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QUASIMONOTONICITY AND FUNCTIONAL INEQUALITIES

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Abstract. A comparison theorem for functional equations in ordered topological vector spaces will be given, which generalizes the results from [3], [4]. Quasimonotonicity is fundamental for these investigations.

1. Introduction

Let E be a real Hausdorff topological vector space and let K be a wedge in E, i.e.

$$\emptyset \neq K \subseteq E \text{ and}$$
$$\lambda \geq 0;\ \xi, \eta \in K \Rightarrow \lambda(\xi + \eta) \in K.$$

We suppose $K = \overline{K},\ \mathrm{Int} K \neq \emptyset$, and for $\xi, \eta \in E$ we define

$$\xi \leq \eta \quad \Leftrightarrow \quad \eta - \xi \in K,$$
$$\xi \ll \eta \quad \Leftrightarrow \quad \eta - \xi \in \mathrm{Int} K.$$

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E^* denotes the topological dual of E, and

$$K^* = \{\varphi | \varphi \in E^*,\ \varphi(\xi) \geq 0 \quad (\xi \in K)\}.$$

Definition 1.1. (Cf. [2].) For $D \subseteq E$, a function $g : D \to E$ is called *quasi-monotone increasing* (qmi) if the following holds:

$$\xi, \eta \in D,\ \xi \leq \eta,\ \varphi \in K^*,\ \varphi(\xi) = \varphi(\eta) \Rightarrow \varphi(g(\xi)) \leq \varphi(g(\eta)).$$

Furthermore, let X be a compact topological Hausdorff space, $Y \subseteq X$, $T > 0$, $R = X \times [0, T]$, and let $C(R, E)$ be the space of continuous functions $u : R \to E$.

Definition 1.2. If $M \subseteq C(R, E)$, then an operator

$$\Phi : Y \times]0, T] \times M \to E$$

has *property* (P), provided

$$\left.\begin{array}{l} 0 < t \leq T,\ x \in Y,\ \varphi \in K^*,\ v, w \in M \\ v(y,s) \ll w(y,s)\ (y \in X,\ 0 \leq s < t) \\ \varphi(v(x,t)) = \varphi(w(x,t)) \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \varphi(\Phi(x,t,v)) \leq \varphi(\Phi(x,t,w)).$$

The following example and remark are easy consequences of the definitions.

Example 1.1. Let $f : Y \times]0, T] \times E \to E$ be such that $f(x, t, \xi)$ is qmi w.r. to $\xi \in E$. Then

$$\Phi(x,t,u) = f(x,t,u(x,t))\ (x \in Y,\ 0 < t \leq T,\ u \in M)$$

fulfils (P), where $M \subseteq C(R, E)$ can be arbitrary.

Remark 1.1. If $\Phi_1, \Phi_2 : Y \times]0, T] \times M \to E$ have property (P), then also $\Phi = \Phi_1 + \Phi_2$ has property (P).

2. Comparison-Theorem

Theorem 2.1. *Let Φ fulfil* (P)*, and let $v, w \in M$ satisfy*

$$I) \quad \begin{cases} v(x,0) \ll w(x,0) & (x \in X), \\ v(x,t) \ll w(x,t) & (0 < t \leq T,\ x \in X \setminus Y), \end{cases}$$

$$II) \quad \Phi(x,t,w) \ll \Phi(x,t,v)\ (0 < t \leq T,\ x \in Y).$$

Then $v(x,t) \ll w(x,t)$ in R.

Proof. Otherwise

$$w(x_0, t_0) - v(x_0, t_0) \in E \setminus \mathrm{Int}K$$

would be possible (where $t_0 > 0,\ x_0 \in Y$; cf. I)). Let t_0 be minimal, then

$$w(x_0, t) - v(x_0, t) \in \mathrm{Int}K \ (0 \leq t < t_0),$$

and $t \to t_0$ leads to

$$w(x_0, t_0) - v(x_0, t_0) \in \partial K.$$

Now the theorem of Hahn/Banach (cf. Walter Rudin [1]) yields a $\varphi \in K^* \setminus \{0\}$ such that

$$\varphi(w(x_0, t_0) - v(x_0, t_0)) = 0.$$

Condition (P) implies

$$\varphi(\Phi(x_0, t_0, v)) \leq \varphi(\Phi(x_0, t_0, w)),$$

which contradicts II) (for $t = t_0,\ x = x_0$). ■

Remark 2.1. By using our Comparison-Theorem and Remark 1.1, new proofs can be given to the results in [3], [4]. In [3] for instance a comparison result is obtained for a certain operator $\Phi = \Phi_1 + ... + \Phi_5$, where now it is sufficient to show that all the operators $\Phi_1, ..., \Phi_5$ have property (P).

3. Application

To give an application, besides the examples cited above, we consider $M = C(R, E)$ in the setting of Definition 1.2. Let $f : Y\times]0, T] \times E \to E$ be as in Example 1.1, and let $h : Y\times]0, T] \to X$ and $\alpha : Y\times]0, T] \to \mathbb{R}$ be such that

$$(x, t) \in Y\times]0, T],\ h(x, t) \neq x \ \Rightarrow \ \alpha(x, t) \geq 0.$$

Then $\Phi : Y\times]0, T] \times C(R, E) \to E$ defined as

$$\Phi(x, t, u) = \alpha(x, t)u(h(x, t), t) + f(x, t, u(x, t))$$

fulfils (P), and hence is admissible in Theorem 2.1. Choosing for example $f(x,t,\xi) = -\xi$ we obtain for continuous $w : R \to E$ the following:

$$I) \quad \begin{cases} 0 \ll w(x,0) & (x \in X), \\ 0 \ll w(x,t) & (0 < t \leq T,\ x \in X \setminus Y), \end{cases}$$

$$II) \quad \alpha(x,t)w(h(x,t),t) \ll w(x,t)\ (0 < t \leq T,\ x \in Y)$$

$$\Rightarrow \quad 0 \ll w(x,t)\ ((x,t) \in R).$$

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