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ON SUMS OF SQUARES OF $\left|\zeta\left(\frac{1}{2}+i\gamma\right)\right|$ OVER SHORT INTERVALS

Aleksandar Ivić (Belgrade, Serbia)

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Abstract. A discussion involving the evaluation of the sum

$$\sum_{T<\gamma\leqslant T+H}|\zeta(\tfrac{1}{2}+i\gamma)|^2$$

and some related integrals is presented, where $\gamma\,(>0)$ denotes imaginary parts of complex zeros of the Riemann zeta-function $\zeta(s)$. It is shown unconditionally that the above sum is $\ll H\log^2 T\log\log T$ for $T^{2/3}\log^4 T\ll H\leqslant T$.

1. Introduction and statement of results

Let $\gamma\,(>0)$ denote ordinates of complex zeros of the Riemann zeta-function $\zeta(s)$. Consider

$$F(T,H):=\sum_{T<\gamma\leqslant T+H}|\zeta(\tfrac{1}{2}+i\gamma)|^2 \qquad (1\ll H=H(T)\leqslant T), \tag{1.1}$$

so that the interval $[T,T+H]$ may be called "short" if $H=o(T)$ as $T\to\infty$.

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A more general sum than the one in (1.1), when $H = T$, was treated by S.M. Gonek [1]. He proved, under the Riemann hypothesis (RH, that all complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfy $\beta = \frac{1}{2}$) that

$$\sum_{0<\gamma\leqslant T}\left|\zeta\left(\frac{1}{2}+i\left(\gamma+\frac{\alpha}{L}\right)\right)\right|^2 = \left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T + O(T\log T)$$

holds uniformly for $|\alpha| \leqslant \frac{1}{2}L$, where $L = \frac{1}{2\pi}\log(\frac{T}{2\pi})$. It would be interesting to recover this result unconditionally, but our method of proof does not seem capable of achieving this.

If the RH holds, then $F(T,H) \equiv 0$ for $H > 0$, and there is nothing more to say. However, the RH is not known yet to hold, so that one may ask: what if RH fails, but $F(T,H) = 0$? It follows that there exists a zeta zero $\beta + i\gamma$ $(T < \gamma \leqslant T + H)$ such that $\beta \neq \frac{1}{2}$. If for such a zero one has $\frac{1}{2} < \beta < 1$, then $1 - \beta + i\gamma$ is also a zero, which follows from $\overline{\zeta(s)} = \zeta(\bar{s})$ and the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) := \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\pi^{s-1/2} \quad (\forall s \in \mathbb{C}).$$

Therefore one may consider only the case when $\frac{1}{2} < \beta < 1$ and define, for a given $\gamma\,(> 0)$,

$$A(\gamma) := \sum_{\substack{\frac{1}{2}<\beta<1 \\ \zeta(\beta+i\gamma)=\zeta(\frac{1}{2}+i\gamma)=0}} 1, \tag{1.2}$$

where the multiplicities of the zeros $\zeta(\beta + i\gamma)$ are counted. It is clear that

$$0 \leqslant A(\gamma) \leqslant N(\gamma + \tfrac{1}{2}) - N(\gamma - \tfrac{1}{2}) \ll \log\gamma. \tag{1.3}$$

It is reasonable to expect that $A(\gamma) = 0$ for almost all γ, but this is not easy to prove.

As is customary, the function

$$N(T) = \sum_{0<\gamma\leqslant T} 1$$

counts, with multiplicities, the number of zeta zeros, whose positive imaginary parts do not exceed T. We have (see Chapter 1 of [1] or Chapter 9 of [11])

$$\begin{gathered} N(T) = \sum_{0<\gamma\leqslant T} 1 = \frac{1}{\pi}\vartheta(T) + 1 + S(T), \\ \vartheta(T) = \operatorname{Im}\left\{\log\Gamma(\tfrac{1}{4} + \tfrac{1}{2}iT)\right\} - \tfrac{1}{2}T\log\pi, \end{gathered} \tag{1.4}$$

whence $\vartheta(T)$ is real and continuously differentiable. In fact, by using Stirling's formula for the gamma-function, it is found that

$$\vartheta(T) = \frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + O\left(\frac{1}{T}\right), \quad \vartheta'(T) = \frac{1}{2}\log\frac{T}{2\pi} + O\Big(\frac{1}{T^2}\Big).$$

Moreover,

$$S(T) = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2}+iT) = \tfrac{1}{\pi}\mathrm{Im}\Big\{\log\zeta(\tfrac{1}{2}+iT)\Big\} \ll \log T. \tag{1.5}$$

Thus (1.3) follows from (1.4) and (1.5). Here for $T \neq \gamma$ the argument of $\zeta(\frac{1}{2}+iT)$ is obtained by continuous variation along the straight lines joining the points 2, $2+iT$, $\frac{1}{2}+iT$, starting with the value 0. If T is an ordinate of a zeta-zero, then we define $S(T) = S(T+0)$. Clearly when $T \neq \gamma$ we can differentiate $S(T)$ by using (1.5). If $T = \gamma$, then by (1.4) it is seen that $S(T)$ has a jump discontinuity which counts the number of zeros ρ with $\gamma = \mathrm{Im}\rho = T$. For a comprehensive account on $\zeta(s)$, the reader is referred to the monographs of E.C. Titchmarsh [11] and the author [1].

There are some results for $F(T,T)$, defined by (1.1). The author [2] proved that unconditionally

$$F(T,T) = \sum_{T<\gamma\leqslant 2T} |\zeta(\tfrac{1}{2}+i\gamma)|^2 \ll_\varepsilon T\log^2 T(\log\log T)^{3/2+\varepsilon}, \tag{1.6}$$

where ε denotes arbitrarily small positive numbers, not necessarily the same ones at each occurrence, and $\ll_\varepsilon$ means that the implied $\ll$-constant depends only on ε. K. Ramachandra [9] used a different method to obtain a result which easily implies that the right-hand side of (1.6) is unconditionally $\ll T\log^2 T\log\log T$. The same bound was obtained by the author [3], by another method. It was also used to obtain several other results, among which are the bounds

$$\begin{aligned}\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 S(t)\,\mathrm{d}t &\ll T\log T\log\log T,\\ \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 S^2(t)\,\mathrm{d}t &\ll T\log T(\log\log T)^2,\end{aligned} \tag{1.7}$$

while under the Riemann Hypothesis one has

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 S(t)\,\mathrm{d}t \ll T\log T. \tag{1.8}$$

By a variant of the method used in]3] one can generalize these results to short intervals and obtain the following unconditional results.

Theorem 1. *If* $BT^{2/3}\log^4 T \leqslant H = H(T) \leqslant T$ *for a suitable* $B > 0$, *then we have*

$$F(T,H) = \sum_{T<\gamma\leqslant T+H} |\zeta(\tfrac12 + i\gamma)|^2 \ll H(\log T)^2 \log\log T. \tag{1.9}$$

Theorem 2. *If* $BT^{2/3}\log^4 T \leqslant H = H(T) \leqslant T$ *for a suitable* $B > 0$, *then we have*

$$\begin{aligned}&\int_T^{T+H} |\zeta(\tfrac12 + it)|^2 S(t)\,\mathrm{d}t \ll H\log T\log\log T,\\ &\int_T^{T+H} |\zeta(\tfrac12 + it)|^2 S^2(t)\,\mathrm{d}t \ll H\log T(\log\log T)^2.\end{aligned} \tag{1.10}$$

2. The necessary lemmas

If one defines

$$R(t) := S(t) + \frac{1}{\pi}\sum_{p\leqslant y} p^{-1/2}\sin(t\log p) \qquad (T \leqslant t \leqslant 2T), \tag{2.1}$$

where p denotes primes, $y = T^\delta$, and $\delta > 0$ is a small positive number, then it is a classical result of A. Selberg [10] that $R(t)$ is small on the average. This was also later elaborated by K.-M. Tsang [12]. What is needed here is

Lemma 1. *Let* $m > 1$ *be an integer*, $1 < m \leqslant (\log x)/192$, $x^{1/(4m)} < y \leqslant x^{1/m}$ *and* $\log T \ll \log x \ll \log T$. *Then we have, for* $T \geqslant T_0(\varepsilon)$,

$$\int_T^{T+H} R^{2m}(t)\,\mathrm{d}t < \left(\mathrm{e}^{37}\pi^{-2}\varepsilon^{-3}m^2\right)^m H \qquad (H = T^{27/82+\varepsilon}). \tag{2.2}$$

This is Lemma 7 of the paper of A.A. Karatsuba and M.A. Korolev [6]. Its good features are that (2.2) is quite explicit, and moreover the range of H is wide.

Lemma 2. *Let* $BT^{2/3}\log^4 T \leqslant H \leqslant T$ *for a suitable* $B > 0$. *Then*

$$\begin{aligned}&\int_T^{T+H} |\zeta(\tfrac12 + it)|^4\,\mathrm{d}t \ll H\log^4 T,\\ &\int_T^{T+H} |\zeta'(\tfrac12 + it)|^4\,\mathrm{d}t \ll H\log^8 T.\end{aligned} \tag{2.3}$$

The first bound in (2.3) follows from the asymptotic formula

$$\int_0^T |\zeta(\tfrac12+it)|^4\,\mathrm{d}t = TP_4(\log T) + O(T^{2/3}\log^C T), \tag{2.4}$$

where $P_4(x)$ is a well-defined polynomial of degree four in x, with leading coefficient $1/(2\pi^2)$. The proof of (2.4), with $C = 53/6$, was given by Y. Motohashi and the author [5]. The value $C = 8$ was given later by Y. Motohashi [8]. The second bound in (2.3) follows from (4.1), (4.2) and (4.9) of the author's paper [4] and the first bound in (2.3). It is clearly the range for H in Lemma 2 which sets the limit to the range for H in Theorem 1 and Theorem 2.

Lemma 3. *Let* $A(s) = \sum_{m\leqslant M} a(m)m^{-s}$, $a(m) \ll_\varepsilon m^\varepsilon$. *Then*

$$\begin{aligned}&\int_0^T |\zeta(\tfrac12+it)A(\tfrac12+it)|^2\,\mathrm{d}t = \\ &= T\sum_{k,\ell\leqslant M}\frac{a(k)\overline{a(\ell)}}{[k,\ell]}\left(\log\frac{T(k,\ell)^2}{2\pi k\ell} + 2C_0 - 1\right) + E(T,A),\end{aligned} \tag{2.5}$$

where $C_0 = -\Gamma'(1)$ *is Euler's constant, and* $E(T,A) \ll_\varepsilon T^{1/3+\varepsilon}M^{4/3}$.

This is a version of the mean value theorem for a Dirichlet polynomial weighted by $|\zeta(\frac12+it)|$, and (2.5) is due to Y. Motohashi [7]. As usual, (k,ℓ) is the greatest common divisor of k and ℓ, and $[k,\ell]$ is their least common multiple.

3. Proof of Theorem 1

Let henceforth $BT^{2/3}\log^4 T \leqslant H \leqslant T$, and let $f(t)$ be a smooth function on $[T, T+H]$. Then in view of (1.4) one has

$$\begin{aligned}&\sum_{T<\gamma\leqslant T+H} f(\gamma) = \int_T^{T+H} f(t)\,\mathrm{d}N(t) = \\ &= \int_T^{T+H} f(t)\frac{1}{2\pi}\log\Big(\frac{t}{2\pi}\Big)\,\mathrm{d}t + \int_T^{T+H} f(t)\,\mathrm{d}\Big(S(t) + O\Big(\frac1t\Big)\Big) = I_1 + I_2,\end{aligned} \tag{3.1}$$

say. The integral I_1 is usually not difficult to evaluate, and so is the integral with $O(1/t)$, which is a continuously differentiable function. The main problem is the evaluation of the integral in (3.1) with $S(t)$, which we write as

$$I_2 = \int_T^{T+H} f(t)\,\mathrm{d}R(t) - \int_T^{T+H} f(t)\frac{1}{\pi}\sum_{p\leqslant T^\delta} p^{-1/2}\log p\cdot\cos(t\log p)\,\mathrm{d}t, \tag{3.2}$$

where (2.1) was used ($\delta > 0$ is sufficiently small). In the case of

$$f(t) \equiv |\zeta(\tfrac12 + it)|^2,$$

which is needed for Theorem 1, we easily see that

$$I_1 \ll H(\log T)^2, \tag{3.3}$$

since $\int_T^{T+H} |\zeta(\frac12+it)|^2\,\mathrm{d}t \ll H\log T$ for $T^{1/3} \leqslant H \leqslant T$ (see e.g., Chapter 15 of [1]). To deal with I_2, let

$$\mathcal{A}(T,H;V) := \Big\{t\,:\,(T\leqslant t\leqslant T+H)\,\wedge\,(|R(t)|\geqslant V)\Big\},$$

where we suppose that $V = V(t) \geqslant 0$ and $\lim\limits_{T\to\infty} V(T) = +\infty$. If $\mu(\cdot)$ denotes measure, then by Lemma 1 we obtain

$$\begin{aligned}\mu\Big(\mathcal{A}(T,H;V)\Big) &= \int\limits_{\mathcal{A}(T,H;V)} 1\,\mathrm{d}t \leqslant V^{-2m}\int_T^{T+H} R^{2m}(t)\,\mathrm{d}t \ll\\ &\ll \left(C\varepsilon^{-3}\left(\frac{m}{V}\right)^2\right)^m H,\end{aligned}$$

where $C, C_j, \ldots$ denote positive, absolute constants. If $m = [AV]$ for a sufficiently small constant $A > 0$, then

$$\begin{aligned}\left(C\varepsilon^{-3}\left(\frac{m}{V}\right)^2\right)^m &\leqslant \left(C\varepsilon^{-3}A^2\right)^{[AV]} \leqslant\\ &\leqslant \exp\left(-[AV]\log\frac{\varepsilon^3}{CA^2}\right) \leqslant\\ &\leqslant \exp\left(-(AV-1)\log\frac{\varepsilon^3}{CA^2}\right) \leqslant C_1\mathrm{e}^{-C_2V}\end{aligned}$$

for suitable C_1, C_2. If we choose

$$V = \frac{100}{C_2}\log\log T \qquad (T \geqslant T_0 > 0), \tag{3.4}$$

then we see that

$$\mu\Big(\mathcal{A}(T,H;V)\Big) \ll H\mathrm{e}^{-C_2V} = H(\log T)^{-100}. \tag{3.5}$$

Now we use (3.1) and (3.2) with $f(t) \equiv |\zeta(\frac{1}{2}+it)|^2 = \zeta(\frac{1}{2}+it)\zeta(\frac{1}{2}-it)$. This is needed since integration by parts yields

$$(3.6)\qquad \begin{gathered}\int_T^{T+H} f(t)\,\mathrm{d}R(t) = O(T^{1/3+\delta}) - \\ -\int_T^{T+H} R(t)\Big(\zeta'(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}-it) - i\zeta(\tfrac{1}{2}+it)\zeta'(\tfrac{1}{2}-it)\Big)\,\mathrm{d}t.\end{gathered}$$

Here we used the classical bound $\zeta(\frac{1}{2}+it) \ll |t|^{1/6}$ (e.g, see Chapter 7 of [1]). Consider now the portion of the integral on the right-hand side of (3.6) for which $|R(t)| \geqslant V$, where V is given by (3.4). By Hölder's inequality for integrals, this integral does not exceed

$$(3.7)\qquad \begin{gathered}\Big\{\mu\Big(\mathcal{A}(T,H;V)\Big)\times \\ \times\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t\int_T^{T+H} |\zeta'(\tfrac{1}{2}+it)|^4\,\mathrm{d}t\int_T^{T+H} R^4(t)\,\mathrm{d}t\Big\}^{1/4} \ll H,\end{gathered}$$

on using (2.2) of Lemma 1 (with $m=2$), (2.3) of Lemma 2 and (3.5). The portion of the integral over $[T,T+H]\setminus\mathcal{A}(T,H;V)$ is

$$\begin{aligned}&\ll \log\log T\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)||\zeta'(\tfrac{1}{2}+it)|\,\mathrm{d}t \leqslant \\ &\leqslant \log\log T\Big\{\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\int_T^{T+H} |\zeta'(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\Big\}^{1/2} \ll \\ &\ll \log\log T\Big(H\log T\cdot H\log^3 T\Big)^{1/2} = H(\log T)^2\log\log T.\end{aligned}$$

The bounds for the mean square of ζ,ζ' in short intervals follow similarly, but with less difficulty, as the bound for the corresponding fourth moments in Lemma 2 (see e.g., Chapter 15 of [1]). It is also easily seen that (3.3) holds in our case. Thus it remains to estimate the second integral in (3.2), namely

$$(3.8)\qquad I_3 := \frac{1}{\pi}\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^2\sum_{p\leqslant T^\delta} p^{-1/2}\log p\cdot\cos(t\log p)\,\mathrm{d}t.$$

The integral in (3.8), by the Cauchy-Schwarz inequality, does not exceed

$$(3.9)\qquad \begin{gathered}\Big\{\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\times \\ \times\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^2\Big|\sum_{p\leqslant T^\delta} p^{-1/2-it}\log p\Big|^2\,\mathrm{d}t\Big\}^{1/2}.\end{gathered}$$

As remarked above, the first integral in (3.9) is $\ll H\log T$, and for the second one we apply (2.5) of Lemma 3, once with $T+H$ and once with T, and we subtract the results. In our interval for H we shall have $E(T+H,A)-E(T,A) \ll H$ for $M = T^\delta$ and sufficiently small $\delta > 0$, where

$$A(s) \;:=\; \sum_{p\leqslant T^\delta} \log p\cdot p^{-1/2-it}.$$

Further

$$\begin{aligned}
&\sum_{p_1,p_2\leqslant M} \frac{\log p_1 \log p_2}{[p_1,p_2]}\left(\log\Big(\frac{T(p_1,p_2)^2}{2\pi p_1p_2}\Big) + 2C_0 - 1\right) = \\
&= \sum_{p\leqslant M} \frac{\log^2 p}{p}\Big(\log\frac{T}{2\pi} + 2C_0 - 1\Big) + \\
&+ \sum_{\substack{p_1,p_2\leqslant M\\ p_1\neq p_2}} \frac{\log p_1\log p_2}{p_1p_2}\Big(\log\Big(\frac{T}{2\pi p_1p_2}\Big) + 2C_0 - 1\Big).
\end{aligned}\tag{3.10}$$

The last expression is $\ll \log T\cdot\log^2 M \ll \log^3 T$, if one uses the elementary bound

$$\sum_{p\leqslant x}\frac{\log p}{p} \ll \log x.$$

Therefore the expression in (3.8) is $\ll H\log^2 T$, which finishes the proof of Theorem 1. ■

4. Proof of Theorem 2

The proof of Theorem 2 is based on the same ideas as the proof of Theorem 1, so only its salient points will be mentioned. To prove the first bound in (1.10) we use (2.1). The integral with $R(t)$, similarly as in the proof of Theorem 1, will be $\ll H\log T\log\log T$. Let

$$\sum(T) := \sum_{p\leqslant T^\delta} p^{-1/2}\sin(t\log p).$$

The contribution of $\sum(T)$ for which $|\sum(T)| \leqslant \log\log T$ is trivially

$$\ll H\log T\log\log T.$$

The remaining contribution is bounded by

$$\frac{1}{\log\log T}\int_T^{T+H}|\zeta(\tfrac12+it)|^2\Bigl|\sum_{p\leqslant T^\delta}p^{-1/2-it}\Bigr|^2\,\mathrm{d}t. \tag{4.1}$$

The integral in (4.1) is estimated by Lemma 3, with the preceding $A(s)$ replaced by

$$A_1(s) \;:=\; \sum_{p\leqslant T^\delta}p^{-1/2-it}.$$

This leads to an expression similar to the one in (3.10), namely

$$\begin{aligned}
&\sum_{p_1,p_2\leqslant M}\frac{1}{[p_1,p_2]}\left(\log\Bigl(\frac{T(p_1,p_2)^2}{2\pi p_1p_2}\Bigr)+2C_0-1\right)=\\
&=\sum_{p\leqslant M}\frac1p\Bigl(\log\frac{T}{2\pi}+2C_0-1\Bigr)+\\
&+\sum_{\substack{p_1,p_2\leqslant M\\ p_1\ne p_2}}\frac{1}{p_1p_2}\Bigl(\log\Bigl(\frac{T}{2\pi p_1p_2}\Bigr)+2C_0-1\Bigr)\ll\log T(\log\log T)^2,
\end{aligned}\tag{4.2}$$

since

$$\sum_{p\leqslant x}\frac1p \;=\; \log\log x \;+\; O(1).$$

The bound in (4.2), combined with (4.1) leads then to

$$\int_T^{T+H}|\zeta(\tfrac12+it)|^2S(t)\,\mathrm{d}t \;\ll\; H\log T\log\log T.$$

To prove the remaining bound in (1.10) we use $S^2(t)\ll R^2(t)+\sum^2(t)$. The integral with $R^2(t)$ is

$$\ll H\log T(\log\log T)^2, \tag{4.3}$$

if we consider separately the cases $|R(t)|\leqslant V$ and $|R(t)|\geqslant V$, where V is as in (3.4). The integral with $\sum^2(t)$ is estimated as the integral in (4.1). With the aid of (4.2) we arrive again at the bound in (4.3), thereby completing the proof. ∎

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A. Ivić
Serbian Academy of Sciences and Arts
Knez Mihailova 35, 11000 Beograd
Serbia
aivic_2000@yahoo.com
aleksandar.ivic@rgf.bg.ac.rs