

JOINT DISCRETE MIXED UNIVERSALITY  
OF  $L$ -FUNCTIONS OF NEW FORMS  
AND PERIODIC HURWITZ ZETA-FUNCTIONS

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**Abstract.** In the paper, we study the analytic properties of the pair consisting of the  $L$ -functions of new forms and periodic Hurwitz zeta-functions with a transcendental parameter, i.e., we investigate the mixed joint discrete value distribution of this tuple in two cases. For this purpose, certain arithmetical conditions are used. Also, we give some applications of these limit theorems.

## 1. Introduction

In the theory of zeta- and  $L$ -functions, one interesting problem is so called mixed joint value distribution problem, especially its discrete case.

In 2007, H. Mishou obtained the first result related to the mixed joint value-distribution of the Riemann zeta-function  $\zeta(s)$  and the Hurwitz zeta-function  $\zeta(s, \alpha)$  with transcendental parameter  $\alpha$  (see [9]). Discrete versions of this result were proved by E. Buivydas and A. Laurinćikas in 2015 (see [3], [2]) (recall that in the discrete case, the imaginary part of complex variable takes values from some specially constructed set, and we study the cardinality of

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the set). In 2016, A. Laurinčikas obtains more general discrete result for the periodic zeta-function  $\zeta(s; \mathfrak{A})$  and periodic Hurwitz zeta-function  $\zeta(s, \alpha; \mathfrak{B})$  (see [7]).

Let  $\mathbb{N}, \mathbb{N}_0, \mathbb{P}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  be the set of all positive integers, non-negative integers, prime numbers, rational numbers, real numbers and complex numbers, respectively, and by  $s = \sigma + it$  denote the complex variable. Let  $\mathfrak{A} = \{a_m : m \in \mathbb{N}\}$  and  $\mathfrak{B} = \{b_m : m \in \mathbb{N}_0\}$  be the periodic sequences of complex numbers  $a_m$  and  $b_m$  with minimal positive periods  $k \in \mathbb{N}$  and  $l \in \mathbb{N}$ , respectively. The periodic zeta-function  $\zeta(s; \mathfrak{A})$  and the periodic Hurwitz zeta-function  $\zeta(s, \alpha; \mathfrak{B})$  with parameter  $\alpha$ ,  $0 < \alpha \leq 1$ , for  $\sigma > 1$ , are defined by the Dirichlet series

$$\zeta(s; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \zeta(s, \alpha; \mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s},$$

respectively. The periodicity of the sequences  $\mathfrak{A}$  and  $\mathfrak{B}$  implies that, for  $\sigma > 1$ ,

$$\zeta(s; \mathfrak{A}) = \frac{1}{k^s} \sum_{q=1}^k a_q \zeta\left(s, \frac{q}{k}\right) \quad \text{and} \quad \zeta(s, \alpha; \mathfrak{B}) = \frac{1}{l^s} \sum_{r=0}^{l-1} b_r \zeta\left(s, \frac{r + \alpha}{l}\right).$$

Thus, combining with the properties of the classical Hurwitz zeta-function  $\zeta(s, \alpha)$ , can be shown that these equalities give an analytic continuation for both functions to the whole complex plane, except possibly, for simple pole at the point  $s = 1$  with residues

$$a := \frac{1}{k} \sum_{q=1}^k a_q \quad \text{and} \quad b := \frac{1}{l} \sum_{r=0}^{l-1} b_r,$$

respectively. If  $a = 0$  and  $b = 0$ , the functions  $\zeta(s; \mathfrak{A})$  and  $\zeta(s, \alpha; \mathfrak{B})$  are entire.

Denote by  $\gamma$  the unit circle on the complex plane  $\mathbb{C}$ , i.e.,  $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ , and

$$\Omega = \Omega_1 \times \Omega_2,$$

where  $\Omega_1$  and  $\Omega_2$  are two tori given by

$$\Omega_1 := \prod_{p \in \mathbb{P}} \gamma_p \quad \text{and} \quad \Omega_2 := \prod_{m=0}^{\infty} \gamma_m$$

with  $\gamma_p = \gamma$  for all  $p \in \mathbb{P}$  and  $\gamma_m = \gamma$  for all  $m \in \mathbb{N}_0$ , respectively. Since, by the classical Tikhonov theorem, both tori  $\Omega_1$  and  $\Omega_2$  are compact topological Abelian groups with respect to the product topology and pointwise

multiplication, then the torus  $\Omega$  is a compact topological Abelian group also. Therefore, on  $(\Omega, \mathcal{B}(\Omega))$ , a probability Haar measure  $m_H$  can be defined as a product of respective Haar measures  $m_{H_1}$  and  $m_{H_2}$  on the spaces  $(\Omega_1, \mathcal{B}(\Omega_1))$  and  $(\Omega_2, \mathcal{B}(\Omega_2))$  (here, by standard notion,  $\mathcal{B}(S)$  denotes the Borel  $\sigma$ -field of the topological space  $S$ ). This leads to the probability space  $(\Omega, \mathcal{B}(\Omega), m_H)$ .

For  $p \in \mathbb{P}$ , denote by  $\omega_1(p)$  the projection of  $\omega_1 \in \Omega_1$  to the coordinate space  $\gamma_p$ , and (taking into account the decomposition of  $m \in \mathbb{N}$  into prime divisors  $p^{\beta_p}$ ) define

$$\omega_1(m) = \prod_{p \in \mathbb{P}} \omega_1(p)^{\beta_p},$$

while, for an element  $\omega_2 \in \Omega_2$ , its projection to the coordinate space  $\gamma_m$  denote by  $\omega_2(m)$ ,  $m \in \mathbb{N}_0$ . Both  $\omega_1 \in \Omega_1$  and  $\omega_2 \in \Omega_2$  are complex-valued random elements defined on the spaces  $(\Omega_1, \mathcal{B}(\Omega_1), m_{H_1})$  and  $(\Omega_2, \mathcal{B}(\Omega_2), m_{H_2})$ , respectively.

Denote by  $H(G)$  the space of analytic on certain region  $G$  functions equipped with the product topology of uniform convergence on compacta. Let  $D_\zeta := \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ . Moreover, for  $s \in D_\zeta$  and  $\omega := (\omega_1, \omega_2) \in \Omega$ , on the probability space  $(\Omega, \mathcal{B}(\Omega), m_H)$ , denote an  $H^2(D_\zeta)$ -valued random element  $\underline{\zeta}^*(s, \alpha, \omega; \mathfrak{A}, \mathfrak{B})$  by the formula

$$\underline{\zeta}^*(s, \alpha, \omega; \mathfrak{A}, \mathfrak{B}) = (\zeta(s, \omega_1; \mathfrak{A}), \zeta(s, \alpha, \omega_2; \mathfrak{B}))$$

with

$$\zeta(s, \omega_1; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m)}{m^s}$$

and

$$(1.1) \quad \zeta(s, \alpha, \omega_2; \mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m + \alpha)^s}.$$

Let  $P_{\underline{\zeta}^*}$  be the distribution of this element, i.e.,

$$P_{\underline{\zeta}^*}(A) = m_H \{ \omega \in \Omega : \underline{\zeta}^*(s, \alpha, \omega; \mathfrak{A}, \mathfrak{B}) \in A \}, \quad A \in \mathcal{B}(H^2(D_\zeta)).$$

Let  $N > 0$ . On  $(H^2(D_\zeta), \mathcal{B}(H^2(D_\zeta)))$ , define

$$P_N^*(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \underline{\zeta}^*(s + ik\underline{h}, \alpha; \mathfrak{A}, \mathfrak{B}) \in A \}$$

for  $A \in \mathcal{B}(H^2(D_\zeta))$  with  $\underline{h} := (h_1, h_2)$ ,  $h_1 > 0$ ,  $h_2 > 0$ , and

$$\underline{\zeta}^*(s + ik\underline{h}, \alpha; \mathfrak{A}, \mathfrak{B}) := (\zeta(s + ikh_1; \mathfrak{A}), \zeta(s + ikh_2, \alpha; \mathfrak{B})).$$

Let

$$\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi) := \{h_1 \log p : p \in \mathbb{P}\} \cup \{h_2 \log(m + \alpha) : m \in \mathbb{N}_0\} \cup \{\pi\}.$$

Then, for the pair  $(\zeta(s; \mathfrak{A}), \zeta(s, \alpha; \mathfrak{B}))$ , following statement is proved.

**Theorem 1.1** (Theorem 3.1, [7]). *Suppose that the set  $\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi)$  is linearly independent over the field of rational numbers  $\mathbb{Q}$ , and the sequence  $\mathfrak{A}$  is multiplicative. Then  $P_N^*$  converges weakly to the probability measure  $P_{\underline{\zeta}^*}$  as  $N \rightarrow \infty$ .*

**Remark 1.1.** The continuous limit theorem in the sense of the weakly convergent probability measures for the tuple  $(\zeta(s; \mathfrak{A}), \zeta(s, \alpha; \mathfrak{B}))$  was obtained by the author and A. Laurinćikas (see [4], Theorem 6).

## 2. Statement of results

The aim of this note is to prove joint mixed limit theorems for the pair consisting of the  $L$ -function for new forms and the periodic Hurwitz zeta-functions.

Let  $SL(2, \mathbb{Z})$  be the full modular group, and  $F(z)$  be a holomorphic cusp form of weight  $\kappa$ ,  $\kappa \in \mathbb{N}$ , for the congruence subgroup

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 \pmod{q} \right\}.$$

Also suppose additionally that  $F(z)$  is normalized Hecke eigenform. An analytic in the upper half-plane  $\Im z > 0$  function  $F(z)$  is called a parabolic form of the weight  $\kappa$  and level  $q$  if

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^\kappa F(z)$$

for all matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ , and  $F(z)$  is analytic and vanishes at the parabolic points. Then the function  $F(z)$  has at  $\infty$  the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) \exp\{2\pi imz\}.$$

We say that  $F(z)$  is a new form if it is not a parabolic form of level less than  $q$  and is an eigenfunction of all Hecke operators. Then  $c(1) \neq 0$ , and it can be assumed that  $c(1) = 1$ .

We consider the associated  $L$ -function  $L(s, F)$  given by the Dirichlet series

$$L(s, F) := \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

This series converges absolutely for  $\sigma > \frac{\kappa+1}{2}$ . Moreover, the function  $L(s, F)$  can be analytically continued to whole complex plane as an entire function.

It is well known that, in the half-plane  $\sigma > \frac{\kappa+1}{2}$ , the function  $L(s, F)$  can be expressed via the Euler product over primes, i.e.,

$$L(s, F) = \prod_{p|q} \left(1 - \frac{c(p)}{p^s}\right)^{-1} \prod_{p \nmid q} \left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s-\kappa+1}}\right)^{-1}.$$

Let  $D_L := \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$ . On the probability space  $(\Omega_1, \mathcal{B}(\Omega_1), m_{H1})$ , define an  $H(D_L)$ -valued random element  $L(s, \omega_1, F)$  by the formula

$$(2.1) \quad L(s, \omega_1, F) = \prod_{p|q} \left(1 - \frac{c(p)\omega_1(p)}{p^s}\right)^{-1} \prod_{p \nmid q} \left(1 - \frac{c(p)\omega_1(p)}{p^s} + \frac{\omega_1^2(p)}{p^{2s-\kappa+1}}\right)^{-1}.$$

Now let  $H_{L,\zeta} := H(D_L) \times H(D_\zeta)$ . On the probability space  $(\Omega, \mathcal{B}(\Omega), m_H)$ , define the  $H_{L,\zeta}$ -valued random element  $\zeta(s, \alpha, \omega, F, \mathfrak{B})$  by the formula

$$\zeta(s, \alpha, \omega, F, \mathfrak{B}) = (L(s, \omega_1, F), \zeta(s, \alpha, \omega_2; \mathfrak{B})),$$

where  $L(s, \omega_1, F)$  and  $\zeta(s, \alpha, \omega_2; \mathfrak{B})$  are defined by the formulas (2.1) and (1.1), respectively. Note that the series for  $L(s, \omega_1, F)$  converges uniformly on compact subsets of the strip  $D_L$  for almost all  $\omega_1 \in \Omega_1$ , while for  $\zeta(s, \alpha, \omega_2; \mathfrak{B})$  we have similar situation on the strip  $D_\zeta$  for almost all  $\omega_2 \in \Omega_2$ . Thus both  $L(s, \omega_1, F)$  and  $\zeta(s, \alpha, \omega_2; \mathfrak{B})$  define respectively  $H(D_L)$ - and  $H(D_\zeta)$ -valued random elements on the probability spaces  $(\Omega_1, \mathcal{B}(\Omega_1), m_{H1})$  and  $(\Omega_2, \mathcal{B}(\Omega_2), m_{H2})$ . These facts allow us to study  $H_{L,\zeta}$ -valued random element  $\zeta(s, \alpha, \omega, F, \mathfrak{B})$ . Suppose that  $P_{L,\zeta}$  is the distribution of above constructed random element  $\zeta(s, \alpha, \omega, F, \mathfrak{B})$  or in other words  $P_{L,\zeta}$  is a probability Haar measure, for  $A \in \mathcal{B}(H_{L,\zeta})$ , given by

$$P_{L,\zeta}(A) := m_H(\omega \in \Omega : \zeta(s, \alpha, \omega, F, \mathfrak{B}) \in A).$$

The first result of this paper is the joint mixed discrete limit theorem for  $P_N^h$  defined by

$$P_N^h(A) := \frac{1}{N+1} \# \{0 \leq k \leq N : \zeta(s + ikh, \alpha, F, \mathfrak{B}) \in A\}, \quad A \in \mathcal{B}(H_{L,\zeta}),$$

where we take the same positive value of the step  $h$  of arithmetical progression for both functions, and

$$\underline{\zeta}(s + ikh, \alpha, F, \mathfrak{B}) := (L(s + ikh, F), \zeta(s + ikh, \alpha; \mathfrak{B})).$$

We slightly modify the set  $\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi)$  also, i.e., we put

$$\mathcal{L}_1(\mathbb{P}, \alpha, h, \pi) := \{\log p : p \in \mathbb{P}\} \cup \{\log(m + \alpha) : m \in \mathbb{N}_0\} \cup \left\{ \frac{\pi}{h} \right\}.$$

**Theorem 2.1.** *Suppose that the set  $\mathcal{L}_1(\mathbb{P}, \alpha, h, \pi)$  is linearly independent over  $\mathbb{Q}$ . Then  $P_N^h$  converges weakly to the probability measure  $P_{L, \zeta}$  as  $N \rightarrow \infty$ .*

The second and main result of this paper is more general as above presented theorem, i.e., we choose two different steps  $h_1$  and  $h_2$  of arithmetical progression for both functions  $L(s, F)$  and  $\zeta(s, \alpha; \mathfrak{B})$ , respectively, and the vector  $\underline{h}$  define as for the functions  $\zeta(s; \mathfrak{A})$  and  $\zeta(s, \alpha; \mathfrak{B})$  in Theorem 1.1. Now, for this purpose, we define

$$P_N(A) := \frac{1}{N+1} \# \{0 \leq k \leq N : \underline{\zeta}(s + ik\underline{h}, \alpha, F, \mathfrak{B}) \in A\}, \quad A \in \mathcal{B}(H_{L, \zeta}),$$

where

$$\underline{\zeta}(s + ik\underline{h}, \alpha, F, \mathfrak{B}) := (L(s + ikh_1, F), \zeta(s + ikh_2, \alpha; \mathfrak{B})).$$

**Theorem 2.2.** *Suppose that the set  $\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi)$  is linearly independent over  $\mathbb{Q}$ . Then  $P_N$  converges weakly to the probability measure  $P_{L, \zeta}$  as  $N \rightarrow \infty$ .*

**Remark 2.1.** The continuous joint mixed limit theorem for the tuple  $(L(s, F), \zeta(s, \alpha; \mathfrak{B}))$  follows from the more general joint theorem for the class of Matsumoto zeta-functions and periodic Hurwitz zeta-functions. It is proved by the author and K. Matsumoto (see [5], Theorem 2.1).

### 3. Proof of Theorems 2.1 and 2.2

We start with the proof of Theorem 2.2. It is similar to that of Theorem 1.1, i.e., Theorem 3.1 from [7]. Therefore, so we just give a sketch of the proof, and only separate two auxiliary results which play an essential role.

First of them is a joint discrete mixed limit theorem on the torus.

**Lemma 3.1.** *Suppose that the set  $\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi)$  is linearly independent over  $\mathbb{Q}$ . Then, for  $A \in \mathcal{B}(\Omega)$ , the measure*

$$\frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \left( (p^{-ikh_1} : p \in \mathbb{P}), ((m+\alpha)^{-ikh_2} : m \in \mathbb{N}_0) \right) \in A \right\}$$

converges weakly to the Haar measure  $m_H$  as  $N \rightarrow \infty$ .

Next result is devoted to some ergodic transformation, and will be used for the identification of the explicit form of the limit measure. Let  $a_{\alpha, h_1, h_2} := \{(p^{-ih_1} : p \in \mathbb{P}), ((m+\alpha)^{-ih_2} : m \in \mathbb{N}_0)\}$  be an element of  $\Omega$ . It is known that the Haar measure  $m_H$  is an invariant with the respect to translations by points from  $\Omega$ . Then, for  $\omega \in \Omega$ , on  $(\Omega, \mathcal{B}(\Omega), m_H)$ , we define the measurable measure preserving transformation of the torus  $\Omega$  by the formula

$$\Phi_{\alpha, h_1, h_2}(\omega) := a_{\alpha, h_1, h_2} \omega.$$

The set  $A \in \mathcal{B}(\Omega)$  is invariant with respect to the transformation  $\Phi_{\alpha, h_1, h_2}$  if the sets  $A$  and  $\Phi_{\alpha, h_1, h_2}(A)$  differ one from other at most by the set of zero  $m_H$ -measure, and the transformation  $\Phi_{\alpha, h_1, h_2}$  is ergodic if its  $\sigma$ -field of invariant sets consists of sets having  $m_H$ -measure 0 or 1.

**Lemma 3.2.** *Suppose that the set  $\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi)$  is linearly independent over  $\mathbb{Q}$ . Then the transformation  $\Phi_{\alpha, h_1, h_2}$  is ergodic.*

For the proof of Lemmas 3.1 and 3.2, see [7].

Now lets back to main steps in the proof of Theorem 2.2.

1. For the fixed  $\sigma_0 > \frac{1}{2}$ , let

$$v_n(m, n) = \exp\left(-\left(\frac{m}{n}\right)^{\sigma_0}\right) \quad \text{and} \quad v_n(m, n, \alpha) = \exp\left(-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_0}\right)$$

with  $m, n \in \mathbb{N}$  and  $m, n \in \mathbb{N}_0$ , respectively. For  $n \in \mathbb{N}$ , define the function

$$\zeta_n(s + ikh, \alpha, F, \mathfrak{B}) = (L_n(s + ikh_1, F), \zeta_n(s + ikh_2, \alpha; \mathfrak{B}))$$

with

$$L_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m, n)}{m^s}, \quad s \in D_L,$$

and

$$\zeta_n(s, \alpha; \mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m v_n(m, n, \alpha)}{(m+\alpha)^s}, \quad s \in D_\zeta.$$

These series converge absolutely for  $\sigma > \frac{\kappa}{2}$  and  $\sigma > \frac{1}{2}$ , respectively. Moreover, for  $\widehat{\omega} := (\widehat{\omega}_1, \widehat{\omega}_2) \in \Omega$ , define the function

$$\underline{\zeta}_n(s + ik\underline{h}, \alpha, \widehat{\omega}, F, \mathfrak{B}) = (L_n(s + ikh_1, \widehat{\omega}_1, F), \zeta_n(s + ikh_2, \alpha, \widehat{\omega}_2; \mathfrak{B}))$$

with

$$L_n(s, \widehat{\omega}_1, F) = \sum_{m=1}^{\infty} \frac{c(m)\widehat{\omega}_1(m)v_n(m, n)}{m^s}, \quad s \in D_L,$$

and

$$\zeta_n(s, \alpha, \widehat{\omega}_2; \mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m\widehat{\omega}_2(m)v_n(m, n, \alpha)}{(m + \alpha)^s}, \quad s \in D_\zeta.$$

The series for  $L(s, \widehat{\omega}_1, F)$  and  $\zeta(s, \alpha, \widehat{\omega}_2; \mathfrak{B})$  also converges for  $\sigma > \frac{\kappa}{2}$  and  $\sigma > \frac{1}{2}$ , respectively.

Now, using Lemma 3.1, it can be proved that the measures

$$\frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \underline{\zeta}_n(s + ik\underline{h}, \alpha, F, \mathfrak{B}) \in A \right\}, \quad A \in \mathcal{B}(H_{L, \zeta}),$$

and

$$\frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \underline{\zeta}_n(s + ik\underline{h}, \alpha, \widehat{\omega}, F, \mathfrak{B}) \in A \right\}, \quad A \in \mathcal{B}(H_{L, \zeta}),$$

both converge weakly to the same probability measure  $P_n$  on  $(H_{L, \zeta}, \mathcal{B}(H_{L, \zeta}))$  as  $N \rightarrow \infty$ .

2. Next we approximate by mean the tuples  $\underline{\zeta}(s + ik\underline{h}, \alpha, F, \mathfrak{B})$  and  $\underline{\zeta}(s + ik\underline{h}, \alpha, \omega, F, \mathfrak{B})$  by the tuples  $\underline{\zeta}_n(s + ik\underline{h}, \alpha, F, \mathfrak{B})$  and  $\underline{\zeta}_n(s + ik\underline{h}, \alpha, \omega, F, \mathfrak{B})$ , respectively. As usual, for this we need a metric  $\varrho_{L, \zeta}$  on  $H_{L, \zeta}$ , which induces its topology of uniform convergence on compacta, as well the metrics  $\varrho_L$  on  $H(D_L)$  and  $\varrho_\zeta$  on  $H(D_\zeta)$  (for the definitions, see [5], p. 1906).

Then, using results for  $L(s + ikh_1, F)$ ,  $L(s + ikh_1, \omega_1, F)$  (with  $s \in D_L$ ) and  $\zeta(s + ikh_2, \alpha; \mathfrak{B})$ ,  $\zeta(s + ikh_2, \alpha, \omega_2; \mathfrak{B})$  (with  $s \in D_\zeta$ ) from [8] and [7]

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varrho_L(L(s + ikh_1, F), L_n(s + ikh_1, F)) = 0$$

and

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varrho_\zeta(\zeta(s + ikh_2, \alpha; \mathfrak{B}), \zeta_n(s + ikh_2, \alpha; \mathfrak{B})) = 0,$$



we deduce that

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varrho_{L,\zeta}(\zeta(s+ikh, \alpha, F, \mathfrak{B}), \zeta_n(s+ikh, \alpha, F, \mathfrak{B})) = 0.$$

Analogous we can prove that, for almost all  $\omega_1 \in \Omega_1$ ,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varrho_L(L(s+ikh_1, \omega_1, F), L_n(s+ikh_1, \omega_1, F)) = 0$$

and, for almost all  $\omega_2 \in \Omega_2$ ,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varrho_\zeta(\zeta(s+ikh_2, \alpha, \omega_2; \mathfrak{B}), \zeta_n(s+ikh_2, \alpha, \omega_2; \mathfrak{B})) = 0.$$

The last two relations give that, for almost all  $\omega \in \Omega$ ,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \varrho_{L,\zeta}(\zeta(s+ikh, \alpha, \omega, F, \mathfrak{B}), \zeta_n(s+ikh, \alpha, \omega, F, \mathfrak{B})) = 0.$$

Note that here the linear independence of the set  $\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi)$  over  $\mathbb{Q}$  plays an important role.

3. For almost all  $\omega \in \Omega$ , we define one more probability measure

$$P_{N,\omega}(A) := \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \zeta(s+ikh, \alpha, \omega, F, \mathfrak{B}) \in A \right\}, \quad A \in \mathcal{B}(H_{L,\zeta}).$$

Using the weak convergence of probability measures studied in the second step and results on approximation by the mean, we can prove that the measures  $P_N$  and  $P_{N,\omega}$  both converge weakly to the same probability measure  $P$  on  $(H_{L,\zeta}, \mathcal{B}(H_{L,\zeta}))$  as  $N \rightarrow \infty$ .

4. It remains to identify the above mentioned measure  $P$ . For this, Lemma 3.2, result obtained in third step and Birkhoff–Khinchine theorem are used. It allows us to prove that  $P = P_{L,\zeta}$ .

The proof of Theorem 2.1 can be obtained if we take  $h_1 = h_2 = h$  (see the proof of Theorem 4 from [6] also).

#### 4. Concluding remarks

Following B. Bagchi, the functional limit theorems can be applied to the proof of famous universality property (see [1]), which states that analytic func-

tion (or vector of analytic functions) can be approximated by shifts of certain zeta-function (or vector consisting of zeta- and  $L$ -functions). It means that it is possible to prove the joint discrete mixed universality for the tuple  $(L(s, F), \zeta(s, \alpha; \mathfrak{B}))$  in both cases of discreteness: with the same step of arithmetic progression  $h$  for both shifts  $L(s + ikh, F)$  and  $\zeta(s + ikh, \alpha; \mathfrak{B})$ , and more general with different steps  $h_1$  and  $h_2$  for different shifts  $L(s + ikh_1, F)$  and  $\zeta(s + ikh_2, \alpha; \mathfrak{B})$ .

In this section, we will give only statements both cases of joint discrete mixed universality for the pair  $(L(s, F), \zeta(s, \alpha; \mathfrak{B}))$ .

**Theorem 4.1.** *Suppose that the set  $\mathcal{L}_1(\mathbb{P}, \alpha, h, \pi)$  is linearly independent over  $\mathbb{Q}$ . Let  $K_1$  and  $K_2$  be compact subsets of  $D_L$  and  $D_\zeta$ , respectively, with connected complements, and  $f_1(s)$  and  $f_2(s)$  be continuous functions on  $K_1$  and  $K_2$  and analytic in the interior of  $K_1$  and  $K_2$ , respectively. Suppose that the function  $f_1(s)$  is a non-vanishing function on  $K_1$  also. Then, for every  $\varepsilon > 0$ , it holds the inequality*

$$\liminf_{N \rightarrow \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |L(s + ikh, F) - f_1(s)| < \varepsilon, \right. \\ \left. \sup_{s \in K_2} |\zeta(s + ikh, \alpha; \mathfrak{B}) - f_2(s)| < \varepsilon \right\} > 0.$$

The proof of the theorem coincides with the proof of Theorem 3 from [6] since  $L$ -functions of new forms are elements of the Steuding class.

**Theorem 4.2.** *Suppose that the set  $\mathcal{L}(\mathbb{P}, \alpha, h_1, h_2, \pi)$  is linearly independent over  $\mathbb{Q}$ , and hypotheses for  $f_1(s)$ ,  $f_2(s)$ ,  $K_1$  and  $K_2$  are the same as in Theorem 4.1. Then, for  $\varepsilon > 0$ , it holds the inequality*

$$\liminf_{N \rightarrow \infty} \frac{1}{N+1} \# \left\{ 0 \leq l \leq N : \sup_{s \in K_1} |L(s + ikh_1, F) - f_1(s)| < \varepsilon, \right. \\ \left. \sup_{s \in K_2} |\zeta(s + ikh_2, \alpha; \mathfrak{B}) - f_2(s)| < \varepsilon \right\} > 0.$$

The proof of this theorem goes in analogous way as of Theorem 1.3 from [7].

Finally, we should mention that probably we can extend a collection of periodic Hurwitz zeta-functions taking instead of one mentioned zeta-function several of them. This could give us more general result for joint mixed discrete limit theorems as well universality property than in the paper are presented. But it seems that for such extension we need certain independence condition on set of primes, the different steps for the shifts of all periodic Hurwitz zeta-functions of the tuple and parameters which occurred in the definitions of these functions.

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