valid. Clearly, (ii) yields (iii).

In the rest of the proof, we show that (iii) implies (i). Assume, on the contrary, that (iii) is valid, but (i) does not hold. Then there exist $x_0 \in D \setminus \{w\}$ and $\lambda_0 \in]0, 1[$ such that

$$f(\lambda_0 x_0 + (1 - \lambda_0)w) > \lambda_0 f(x_0) + (1 - \lambda_0)f(w). \tag{2.2}$$

Define the function $g : [0, 1] \to \mathbb{R}$ by

$$g(t) := f(tx_0 + (1 - t)w) - tf(x_0) - (1 - t)f(w).$$

Then, by our regularity assumption on f, the function g is upper semicontinuous on $]0, 1]$, and by (2.2), $g(\lambda_0) > 0$. Therefore, g attains its maximum on the compact interval $[\lambda_0, 1]$ at a point t_0, and $t_0 \neq 1$, because $g(1) = 0$. Then we have

$$g(t) \leq g(t_0) \qquad (t_0 \leq t \leq 1) \tag{2.3}$$

and

$$g(t_0) > 0. \tag{2.4}$$

In view of (2.3), for $t_0 < t \leq 1$, we get

$$0 \geq \frac{g(t) - g(t_0)}{t - t_0} = \frac{f(tx_0 + (1 - t)w) - f(t_0 x_0 + (1 - t_0)w)}{t - t_0} + f(w) - f(x_0),$$

that is, with the notation $x_* := t_0 x_0 + (1 - t_0)w$,

$$\frac{f(x_* + (t_0 - t)(w - x_0)) - f(x_*)}{t_0 - t} \geq f(w) - f(x_0),$$

if $t_0 < t \leq 1$. Taking the liminf as $t \to t_0+$ (then $t - t_0 \to 0-$), we obtain that

$$d_- f(x_*, w - x_0) \geq f(w) - f(x_0). \tag{2.5}$$

It follows from the choice of x_* that $w-x_*=t_0(w-x_0)$, hence, by the positive homogeneity of the Dini derivatives, (2.5) yields

$$d_- f(x_*, w-x_*) \geq t_0[f(w)-f(x_0)]. \tag{2.6}$$

On the other hand, by (2.4),

$$f(t_0x_0+(1-t_0)w)-f(w) > t_0\big(f(x_0)-f(w)\big).$$

Hence

$$t_0[f(w)-f(x_0)] > f(w)-f(x_*) \tag{2.7}$$

Combining the two inequalities (2.6) and (2.7), we get

$$d_- f(x_*, w-x_*) \geq t_0[f(w)-f(x_0)] > f(w)-f(x_*)$$

which contradicts (iii). The contradiction obtained shows that (iii) implies (i). ■

Remark 4. We note that the regularity assumption on f was used only in the proof of the implication (iii)⇒(i). It can also be shown that if (i) is valid then the function f also has the following properties.

(iv) $d^+ f(x, w-x) \leq f(w)-f(x)$ for all $x \in D$;

(v) $d_+ f(x, w-x) \leq f(w)-f(x)$ for all $x \in D$;

However, it is not clear if (iv) or (v) is sufficient for (i) to hold.

Remark 5. The inequality which is reversed to (1.1) can also be characterized. One has to apply Theorem 3 for the function $(-f)$ instead of f. Then, using the easy-to-see identities $d_-(-f)(x;v) = -d^+ f(x;v)$ and $d^-(-f)(x;v) = -d_+ f(x;v)$, we can see that this charaterization is made in terms of the right hand side Dini derivatives $d_+ f$ and $d^+ f$ via reversed inequalities in conditions (ii) and (iii) of Theorem 3, respectively.

The result obtained in Theorem 3, allows us to present a new characterization of convexity.

Corollary 6. *Let X be a Hausdorff topological linear space and D be an open convex subset of X. Assume that $f: D \to \mathbb{R}$ is upper semicontinuous on $[x,y]$ if $x, y \in D$. Then the following conditions are pairwise equivalent:*

(i) *f is convex;*

(ii) *$d^- f(x, w-x) \leq f(w)-f(x)$ for all $x, w \in D$;*

(iii) $d_-f(x, w-x) \le f(w) - f(x)$ *for all* $x, w \in D$.

Applying this corollary to the function $(-f)$, we can also get

Corollary 7. *Let* X *be a Hausdorff topological linear space and* D *be an open convex subset of* X*. Assume that* $f : D \to \mathbb{R}$ *is lower semicontinuous on* $[x, y]$ *if* $x, y \in D$*. Then the following conditions are pairwise equivalent:*

(i) f *is concave;*

(ii) $d_+f(x, w-x) \ge f(w) - f(x)$ *for all* $x, w \in D$*;*

(iii) $d^+f(x, w-x) \ge f(w) - f(x)$ *for all* $x, w \in D$.

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