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TURÁN–KUBILIUS' INEQUALITY ON PERMUTATIONS

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Abstract. We find the exact constant in the second moment estimate for strongly additive functions defined on random permutations. The result draws some distinction comparing to the number-theoretical Turán–Kubilius inequality when its sharp form is taken into account.

1. Introduction and the result

The purpose of this note is to demonstrate the parallelism and contrast of probabilistic number theory and probabilistic combinatorics just considering the second moment for additive functions defined on random permutations. The starting point is the Turán–Kubilius inequality which, for strongly additive number-theoretic functions $h : \mathbf{N} \to \mathbf{R}$, reads as follows

$$\frac{1}{n}\sum_{m=1}^{n}\Big(\sum_{p|m} h(p) - \sum_{p\le n}\frac{h(p)}{p}\Big)^2\Big(\sum_{p\le n}\frac{h^2(p)}{p}\Big)^{-1} =: K_n(h) \le C.$$

Here p denotes a prime number, $h(p) \ne 0$ for at least one p, $n \in \mathbf{N}\setminus\{1\}$, and $C > 0$ is an absolute constant. Refining his earlier results, in 1985 J. Kubilius

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[3] established that C can be substituted by $3/2+O\big(1/\log n\big)$ with an absolute constant in the symbol $O(\cdot)$. The result is sharp. Another approach estimating the variance giving also the optimal constant was proposed by A. Hildebrand [2]. An estimate with the $O((\log n)^{-1/2})$ remainder is given in the technical report [6]. The inequality above with $C=32$ holds for all $n\in\mathbf{N}$, as it has been observed by P.D.T.A. Elliott (see Supplement in [1] for an exhaustive historical account). A result by J. Lee deserves a special mentioning. He proved in [4] that there exist absolute positive constants c and d such that

$$\frac{3}{2}-\frac{c}{\log n}\le \sup\big\{K_n(h):\ h\neq 0\big\}\le \frac{3}{2}-\frac{d}{\log n} \tag{1.1}$$

for all sufficiently large n. Moreover, (1.1) holds for any $d\le 0,644....$ Having this in mind, the second author [5] did an attempt to find an asymptotic constant in the second moment estimate for additive functions defined on random permutations.

Let $\mathbf{S}_n$ be the symmetric group of permutations σ acting on $n\ge 1$ letters. Each $\sigma\in\mathbf{S}_n$ has a unique representation (up to the order) by the product of independent cycles κ_i:

$$\sigma=\kappa_1\cdots\kappa_w, \tag{1.2}$$

where $w=w(\sigma)$ denotes the number of cycles. Let ν_n be the uniform probability measure on $\mathbf{S}_n$ (Haar measure). Set $\ell(\bar k)=1k_1+\cdots+nk_n$ for a vector $\bar k=(k_1,\dots,k_n)\in\mathbf{Z}_+^n$. Let $k_j(\sigma)$ denote the number of cycles of length j in (1.2) and $\bar k(\sigma)\colon=(k_1(\sigma),\dots,k_n(\sigma))$ be the *cycle structure vector* of a random permutation σ. The latter satisfies the relation

$$\ell(\bar k(\sigma))=n \tag{1.3}$$

involving dependence of the random variables $k_j(\sigma)$, $1\le j\le n$.

For simplicity, we will only deal with the *linear statistics* (or *completely additive function*)

$$h(\sigma)\colon=\sum_{j=1}^{n}a_jk_j(\sigma), \tag{1.4}$$

where $\bar a\colon=(a_1,\dots,a_n)\in\mathbf{R}^n$. By (1.3), we see that the function $h(\sigma)$ is a sum of dependent r.vs.

If $\mathbf{E}_n$ and $\mathbf{V}_n$ denote the mean value and the variance with respect to the frequency ν_n, then [5]

$$\mathbf{E}_nh(\sigma)=\sum_{j\le n}\frac{a_j}{j},$$

$$D_n(\bar{a}) :\, = \mathbf{V}_n h(\sigma) = \sum_{j\le n} \frac{a_j^2}{j} - \sum_{i,j\le n} \frac{a_i a_j}{ij} \mathbf{1}\{i+j>n\} =$$

$$=: B_n(\bar{a}) - \Delta_n(\bar{a}). \tag{1.5}$$

Here $\mathbf{1}\{\cdot\}$ is the indicator function. The term $\Delta_n(\bar{a})$ is nonnegative for $\bar{a} \in \mathbf{R}_+^n$ or $\bar{a} \in \mathbf{R}_-^n$. Thus splitting the summands into nonnegative and negative parts and using the inequality $(x+y)^2 \le 2(x^2+y^2)$ if $x, y \in \mathbf{R}$, we easily derive that

$$D_n(\bar{a}) \le 2B_n(\bar{a}), \quad \bar{a} \in \mathbf{R},\ n \in \mathbf{N}.$$

Adopting the approach developed in aforementioned number-theoretical papers, the second of the authors established in [5] that

$$\tau_n :\, = \sup\left\{ \frac{D_n(\bar{a})}{B_n(\bar{a})} :\ \bar{a} \in \mathbf{R}^n \setminus \{\bar{0}\} \right\} \ge \frac{3}{2} - \frac{3}{n^2+n} \tag{1.6}$$

for all $n \ge 1$ and

$$\tau_n \le \frac{3}{2} + \frac{3}{\sqrt{2(n^2+n)}} \tag{1.7}$$

if n is sufficiently large. A motivation for taking $B_n(\bar{a})$ to evaluate the variance $\mathbf{V}_n h(\sigma)$ is seen from the inequalities

$$B_n(\bar{a})(1-2/n) \le \sum_{j\le n} \mathbf{V}_n(a_j k_j(\sigma)) = B_n(\bar{a}) - \sum_{n/2<j\le n} \frac{a_j^2}{j^2} \le B_n(\bar{a}).$$

Improving (1.6) and (1.7), we have reached a final answer.

Theorem 1.1. *For each $n \ge 2$, we have $\tau_n = \frac{3}{2}$.*

Comparing this with Lee's approximation (1.1) for the number-theoretic functions, we see some declination. Let us include the dual inequality which follows from 1.1 and Lemma 5.1 on page 81 of the book [1].

Theorem 1.2. *Let $y(\sigma) \in \mathbf{R}$, where $\sigma \in \mathbf{S}_n$, be artitrary. Then*

$$\sum_{j=1}^{n} j \left(\sum_{\sigma \in \mathbf{S}_n} y(\sigma) \Big(k_j(\sigma) - \frac{1}{j} \Big) \right)^2 \le \frac{3}{2} n! \sum_{\sigma \in \mathbf{S}_n} y(\sigma)^2$$

if $n \ge 2$.

In fact, our problem concerns quadratic forms. The substitution $a_j = x_j\sqrt{j}$, $1 \le j \le n$, reduces $B_n(\bar{a})$ to the Euclidean norm $||x||^2 = x_1^2 + \cdots + x_n^2$, and

$\Delta_n(\bar{a})$ becomes a quadratic form $\bar{x}Q_n\bar{x}'$ of rank n and the matrix $Q_n = ((q_{ij}))$, $1 \le i, j \le n$, where

$$q_{ij} = \frac{\mathbf{1}\{i + j > n\}}{\sqrt{ij}}. \tag{1.8}$$

Now, by the definition of τ_n,
(1.9)
$$\tau_n = 1 - \inf\left\{\Big(||\bar{x}||^{-2}\mathcal{Q}_n(\bar{x})\Big) : \ \bar{x} \ne \bar{0}\right\} = 1 - \min\{\mu : \ \mu \,\text{is eigenvalue of}\ \ Q_n\}.$$

This idea to search the minimal eigenvalue of Q_n is implemented in the next Section. Instead of its approximation by an eigenvalue of the appropriate integral operator used in [5], we now involve discrete orthogonal polynomial sequences and find this minimal value.

2. Proof of Theorem

1. We firstly find four eigenvalues of the matrix Q_n. Denote $\bar{a}_r = (a_{r1}, \dots, a_{rn})$ and $\bar{x}_r = (x_{r1}, \dots, x_{rn})$, where $a_{rj} = \sqrt{j}x_{rj}$ and $k = 1, 2, \dots$

The additive function $\ell(\sigma)$ defined via $a_{1j} = j$, where $1 \le j \le n$, equals n for every $\sigma \in \mathbf{S}_n$. Hence $\mathbf{V}_n\ell(\sigma) = 0$ and $B_n(\bar{a}_1) = \Delta_n(\bar{a}_1)$. This gives a hint that $\mu_1 = 1$ is an eigenvalue of Q_n corresponding to the eigenvector $\bar{x}_1 = (1, \dots, \sqrt{j}, \dots, \sqrt{n})$. Indeed, by (1.8), we have $\bar{x}_1Q_n = 1 \cdot \bar{x}_1$.

Further we apply the Gram-Schmidt process orthonormalising a set of vectors $\bar{p}_r = \big(p_r(1), p_r(2), \dots, p_r(n)\big) \in \mathbf{R}^n$, where $p_r(y) \in \mathbf{R}[y]$ are polynomials of degree r and $r = 0, 1, \dots$. For the inner product, we take

$$< \bar{p}_r, \bar{p}_s >:= \sum_{j=1}^{n} jp_r(j)p_s(j).$$

A possible polynomial $p_1(y)$ of degree 1 satisfying $< \bar{p}_0, \bar{p}_1 >= 0$ is

$$p_1(y) = 3y - (2n + 1)$$

if $n \ge 1$. Then we define $a_{2j} = jp_1(j) = \sqrt{j}x_{2j}$, where $1 \le j \le n$. With such a choice we obtain $\bar{x}_2Q_n = (-1/2)\bar{x}_2$ since

$$\sum_{j=1}^{n} q_{ij}x_{2j} = \frac{1}{\sqrt{i}} \sum_{n-i<j\le n} \big(3j - (2n + 1)\big) = -\frac{\sqrt{i}}{2}\big(3i - (2n + 1)\big) = -\frac{x_{2i}}{2}$$

for each $1 \le i \le n$. In other words, $\mu_2 = -1/2$ is the eigenvalue of Q_n if $n \ge 2$.

Let $p_2(y) = by^2 + cy + d$ be such that

$$< \bar{p}_0, \bar{p}_2 >= 0, \qquad < \bar{p}_1, \bar{p}_2 >= 0$$

for every $n \ge 3$. The requirements can be reduced to the system of equations

$$\begin{aligned} s_3 b + s_2 c + s_1 d = 0,\\ \big(3s_4 - (2n+1)s_3\big)b + \big(3s_3 - (2n+1)s_2\big)c + \big(3s_2 - (2n+1)s_1\big)d = 0, \end{aligned}$$

where $s_k := 1^k + 2^k + \cdots + n^k$, $k = 1, 2, \ldots$, are the power sums. Applying the formulas

$$(2.1)\quad s := s_1 = \frac{n(n+1)}{2}, \quad s_2 = \frac{2n+1}{3}s, \quad s_3 = s^2, \quad s_4 = \frac{2n+1}{15}s(6s-1),$$

we modify the previous system to

$$\begin{aligned} 3n(n+1)b + 2(2n+1)c + 6d = 0,\\ 3(2n+1)b + 5c = 0. \end{aligned}$$

Simplifying we may take $c = -6(2n+1)$. Then $b = 10$ and $d = 3n^2 + 3n + 2$. This gives

$$p_2(y) = 10y^2 - 6(2n+1)y + 3n^2 + 3n + 2.$$

We now define the vector $\bar{x}_3$ with coordinates $x_{3j} = \sqrt{j}p_2(j)$, $1 \le j \le n$, and check that

$$\begin{aligned} \sum_{n-j<i\le n} q_{ij}x_{3i} &= \sum_{n-j<i\le n} (bi^2 + ci + d) =\\ &= \frac{1}{3}\big(10j^3 - 6(2n+1)j^2 + 3n^2 + 3n + 2\big) = \frac{1}{3}x_{3j} \end{aligned}$$

if $1 \le j \le n$. Thus $\bar{x}_3$ is the eigenvector corresponding to the value $\mu_3 := 1/3$ of the matrix Q_n if $n \ge 3$.

Finally, in the same manner looking for a polynomial $p_3(y) = by^3 + cy^2 + {}$ $+dy + e$ satisfying

$$< \bar{p}_0, \bar{p}_3 >= 0, \quad < \bar{p}_1, \bar{p}_3 >= 0, \quad < \bar{p}_2, \bar{p}_3 >= 0,$$

we discover the needed choice

$$b = 35, \quad c = -30(2n+1), \quad d = 5(6n^2+6n+5), \quad e = 2(2n+1)(4n^2+4n-3).$$

Then we verify that the vector $\bar{x}_4$ with $x_{4i} = \sqrt{i}(bi^3 + ci^2 + di + e)$, $1 \le i \le n$, satisfies

$$\bar{x}_4 Q_n = -\frac{1}{4}\bar{x}_4.$$

if $n \ge 4$. The eigenvalue $\mu_4 = -1/4$ is found.

Luckily, at this point we may stop the involved calculations. We end the first part of proof by

Remark 2.1. Let $n \geq 1$ and $\mathcal{M}_n = \{\mu_r,\ 1 \leq r \leq n\}$ be the spectrum of matrix Q_n. We guess that

$$\mathcal{M}_n = \big\{(-1)^{j-1}/j, \quad 1 \leq j \leq n\big\}.$$

The matrix trace property

$$\mathrm{trace}(Q_n) = \sum_{j=1}^{n} \mu_j = \sum_{n/2<j\leq n} \frac{1}{j} = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}$$

supports the hypothesis but does not prove it. The computations using the Maple program confirm it for all $n \leq 10$. Moreover, we note that they showed the evidence of the next equivalent form:

$$\mathcal{M}_{n+1} \setminus \mathcal{M}_n = \Big\{\frac{(-1)^n}{n+1}\Big\}, \quad n \geq 1,$$

with the initial equality $\mathcal{M}_1 = \{1\}$.

2. Let us examine the remaining eigenvalues μ_r, where $5 \leq r \leq n$. Maybe, they depend on n. In any case, we claim that they are small in absolute value. To prove this, examine the square of matrix

$$Q_n^2 = \big((u_{ij})\big), \quad 1 \leq i, j \leq n.$$

Then

$$u_{ij} = \sum_{k=1}^{n} q_{ik} q_{kj} = \frac{1}{\sqrt{ij}} \sum_{k=1}^{n} \frac{\mathbf{1}\{i+k>n, j+k>n\}}{k}, \quad 1 \leq i, j \leq n,$$

and

$$\mathrm{trace}(Q_n^2) = \sum_{j=1}^{n} u_{jj} = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=1}^{n} \frac{\mathbf{1}\{j+k>n\}}{k} = \sum_{k=1}^{n} \frac{1}{k^2}.$$

The last equality can be proved applying the induction argument. Using the calculations of Part 1, we obtain

$$\mathrm{trace}(Q_n^2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \sum_{5\leq r\leq n} \mu_r^2.$$

Hence

$$\max_{5\leq r\leq n} \mu_r^2 \leq \sum_{5\leq r\leq n} \mu_r^2 \leq \sum_{k=5}^{\infty} \frac{1}{k^2} < \int_4^{\infty} \frac{du}{u^2} = \frac{1}{4}.$$

Consequently,

$$\min_{5\le r\le n} \mu_r > -\frac{1}{2}.$$

In other words, the eigenvalue $\mu_1 = -1/2$ is the minimal one in the whole spectrum $\mathcal{M}_n$.

This completes the proof of Theorem. ∎

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