

FURTHER RESULTS ON A MULTIPLICATIVE TYPE FUNCTIONAL EQUATION

Fruzsina Mészáros (Debrecen, Hungary)

Communicated by Antal Járai

(Received February 28, 2018; accepted April 24, 2018)

Abstract. In the present paper the multiplicative type functional equation

$$f(xy)g(x+y) = h(xy+x)k(y),$$

derived from the pexiderized Davison equation (PD), is considered on different structures.

1. Introduction

The functional equation

$$(D) \quad f(xy) + f(x+y) = f(xy+x) + f(y)$$

was introduced by T. M. K. Davison at the 17th ISFE (Oberwolfach, 1979) (see [2]). During the meeting W. Benz gave the continuous solution $f : \mathbb{R} \rightarrow \mathbb{R}$ of (D) for all $x, y \in \mathbb{R}$.

The general solution of (D) was given in [3] by R. Girgensohn and K. Lajkó:

Theorem 1.1. *The function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies functional equation (D) for all $x, y \in \mathbb{R}$ if and only if f is of the form $f(x) = A(x) + b$, where $A : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function and $b \in \mathbb{R}$ is an arbitrary constant.*

Key words and phrases: Davison equation, multiplicative functional equation satisfied a.e., measurable solutions.

2010 Mathematics Subject Classification: 39B22.

This research has been supported by the Hungarian Scientific Research Fund (OTKA), Grant K111651.

In [3] the authors presented the general solution of the Pexiderized version

$$(PD) \quad f(xy) + g(x+y) = h(xy+x) + k(y)$$

of (D) for all $x, y \in \mathbb{R}$ and for all $x, y \in \mathbb{R}_+ := \{x | x > 0\}$:

Theorem 1.2. *The functions $f, g, h, k : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (PD) for all $x, y \in \mathbb{R}$ if and only if they have the form $f(x) = A(x) + b_1$, $g(x) = A(x) + b_2$, $h(x) = A(x) + b_3$, $k(x) = A(x) + b_4$, where $A : \mathbb{R} \rightarrow \mathbb{R}$ is additive and $b_1, b_2, b_3, b_4 \in \mathbb{R}$ are constants with $b_1 + b_2 = b_3 + b_4$.*

Theorem 1.3. *The functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy (PD) for all $x, y \in \mathbb{R}_+$ if and only if they are of the form*

$$(1.1) \quad \begin{aligned} f(x) &= A(x) + B(\log x) + b_1, & g(x) &= A(x) + b_2, \\ h(x) &= A(x) + B(\log x) + b_3, & k(x) &= A(x) + B\left(\log \frac{x}{x+1}\right) + b_4, \end{aligned}$$

where $A, B : \mathbb{R} \rightarrow \mathbb{R}$ are additive and $b_1, b_2, b_3, b_4 \in \mathbb{R}$ are constants with $b_1 + b_2 = b_3 + b_4$.

Using (1.1) in Theorem 1.3, we easily get that

$$A(x) = g(x) - b_2, \quad B(\log x) = f(x) - g(x) - b_1 + b_2 \quad (x \in \mathbb{R}_+).$$

Thus the continuity (or measurability) of functions f, g implies that A, B are continuous (or measurable) on \mathbb{R}_+ , too. This implies (see [1], [9]) that

$$A(x) = ax, \quad B(x) = bx \quad (x \in \mathbb{R}_+),$$

where $a, b \in \mathbb{R}$ are arbitrary constants.

Using these considerations together with Theorem 1.3, we get the following result.

Theorem 1.4. *The measurable (or continuous) functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy (PD) for all $x, y \in \mathbb{R}_+$ if and only if they are of the form*

$$(1.2) \quad \begin{aligned} f(x) &= ax + b \log x + b_1, & g(x) &= ax + b_2, \\ h(x) &= ax + b \log x + b_3, & k(x) &= ax + b \log \frac{x}{x+1} + b_4, \end{aligned}$$

where $a, b, b_1, b_2, b_3, b_4 \in \mathbb{R}$ are constants with $b_1 + b_2 = b_3 + b_4$.

2. Positive solution of a multiplicative type functional equation stemming from (PD)

Let us write (PD) in the following multiplicative form:

$$(2.1) \quad f(xy)g(x+y) = h(xy+x)k(y)$$

for functions $f, g, h, k : \mathbb{R}$ (or \mathbb{R}_+) $\rightarrow \mathbb{R}_+$ for all $x, y \in \mathbb{R}$ or for all $x, y \in \mathbb{R}_+$.

Taking the logarithm of (2.1), we get the functional equation

$$\log(f(xy)) + \log(g(x+y)) = \log(h(xy+x)) + \log(k(y))$$

for all $x, y \in \mathbb{R}$ or for all $x, y \in \mathbb{R}_+$.

Thus the functions $F, G, H, K : \mathbb{R}$ (or \mathbb{R}_+) $\rightarrow \mathbb{R}$ defined by

$$F = \log \circ f, \quad G = \log \circ g, \quad H = \log \circ h, \quad K = \log \circ k$$

satisfy functional equation (PD).

Using Theorems 1.2, 1.3 and 1.4 and that

$$f = \exp \circ F, \quad g = \exp \circ G, \quad h = \exp \circ H, \quad k = \exp \circ K,$$

we get immediately the following results.

Theorem 2.1. *The functions $f, g, h, k : \mathbb{R} \rightarrow \mathbb{R}_+$ satisfy (2.1) for all $x, y \in \mathbb{R}$ if and only if*

$$\begin{aligned} f(x) &= c_1 \exp(A(x)), & g(x) &= c_2 \exp(A(x)), \\ h(x) &= c_3 \exp(A(x)), & k(x) &= c_4 \exp(A(x)), \end{aligned}$$

where $A : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function and $c_1, c_2, c_3, c_4 \in \mathbb{R}_+$ are constants with $c_1 c_2 = c_3 c_4$.

Theorem 2.2. *The functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy (2.1) for all $x, y \in \mathbb{R}_+$ if and only if they are of the form*

$$(2.2) \quad \begin{aligned} f(x) &= c_1 \exp(A(x) + B(\log x)), & g(x) &= c_2 \exp(A(x)), \\ h(x) &= c_3 \exp(A(x) + B(\log x)), \\ k(x) &= c_4 \exp\left(A(x) + B\left(\log \frac{x}{x+1}\right)\right), \end{aligned}$$

where $A, B : \mathbb{R} \rightarrow \mathbb{R}$ are additive and $c_1, c_2, c_3, c_4 \in \mathbb{R}_+$ are constants with $c_1 c_2 = c_3 c_4$.

Theorem 2.3. *The measurable (or continuous) functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy (2.1) for all $x, y \in \mathbb{R}_+$ if and only if*

$$(2.3) \quad \begin{aligned} f(x) &= c_1 \exp(ax + b \log x), & g(x) &= c_2 \exp(ax), \\ h(x) &= c_3 \exp(ax + b \log x), & k(x) &= c_4 \exp\left(ax + b \log \frac{x}{x+1}\right), \end{aligned}$$

where $a, b \in \mathbb{R}$ and $c_1, c_2, c_3, c_4 \in \mathbb{R}_+$ are constants with $c_1 c_2 = c_3 c_4$.

3. Nonnegative solutions of (2.1)

Now let us assume, that the nonnegative measurable functions f, g, h, k satisfy (2.1) for all $x, y \in \mathbb{R}_+$ and none of the functions h, k are almost everywhere zero. Does it follow that they are positive everywhere on \mathbb{R}_+ ?

In order to give an affirmative answer we will use the following result (see Járai, Lajkó, Mészáros [8], Remark 4).

Theorem 3.1. *Suppose that measurable functions $f_1 : X \rightarrow \mathbb{C}$, $f_2 : Y \rightarrow \mathbb{C}$, $g_1 : U \rightarrow \mathbb{C}$, $g_2 : V \rightarrow \mathbb{C}$ satisfy functional equation*

$$(3.1) \quad f_1(x)f_2(y) = g_1(G_1(x, y))g_2(G_2(x, y))H(x, y)$$

for all $(x, y) \in X \times Y$, where X, Y, U, V are nonvoid open intervals, G_1, G_2 and H are given functions, such that H is nowhere zero on $X \times Y$, the mapping $(x, y) \mapsto G(x, y) = (G_1(x, y), G_2(x, y))$ is a C^1 -diffeomorphism of $X \times Y$ onto $U \times V$ with inverse $(u, v) \mapsto F(u, v) = (F_1(u, v), F_2(u, v))$, such that all the partial derivatives of functions G_1, G_2, F_1, F_2 vanish nowhere on their domain and if on one side none of the functions are almost everywhere zero, then all the functions are everywhere nonzero.

Theorem 3.2. *If the nonnegative measurable functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy functional equation (2.1) for all $x, y \in \mathbb{R}_+$ such that h, k are positive on some subsets of \mathbb{R}_+ with positive Lebesgue measure, then f, g, h, k are positive everywhere on \mathbb{R}_+ .*

Proof. Using the symmetry of the left-hand side of (2.1) in x and y , we get

$$h(xy + x)k(y) = h(xy + y)k(x)$$

for all $x, y \in \mathbb{R}_+$. On the other hand, by replacing x by $\frac{x}{y+1}$, we find that h and k satisfy functional equation

$$(3.2) \quad h(x)k(y) = h\left(\frac{xy}{y+1} + y\right)k\left(\frac{x}{y+1}\right)$$

for all $x, y \in \mathbb{R}_+$, i.e. functional equation (3.1) for the unknown functions $f_1 = g_1 = h$, $f_2 = g_2 = k$, $X = Y = U = V = \mathbb{R}_+$ and for the given functions

$$G_1(x, y) = \frac{xy}{y+1} + y, \quad G_2(x, y) = \frac{x}{y+1}, \quad H(x, y) = 1 \quad ((x, y) \in \mathbb{R}_+^2).$$

Observe that H is nowhere zero on \mathbb{R}_+^2 and the mapping

$$(x, y) \rightarrow G(x, y) = (G_1(x, y), G_2(x, y)) = \left(\frac{xy}{y+1} + y, \frac{x}{y+1}\right)$$

is C^1 -diffeomorphism of \mathbb{R}_+^2 onto \mathbb{R}_+^2 with inverse

$$(u, v) \rightarrow F(u, v) = (F_1(u, v), F_2(u, v)) = \left(\frac{uv}{v+1} + v, \frac{u}{v+1} \right).$$

The partial derivatives

$$\begin{aligned} \frac{\partial G_1}{\partial x} &= \frac{y}{y+1}, & \frac{\partial G_1}{\partial y} &= \frac{x}{(y+1)^2}, & \frac{\partial G_2}{\partial x} &= \frac{1}{y+1}, & \frac{\partial G_2}{\partial y} &= -\frac{x}{(y+1)^2}, \\ \frac{\partial F_1}{\partial u} &= \frac{v}{v+1}, & \frac{\partial F_1}{\partial v} &= \frac{u}{(v+1)^2}, & \frac{\partial F_2}{\partial u} &= \frac{1}{v+1}, & \frac{\partial F_2}{\partial v} &= -\frac{u}{(v+1)^2}, \end{aligned}$$

vanish nowhere on \mathbb{R}_+^2 and further none of the functions h, k are almost everywhere zero. All assumptions of Theorem 3.1 are satisfied, which implies that the functions h, k are everywhere nonzero on \mathbb{R}_+ , then by equation (2.1) we get that functions f, g are everywhere nonzero, too.

Thus the nonnegativity of functions implies that $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ are everywhere positive on \mathbb{R}_+^2 . ■

Now we can easily prove the following result for equation (2.1).

Theorem 3.3. *If the nonnegative measurable (or continuous) functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy (2.1) for all $x, y \in \mathbb{R}_+$ such that h, k are positive on some Lebesgue measurable subsets of positive Lebesgue measure, then they have the form (2.3), i.e.*

$$f(x) = c_1 \exp(ax + b \log x), \quad g(x) = c_2 \exp(ax),$$

$$h(x) = c_3 \exp(ax + b \log x), \quad k(x) = c_4 \exp\left(ax + b \log \frac{x}{x+1}\right),$$

where $a, b \in \mathbb{R}$ and $c_1, c_2, c_3, c_4 \in \mathbb{R}_+$ are constants with $c_1 c_2 = c_3 c_4$.

Proof. Theorem 3.2 implies that functions f, g, h, k are positive everywhere on \mathbb{R}_+ and then Theorem 2.3 gives (2.3) for these functions, which completes the proof. ■

4. Nonnegative solutions of (2.1) satisfying almost everywhere

Now let us assume, that the nonnegative measurable functions f, g, h, k satisfy (2.1) for almost all $x, y \in \mathbb{R}_+$ and none of the functions h, k are almost

everywhere zero. Does it follow that they are positive almost everywhere on \mathbb{R}_+ ?

In order to give an affirmative answer we will use the following result (see Járai, Lajkó, Mészáros [8], Theorem 2).

Theorem 4.1. *Suppose that measurable functions $f_1 : X \rightarrow \mathbb{C}$, $f_2 : Y \rightarrow \mathbb{C}$, $g_1 : U \rightarrow \mathbb{C}$, $g_2 : V \rightarrow \mathbb{C}$ satisfy functional equation*

$$(4.1) \quad f_1(x)f_2(y) = g_1(G_1(x,y))g_2(G_2(x,y))H(x,y)$$

for almost all $(x,y) \in X \times Y$ (with respect to the plane Lebesgue measure), where X, Y, U, V are nonvoid open intervals, G_1, G_2 and H are given functions, such that H is nowhere zero on $X \times Y$, the mapping $(x,y) \mapsto G(x,y) = (G_1(x,y), G_2(x,y))$ is a C^1 -diffeomorphism of $X \times Y$ onto $U \times V$ with inverse $(u,v) \mapsto F(u,v) = (F_1(u,v), F_2(u,v))$, such that all the partial derivatives of functions G_1, G_2, F_1, F_2 vanish nowhere on their domain. Then either one of the functions f_1 and f_2 and one of the functions g_1 and g_2 is zero almost everywhere or all of them are almost everywhere nonzero.

Theorem 4.2. *If the nonnegative measurable functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy functional equation (2.1) for almost all $x, y \in \mathbb{R}_+$ such that h, k are positive on some subsets of \mathbb{R}_+ with positive Lebesgue measure, then f, g, h, k are positive almost everywhere on \mathbb{R}_+ .*

Proof. Similarly to the proof of Theorem 3.2 we can prove that all assumptions of Theorem 4.1 are satisfied. This implies that the functions h, k are almost everywhere nonzero on \mathbb{R}_+ , then by equation (2.1) we get that functions f, g are almost everywhere nonzero, too.

Thus the nonnegativity of functions implies that $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ are almost everywhere positive on \mathbb{R}_+ . ■

To get the nonnegative measurable solutions of (2.1) satisfying almost everywhere, we need the following result of A. Járai (see [4], [5], [6], [7]).

Theorem 4.3. *Let Z be a regular topological space, Z_i ($i = 1, 2, \dots, n$) be topological spaces and T be a first countable topological space. Let Y be an open subset of \mathbb{R}^k , X_i an open subset of \mathbb{R}^{r_i} , $r_i \in \mathbb{Z}$, ($i = 1, 2, \dots, n$) and D an open subset of $T \times Y$. Let furthermore $T' \subset T$ be a dense subset, $F : T' \rightarrow Z$, $g_i : D \rightarrow X_i$ and $H : D \times Z_1 \times \dots \times Z_n \rightarrow Z$. Suppose that the function f_i is almost everywhere defined on X_i (with respect to the r_i -dimensional Lebesgue measure) with values in Z_i ($i = 1, 2, \dots, n$) and the following conditions are satisfied:*

1. for all $t \in T'$ and for almost all $y \in D_t = \{y \in Y | (t, y) \in D\}$

$$F(t) = H(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y)));$$

2. for each fixed y in Y , the function H is continuous in the other variables;
3. f_i is Lebesgue measurable on \mathbb{R}^{r_i} ($i = 1, 2, \dots, n$);
4. g_i and the partial derivative $\frac{\partial g_i}{\partial y}$ are continuous on D ($i = 1, 2, \dots, n$);
5. for each $t \in T$ there exist a y such that $(t, y) \in D$ and the partial derivative $\frac{\partial g_i}{\partial y}$ has the rank r_i at $(t, y) \in D$ ($i = 1, 2, \dots, n$).

Then there exists a unique continuous function \tilde{F} such that $F = \tilde{F}$ almost everywhere on T , and if F is replaced by \tilde{F} then the functional equation is satisfied almost everywhere on D .

Using Theorems 4.2 and 4.3 we can prove the following result.

Theorem 4.4. *If the nonnegative measurable functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy (2.1) for almost all $(x, y) \in \mathbb{R}_+^2$ such that they are positive on some subsets of \mathbb{R}_+ with positive Lebesgue measure, then there exist unique continuous functions $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{k} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\tilde{f} = f, \tilde{g} = g, \tilde{h} = h$ and $\tilde{k} = k$ almost everywhere on \mathbb{R}_+ , and if f, g, h, k are replaced by $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{k}$, respectively, then (2.1) is satisfied everywhere on \mathbb{R}_+^2 .*

Proof. Theorem 4.2 shows that the functions f, g, h, k are positive almost everywhere on \mathbb{R}_+ .

First we prove that there exists a unique continuous function \tilde{h} which is equal to h almost everywhere on \mathbb{R}_+ and replacing h by \tilde{h} , equation (2.1) is satisfied almost everywhere on \mathbb{R}_+^2 .

With the substitution $t = xy + x$ we get from (2.1) the equation

$$(4.2) \quad h(t) = \frac{f\left(\frac{ty}{y+1}\right)g\left(\frac{t}{y+1} + y\right)}{k(y)}$$

which is satisfied for almost all $(t, y) \in \mathbb{R}_+^2$.

By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}_+$ of full measure such that for all $t \in T'$ equation (4.2) is satisfied for almost every $y \in \{y \in \mathbb{R}_+ | (t, y) \in \mathbb{R}_+^2\} = \mathbb{R}_+$.

Let us define the functions g_1, g_2, g_3, H in the following way:

$$g_1(t, y) = \frac{ty}{y+1}, \quad g_2(t, y) = \frac{t}{y+1} + y, \quad g_3(t, y) = y$$

$$H(t, y, z_1, z_2, z_3) = \frac{z_1 z_2}{z_3}$$

and let us now apply Theorem 4.3 of Járai to (4.2) with the following casting: $h(t) = F(t)$, $f(t) = f_1(t)$, $g(t) = f_2(t)$, $k(t) = f_3(t)$, $Z = Z_i = \mathbb{R}_+$, $T = Y = X_i = \mathbb{R}_+$ ($i = 1, 2, 3$).

The first assumption of Theorem 4.3 with respect to (4.2) holds.

In the case of fixed y , the function H is continuous in the other variables, so the second assumption holds too.

Because the functions in (4.2) are measurable, the third assumption is trivial.

The functions g_i and the partial derivatives

$$D_2g_1(t, y) = \frac{t}{(y+1)^2}, \quad D_2g_2(t, y) = -\frac{t}{(y+1)^2} + 1, \quad D_2g_3(t, y) = 1$$

are continuous, so the fourth assumption holds, too.

For each $t \in \mathbb{R}_+$ there exist a $y \in \mathbb{R}_+$ such that $(t, y) \in D$ and the partial derivatives don't equal zero in (t, y) , so they have the rank 1. Thus the last assumption is satisfied in Theorem 4.3.

So we get from Theorem 4.3 that there exists unique continuous function \tilde{h} which is almost everywhere equal to h on \mathbb{R}_+ and f , g , \tilde{h} , k satisfy equation (4.2) almost everywhere, which is equivalent to the equation

$$f(xy)g(x+y) = \tilde{h}(xy+x)k(y)$$

for almost all $(x, y) \in \mathbb{R}_+^2$. Furthermore \tilde{h} is positive for almost all $x \in \mathbb{R}_+$.

By a similar argument we can prove the same for the functions f , g and k , i.e. there exist continuous functions $\tilde{f} : \mathbb{R}_+ \rightarrow \mathbb{R}$, $\tilde{g} : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\tilde{k} : \mathbb{R}_+ \rightarrow \mathbb{R}$ which are almost everywhere equal to f , g and k on \mathbb{R}_+ , respectively, and the functional equation

$$(4.3) \quad \tilde{f}(xy)\tilde{g}(x+y) = \tilde{h}(xy+x)\tilde{k}(y)$$

is satisfied almost everywhere on \mathbb{R}_+^2 .

Both sides of (4.3) define continuous functions on \mathbb{R}_+^2 , which are equal to each other on a dense subset of \mathbb{R}_+^2 , therefore we obtain that (4.3) is satisfied everywhere on \mathbb{R}_+^2 .

Applying Theorem 3.2 for equation (4.3), one can show that if the nonnegative continuous functions \tilde{f} , \tilde{g} , \tilde{h} and $\tilde{k} : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy functional equation (4.3) for all $(x, y) \in \mathbb{R}_+^2$, such that they are positive almost everywhere on \mathbb{R}_+ , then they are positive everywhere on \mathbb{R}_+ . ■

Now we can easily prove the following result for equation (2.1).

Theorem 4.5. *If the nonnegative measurable (or continuous) functions $f, g, h, k : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfy (2.1) for almost all $x, y \in \mathbb{R}_+$ such that they are positive on some Lebesgue measurable subsets of positive Lebesgue measure, then they have the form*

$$f(x) = c_1 \exp(ax + b \log x), \quad g(x) = c_2 \exp(ax) \quad a.a. \ x \in \mathbb{R}_+,$$

$$h(x) = c_3 \exp(ax + b \log x) \quad a.a. \ x \in \mathbb{R}_+,$$

$$k(x) = c_4 \exp\left(ax + b \log \frac{x}{x+1}\right) \quad a.a. \ x \in \mathbb{R}_+,$$

where $a, b \in \mathbb{R}$ and $c_1, c_2, c_3, c_4 \in \mathbb{R}_+$ are constants with $c_1 c_2 = c_3 c_4$.

Proof. Using Theorems 4.4 and 4.2, we get immediately the statement of Theorem 4.5. ■

References

- [1] **Aczél, J. and J. Dhombres**, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.
- [2] **Davison, T.M.K.**, Problem 191, *Aequationes Math.*, **20** (1980), 306.
- [3] **Girgensohn, R. and K. Lajkó**, A functional equation of Davison and its generalization, *Aequationes Math.*, **60** (2000), 219–224.
- [4] **Járai, A.**, Measurable solutions of functional equations satisfied almost everywhere, *Mathematica Pannonica*, **10/1** (1999), 103–110.
- [5] **Járai, A.**, *Regularity Properties of Functional Equations in Several Variables*, Springer, Adv. Math. Dordrecht, Vol. 8, 2005.
- [6] **Járai, A.**, Regularity properties of measurable functions satisfying a multiplicative type functional equation almost everywhere, *Aequationes Math.*, **89** (2015), 367–381.
- [7] **Járai, A.**, A 0-1 law for multiplicative functional equations, *Aequationes Math.*, **90** (2016), 147–161.
- [8] **Járai, A., K. Lajkó and F. Mészáros**, On measurable functions satisfying multiplicative type functional equation almost everywhere, in: *Inequalities and Applications '10*, International Series of Numerical Mathematics, Birkhäuser Verlag, 2012, 241–253.
- [9] **Kuczma, M.**, *Introduction to the Theory of Functional Equations and Inequalities*, Second Edition, Birkhäuser Verlag, Basel, Boston, Berlin, 2009.

F. Mészáros

University of Debrecen

H-4002 Debrecen

P.O. Box 400

Hungary

mefru@science.unideb.hu