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ON SOME RESULTS OF INDLEKOFER FOR MULTIPLICATIVE FUNCTIONS

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Abstract. In this paper we describe some results of Indlekofer for multiplicative functions. Especially we give the definition for the class $\mathcal{F}$ of exp-log functions introduced by Indlekofer in [13]. Further, we compare Indlekofer's results with recent investigations [5, 6] by Granville et. al..

1. Multiplicative function on $\mathbb{N}$

Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function, i.e.

$$f(mn) = f(m)f(n) \text{ for } (m,n) = 1.$$

The mean value of f is defined by

$$M(f) := \lim_{x\to\infty} x^{-1} \sum_{n\le x} f(n)$$

if the limit exists.

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Delange [3] proved in 1961 under the assumption $|f(n)| \leq 1$ for all $n \in \mathbb{N}$ that the mean value $M(f)$ exists and is different from zero if and only if the series

$$\sum_p \frac{1-f(p)}{p} \tag{1.1}$$

converges, and for some positive k, $f(2^k) \neq -1$.

Wirsing [18] showed in 1967, that if f is real-valued and the series (1.1) diverges, then $M(f) = 0$. This implies that $M(f)$ always exists for all real-valued multiplicative function with $|f| \leq 1$.

Halász [7] proved in 1968 the following

Proposition 1.1. *Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative, $|f| \leq 1$. If there exists a real number a_0 so that the series*

$$\sum_p p^{-1}(1 - Ref(p)p^{-ia}) \tag{1.2}$$

converges for $a = a_0$, then, as $x \to \infty$,

$$x^{-1} \sum_{n \leq x} f(n) = \frac{x^{ia_0}}{1+ia_0} \prod_{p \leq x} \left(1 - p^{-1}\right) \left(1 + \sum_{m=1}^{\infty} p^{-m(1+ia_0)} f(p^m)\right) + o(1).$$

If the series (1.2) *diverges for all $a \in \mathbb{R}$, then*

$$x^{-1} \sum_{n \leq x} f(n) = o(1) \quad (x \to \infty).$$

In either case there are constant c, c_0 and a slowly oscillating function $L(u)$ with $|L(u)| = 1$, so that, as $x \to \infty$,

$$x^{-1} \sum_{n \leq x} f(n) = cx^{ia_0} L(\log x) + o(1).$$

The proof of the proposition is based on analytic methods. Elementary proofs of the Halász theorem were given by Daboussi and Indlekofer [1]. A simpler and shorter proof has been shown by Indlekofer in [11].

The wish to abandon the restriction on the size of f led to the investigation of multiplicative functions which belong to the class $\mathcal{L}^\alpha, \alpha \geq 1$. Here, for $1 \leq \alpha < \infty$,

$$\mathcal{L}^\alpha := \{f : \mathbb{N} \to \mathbb{C}, \|f\|_\alpha < \infty\}$$

denotes the linear space of arithmetic functions with bounded seminorm

$$\|f\|_\alpha := \left\{ \limsup_{x\to\infty} x^{-1} \sum_{n\le x} |f(n)|^\alpha \right\}^{1/\alpha}.$$

Obviously the functions considered by Delange, Wirsing and Halász belong to every class $\mathcal{L}^\alpha$.

A characterization of multiplicative functions $f \in \mathcal{L}^\alpha (\alpha > 1)$ which possess a non-zero mean value $M(f)$ was independently given by Elliott and Daboussi in [4] and [2], respectively. Indlekofer [8] introduced the space $\mathcal{L}^*$ of *uniformly summable* functions. $f \in \mathcal{L}^*$ iff $f \in \mathcal{L}^1$ and

$$\lim_{K\to\infty} \sup_{N\ge 1} N^{-1} \sum_{\substack{n\le N\\ |f(n)|>K}} |f(n)| = 0.$$

Obviously

$$\mathcal{L}^\alpha \subsetneqq \mathcal{L}^* \subsetneqq \mathcal{L}^1 \quad if \quad \alpha > 1.$$

The idea of uniform summability turned out to provide the appropriate tools for describing the mean behaviour of multiplicative functions. Indlekofer proved in [8, 9, 10] generalizations of the results of Delange, Wirsing and Halász for multiplicative functions $f \in \mathcal{L}^*$.

In [9] Indlekofer described the connections of uniform summability with the existence of a limit distribution for *real-valued* multiplicative functions and the uniform distribution of *positive valued* multiplicative functions.

To be precise we say that the real-valued f has a *limiting distribution* F_f if the frequencies

$$F_{f,x}(y) := x^{-1} \sum_{\substack{n\le x\\ f(n)\le y}} 1$$

converge to a limiting distribution F_f in the usual probabilistic sense. We call the distribution F_f *degenerate* if $F_f(y) = 0$ for $y < 0$ and $F_f(y) = 1$ for $y \ge 0$, and *nondegenerate* otherwise.

On the other hand, following Erdös, we say that the values of a function $f : \mathbb{N} \to (0, \infty)$ are *uniformly distributed* in $(0, \infty)$ (briefly, f is u.d. in $(0, \infty)$) if $f(n)$ tends to infinity as $n \to \infty$ and if there exists a positive c such that as $y \to \infty$

$$N(y, f) := \sum_{\substack{n\\ f(n)\le y}} 1 \sim cy \quad as \quad y \to \infty.$$

With these notations Indlekofer proved the following three results.

Proposition 1.2. (See [9], Theorem 1.) *Let the real-valued multiplicative function $f \in \mathcal{L}^*$. Then*

(i) *f possesses a limiting distribution F_f if and only if the mean-value $M(|f|)$ exists $\left(\text{and has the value } \int_{-\infty}^{+\infty} |y| dF_f(y) \text{ then}\right)$, and*

(ii) *this limiting distribution is degenerate if and only if $M(|f|) = 0$.*

Proposition 1.3. (See [9], Theorem 2.) *Let $f : \mathbb{N} \to \mathbb{R}$ be multiplicative and uniformly summable. Then the existence of $M(|f|)$ implies the existence of $M(f)$.*

Proposition 1.4. (See [9], Theorem 4.) *Let f be multiplicative and > 0. Then the following assertions are equivalent.*

(i) *$1/f \in \mathcal{L}^*$ and f possesses a non-degenerate limiting distribution.*

(ii) (α) *$f \cdot id$ is uniformly distributed in $(0, \infty)$.*
(β) *There exists a constant $K > 0$ such that*

$$\sum_{\substack{n \le x \\ f(n) \le K}} 1 \gg x \quad for \quad all \quad x > 0.$$

(γ) *For all positive x*

$$\sum_{n \le x} 1/f(n) \ll x.$$

Let us come back to the investigations of $\sum\limits_{n \le x} f(n)$ for multiplicative functions $|f| \le 1$. Indlekofer, Kátai and Wagner [12] used the methods of [11] to compare $\sum\limits_{n \le x} f(n)$ with $\sum\limits_{n \le x} g(n)$ where $g \ge 0$ is multiplicative and $|f| \le g$. They showed

Proposition 1.5. (See [12], Theorem.) *Let g be a multiplicative function which assumes real nonnegative values only. Let*

$$\sum_{p \le x} \frac{\log p}{p} g(p) \sim \tau \log x, \quad x \to \infty,$$

hold with a constant $\tau > 0$. Furthermore, let $g(p) = O(1)$ for all primes p, and let

$$\sum_{p, k \ge 2} p^{-k} g(p^k) < \infty.$$

Besides this, if $\tau \le 1$, then let

$$\sum_{p^k \le x, k \ge 2} g(p^k) = O\left(x(logx)^{-1}\right).$$

Let f be a complex-valued function, which satisfies $|f(n)| \le g(n)$ for every positive integer n. If there exists a real number a_0 such that the series

$$\sum_p p^{-1}(g(p) - Ref(p)p^{-ia}) \tag{1.3}$$

converges for $a = a_0$, then

$$\sum_{n \le x} f(n) = \frac{x^{ia_0}}{1+ia_0} \prod_{p \le x} \left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{m(1+ia_0)}}\right)\left(1 + \sum_{m=1}^{\infty} \frac{g(p^m)}{p^m}\right)^{-1} \times$$
$$\times \sum_{n \le x} g(n) + o\left(\sum_{n \le x} g(n)\right)$$

as $x \to \infty$. If the series (1.3) *diverges for all $a \in \mathbb{R}$, then*

$$\sum_{n \le x} f(n) = o\left(\sum_{n \le x} g(n)\right), \quad x \to \infty.$$

In both cases, there are constants c, a_0 and a slowly oscillating function $\tilde{L}$ with $|\tilde{L}(u)| = 1$ such that, as $x \to \infty$,

$$\sum_{n \le x} f(n) = \left(cx^{ia_0}\tilde{L}(\log x) + o(1)\right) \sum_{n \le x} g(n).$$

As an example let us consider the generalized divisor function d_κ for $\kappa > 0$. Here the multiplicative function d_κ is defined by

$$\sum_{n=1}^{\infty} d_\kappa(n) n^{-s} = \zeta^\kappa(s).$$

It is well known that

$$\sum_{n \le x} d_\kappa(n) \sim c_\kappa x(\log x)^{\kappa-1} \quad \text{as } x \to \infty.$$

Obviously, $g = d_\kappa$ fulfills all conditions of Proposition 1.5. Thus we have

Corollary 1.1. *Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative such that $|f| \leq d_\kappa (\kappa > 0)$. Then, if there exists a real number a_0 such that the series*

$$\sum_p p^{-1}(\kappa - Ref(p)p^{-ia}) \tag{1.4}$$

converges for $a = a_0$, then

$$\sum_{n\leq x} f(n) = \frac{x^{ia_0}}{1+ia_0} \prod_{p\leq x} \left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{m(1+ia_0)}}\right)\left(1-\frac{1}{p}\right)^{\kappa} \sum_{n\leq x} d_\kappa(n) + \\ +o\left(\sum_{n\leq x} d_\kappa(n)\right)$$

as $x \to \infty$. If the series (1.4) diverges for all $a \in \mathbb{R}$, then

$$\sum_{n\leq x} f(n) = o\left(\sum_{n\leq x} d_\kappa(n)\right), \quad x \to \infty.$$

In both cases, there are constants c, a_0 and a slowly oscillating function $\tilde{L}$ with $|\tilde{L}(u)| = 1$ such that, as $x \to \infty$,

$$\sum_{n\leq x} f(n) = \left(cx^{ia_0}\tilde{L}(\log x) + o(1)\right) \sum_{n\leq x} d_\kappa(n).$$

Remark 1.1. In a recent paper Granville et.al. [6] gave upper estimates for $\sum_{n\leq x} f(n)$ where $f : \mathbb{N} \to \mathbb{C}$ is multiplicative and $|f| \leq d_\kappa$.

2. Multiplicative function on additive arithmetical semigroups

Let (G, ∂) be an additive arithmetical semigroup that is, by definition, G is a free abelian semigroup with identity element 1 such that G has a countable free generating set $\mathcal{P}$ of "primes" and $\partial : G \to \mathbb{N} \cup \{0\}$ is a "degree mapping" satisfying

(i) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,

(ii) the total number $G(n)$ of elements of degree n in G is finite for each $n \geq 0$.

In particular, if we assume $G(n) \ll q^n n^\varrho$ with some ϱ and $q > 1$ then

$$\hat{Z}(z) := \sum_{n=0}^{\infty} G(n) z^n = \prod_{m=1}^{\infty} (1 - z^m)^{-P(m)}$$

is the zeta function associated with G, where $P(m)$ denotes the total number of primes of degree m in G. (See for details Knopfmacher [15], Knopfmacher and Zhang [16]).

Obviously

$$\log \prod_{m=1}^{\infty} (1 - z^m)^{-P(m)} = \sum_{m=1}^{\infty} P(m) \sum_{j=1}^{\infty} j^{-1} z^{jm} =$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{d|m} dP(d) z^m = \sum_{m=1}^{\infty} \frac{\bar{\Lambda}(m)}{m} z^m,$$

where

$$\bar{\Lambda}(m) = \sum_{d|m} dP(d).$$

Then, since $P(d) \le G(d) \ll q^d d^\varrho$,

$$\begin{aligned} \bar{\Lambda}(m) &= mP(m) + O\left(mG(\frac{m}{2}) \sum_{r \le m} \frac{1}{r} \right) = \\ &= mP(m) + O\left(mq^{\frac{m}{2}} (\frac{m}{2})^\varrho \log m \right). \end{aligned}$$

Putting $y = qz, \lambda(m) = q^{-m} \bar{\Lambda}(m)$ and $\gamma(n) = q^{-n} G(n)$ leads to

$$Z(y) := \hat{Z}(yq^{-1}) = \sum_{n=0}^{\infty} \gamma(n) y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m \right).$$

Observe

$$\frac{\lambda(m)}{m} = q^{-m} \sum_{\substack{p \in \mathcal{P} \\ \partial(p) = m}} 1 + O\left(q^{-m/2} m^{\varrho \log m} \right).$$

Let $\tilde{f} : G \to \mathbb{C}$ be a multiplicative function on G, i.e. $\tilde{f}(1) = 1$ and

$$\tilde{f}(ab) = \tilde{f}(a) \tilde{f}(b) \;\; \text{for all coprime } a, b \in G.$$

Put

$$f(n) := q^{-n} \sum_{\substack{a \in G \\ \partial(a) = n}} \tilde{f}(a).$$

Then the generating function of f is given by

$$\begin{aligned} F(y) &:= \sum_{n=0}^{\infty} f(n)y^n = \sum_{a\in G} \tilde{f}(a)q^{-\partial(a)}y^{\partial(a)} = \\ &= \prod_p \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k)q^{-k\partial(p)}y^{k\partial(p)}\right) = \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m} y^m\right). \end{aligned}$$

This holds at least in a *formal* sense since $f(0) = 1$ $(\neq 0)$. It is also valid for complex values $y, |y| < 1$ in terms of *ordinary* convergence if, for example, the function $\tilde{f}$ is multiplicative of modulus ≤ 1. Then $|\lambda_f(m)| \leq \lambda(m)$ and $|f(n)| \leq \gamma(n)$ for all $m, n \in \mathbb{N}$.

Furthermore, here we consider additive arithmetical semigroups satisfying

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n)z^n = \frac{H(z)}{(1-qz)^{\delta}} = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} q^m z^m\right) \tag{2.1}$$

where δ is a positive number, and we assume that $0 \leq \lambda(m) \ll 1$, $H(z) = O(1)$ for $|z| < q^{-1}$ and

$$\lim_{z \nearrow q^{-1}} H(z) = A > 0.$$

It is well known [14] that under these conditions

$$q^{-n}G(n) \sim A\frac{n^{\delta-1}}{\Gamma(\delta)} \quad as \quad n \to \infty.$$

Example 2.1. Let $F_q[X]$ denote the polynomial ring in an indeterminate X over the finite Galois field F_q with q elements (q prime power). The subset $G_q = G(q, X)$ consisting of all monic polynomials in $F_q[X]$ forms a semigroup under multiplication. In particular, G_q together with the usual degree mapping on polynomials defines an additive arithmetical semigroup such that

$$G_q(n) = q^n \quad (n = 0, 1, 2, \ldots).$$

The generating function $\hat{Z}_q$ of G_q is given by ($|z| < q^{-1}$)

$$\hat{Z}_q(z) = \sum_{n=0}^{\infty} G_q(n)z^n = \frac{1}{1-qz} = \exp\left(\sum_{m=1}^{\infty} \frac{q^m}{m} z^m\right). \tag{2.2}$$

For more general investigations Indlekofer [13] introduced the class $\mathcal{F}$ of exp-log functions. For this let

$$Z(y) = \sum_{n=0}^{\infty} \gamma(n) y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m \right) \tag{2.3}$$

be holomorphic for $|y| < 1$ where

$$0 \leq \lambda(m) = O(1), \quad m \in \mathbb{N}, \tag{2.4}$$

and

$$|Z(y)| \ll Z(|y|) \left| \frac{1-|y|}{1-y} \right|^{\varepsilon}, \quad (|y| < 1) \tag{2.5}$$

for some $\varepsilon > 0$. Further, putting

$$B(n) = \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m} \right),$$

we assume that

$$n\gamma(n) \asymp B(n) \tag{2.6}$$

and

$$B(m) = o(B(n)) \ \ if \ \ m = o(n), \quad n \to \infty. \tag{2.7}$$

Then we say that the function Z given in (2.3) belongs to te exp-log class $\mathcal{F}$ in case $(2.4), (2.5), (2.6)$ and (2.7) hold.

Example 2.2. The generating functions $Z(y) = \hat{Z}(q^{-1}z)$ (see (2.1)) of additive arithmetical semigroups belong to the class $\mathcal{F}$. Observe that, for $r = 1 - \frac{1}{n}$,

$$\begin{aligned} B(n) = \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m} \right) \quad &\asymp \quad \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m} r^m \right) \\ &\asymp \quad Z(r) \asymp (1-r)^{-\delta} = n^{\delta}, \end{aligned}$$

which implies

$$\frac{B(m)}{B(n)} \asymp (\frac{m}{n})^{\delta} = o(1) \ \ if \ \ m = o(n).$$

As a further example we mention (see [13], Example 4)

$$Z(y) = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m \right)$$

where
$$0 < c_1 \le \lambda(m) \le c_2 < \infty \quad (m \in \mathbb{N}).$$
Then, obviously
$$\begin{aligned}
|Z(y)| &= Z(|y|)\exp\left(\sum_{m=1}^{\infty}\frac{\lambda(m)}{m}|y|^m(\cos(mt)-1)\right) \le \\
&\le Z(|y|)\exp\left(c_1\sum_{m=1}^{\infty}\frac{|y|^m}{m}(\cos(mt)-1)\right) = \\
&= Z(|y|)\left|\frac{1-|y|}{1-y}\right|^{c_1}
\end{aligned}$$
and
$$\frac{B(m)}{B(n)} = \exp\left(-\sum_{m<l\le n}\frac{\lambda(l)}{l}\right) \ll \exp\left(c_1\log\frac{m}{n}\right) = o(1)$$
if $m = o(n)(n \to \infty)$. Elementary estimates immediately yield
$$n\gamma(n) \asymp B(n),$$
where the constants involved in $\asymp$ only depend on c_1 and c_2 (see Manstavičius [17], Lemma 3.1).

Now, if the function

$$F(y) = \sum_{n=0}^{\infty} f(n)y^n = \exp\left(\sum_{m=1}^{\infty}\frac{\lambda_f(m)}{m}y^m\right) \tag{2.8}$$

($|y| < 1$) is given then the following result holds.

Theorem 2.1. *Let Z be an element of the exp-log class $\mathcal{F}$ and let $F(y)$ in (2.8) satisfy $|\lambda_f(m)| \le \lambda(m)$ for all $m \in \mathbb{N}$. Then the following two assertions hold*

(i) *Let*

$$\sum_{m=1}^{\infty}\frac{\lambda(m) - Re\lambda_f(m)e^{ima}}{m} \tag{2.9}$$

converge for some $a \in \mathbb{R}$. Put
$$A_n := \exp\left(-ina + \sum_{m\le n}\frac{\lambda_f(m)e^{ima} - \lambda(m)}{m}\right).$$
Then
$$f(n) = A_n\gamma(n) + o(\gamma(n)) \quad as \quad n \to \infty.$$

(ii) *Let* (2.9) *diverge for all* $a \in \mathbb{R}$. *Then*

$$f(n) = o(\gamma(n)) \quad as \quad n \to \infty.$$

We apply Theorem 2.1 to multiplicative functions on G_q, where G_q is the additive arithmetical semigroup of monic polynomials over the Galois field with q elements.

Define the generalized divisor function $\bar{d}_k$ on G_q by

$$\begin{aligned}\sum_{n=0}^{\infty} d_\kappa(n) z^n &= (\hat{Z}_q(z))^\kappa = \frac{1}{(1-qz)^\kappa} = \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{\kappa q^m}{m} z^m\right),\end{aligned}$$

where $d_\kappa(n) = \sum\limits_{\substack{a\in G_q\\ \partial(a)=n}} \bar{d}_\kappa(a)$. Obviously

$$\begin{aligned} q^{-n} d_\kappa(n) &= \binom{\kappa+n-1}{n} \sim \\ &\sim \frac{n^{\kappa-1}}{\Gamma(\kappa)} \; as \; n \to \infty. \end{aligned}$$

Corollary 2.1. *Let* $\kappa > 0$. *Let* $\tilde{f} : G_q \to \mathbb{C}$ *be multiplicative such that* $|\lambda_f(m)| \leq \kappa$ *for all* $m \in \mathbb{N}$. *Then the following two assertions hold.*

(i) *Let*

$$\sum_{m=1}^{\infty} \frac{\kappa - Re\lambda_f(m) e^{ima}}{m} \tag{2.10}$$

converge for some $a \in \mathbb{R}$. *Put*

$$A_n := \exp\left(-ina + \sum_{m\leq n} \frac{Re\lambda_f(m)e^{ima} - \kappa}{m}\right).$$

Then

$$f(n) := q^{-n} \sum_{\substack{a\in G_q\\ \partial(a)=n}} \bar{f}(a) = A_n \frac{n^{\kappa-1}}{\Gamma(\kappa)} + o(n^{\kappa-1}).$$

(ii) *Let* (2.10) *diverge for all* $a \in \mathbb{R}$. *Then*

$$f(n) = o(n^{\kappa-1}) \quad as \quad n \to \infty.$$

Remark 2.1. In a recent paper Granville et. al. [5] investigated upper estimates for $q^{-n} \sum\limits_{\substack{a\in G_q\\ \partial(a)=n}} \bar{f}(a)$ where $|\lambda_f(m)| \leq \kappa$ for all $m \in \mathbb{N}$.

References

[1] **Daboussi H., and K.-H. Indlekofer,** Two elementary proofs of Halász's theorem, *Math. Z.*, **209** (1990), 43–52.

[2] **Daboussi H.,** Sur les fonctions multiplicatives ayant une valeurmoyenne non nulle, *Bull. Soc. Math. France*, **109** (1981), 183–206.

[3] **Delange, H.,** Sur les fonctions arithmétiques multiplicatives, *Ann. Sci. École* Norm. Sup. (3), **78** (1961), 273–304.

[4] **Elliott, P.D.T.A.,** Mean value theorems for multiplicative functions bounded in mean α-power, $\alpha > 1$, *J. Australian Math. Soc. (Ser. A)*, **29** (1980), 177–205.

[5] **Granville A., A.J. Harper and K. Soundarajan,** Mean values of multiplicative functions over function fields, *Research in number theory* 1:25, (2015), doi:10.1007/s40993-015-0023-5.

[6] **Granville A., A.J. Harper and K. Soundarajan,** A new proof of Halász's theorem, and its consequences, *eprint arXiv:1706.03749, origin Arxiv*, (2017), 1–36.

[7] **Halász G.,** Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta. Math. Acad. Sci. Hung.*, **19** (1968), 365–403.

[8] **Indlekofer, K.-H.,** A mean-value theorem of multiplicative arithmetical function, *Math. Z.*, **172**, (1980), 255–271.

[9] **Indlekofer, K.-H.,** Limiting distributions and mean-values of multiplicative arithmetical functions, *J. Reine Angew. Math.*, **328**, (1981), 116–127.

[10] **Indlekofer, K.-H.,** Properties of uniformly summable multiplicative functions, *Period. Math. Hungar.*, **17**, (1986), 143–161.

[11] **Indlekofer, K.-H.,** Remarks on an elementary proof of Hálasz's theorem, *Liet. Mat. Rink.*, **33**, (1993), 417–423.

[12] **Indlekofer, K.-H., I. Kátai and R. Wagner,** A comparative result for multiplicative functions, *Lithuanian Mathematical Journal*, **41**, No. 2, (2001), 143–157.

[13] **Indlekofer, K.-H.,** Tauberian theorems with applications to arithmetical semigroups and probabilistic combinatorics, *Annales Univ. Sci. Budapest., Sect. Comp.*, **34**, (2011), 135–177.

[14] **Indlekofer, K.-H.,** Remarks on Tauberian theorems for exp-log functions, *Šiauliai Math. Semin.*, **8** 16, (2013), 83–93.

[15] **Knopfmacher, J.,** *Analytic Arithmetic of Algebraic Function Fields*, Marcel Dekker, New York, 1979.

[16] **Knopfmacher, J. and W.-B. Zhang,** *Number Theory Arising from Finite Fields. Analytic and Probabilistic Theory* **241**, Marcel Dekker, New York, 2001.

[17] **Manstavičius, E.,** An analytic method in probabilistic combinatorics, *Osaka J. Math.*, **46**, (2009), 273–290.

[18] **Wirsing, E.,** Das asymptotische Verhalten von Summen über multiplikative Funktionen II, *Acta. math. Sci. Hungar.*, **18**, (1967), 411–467.

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