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SOLVING ALTERNATIVE FUNCTIONAL EQUATIONS: WHAT FOR?

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Abstract. To make it less abstract, answering the title question let us twist British mountaineer George Mallory's famous dictum: "Because they're there." The article yields a counterpart of Mihály Bessenyei and Gréta Szabó's paper [3] dealing with addition formula characterizing the tanh function. The emphasis is given on proving that a natural understanding of a solution of a functional in question requires to view that equation as an alternative one. For the most part the paper has a survey character.

Our considerations may be viewed as a counterpart of Mihály Bessenyei and Gréta Szabó's paper [3] containing, as main result, the following

Theorem (BS). *A vanishing at zero function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a solution to equation*

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)} \tag{1}$$

if and only if there is an exponential function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) = \frac{g(x)-1}{g(x)+1}\,.$$

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However, at the beginning of the *Prelude: preliminary experiments* section they wrote: *we assume* $f(x)f(y) \neq -1$ *for all* $x, y \in \mathbb{R}$*, so equation* (1) *makes sense.*

At this moment some German people (say, mathematicians) would call out: *Da liegt der Hund begraben!* In Section 4 entitled *Singularities admissible* of paper [4] written jointly with Katarzyna Domańska we wrote:

Definitely much more interesting seems to be the case where by a solution of equation (1) *we understand any function* $f : G \longrightarrow \mathbb{R}$ (*with no restrictions on its range*) *that satisfies equality* (1) *for every pair* $(x, y) \in G^2$ *such that* $f(x)f(y) \neq -1$*; here* $(G, +)$ *stands for a group* (*not necessarily commutative*)*.*

Remark 1. Since any exponential function $g : \mathbb{R} \longrightarrow \mathbb{R}$ may be written in the form $g(x) = e^{2A(x)}, x \in \mathbb{R}$, where $A : \mathbb{R} \longrightarrow \mathbb{R}$ is additive, the formula

$$f(x) = \frac{g(x) - 1}{g(x) + 1}$$

may equivalently be written as

$$f(x) = \tanh A(x)\,, \quad x \in \mathbb{R}\,.$$

In what follows we summarize the corresponding results from [4] by formulating Proposition (DG) and Theorem (DG) below.

Proposition (DG). *Let* $(G, *)$ *be a groupoid. A function* $f : G \longrightarrow (-1, 1)$ *yields a solution to the functional equation*

$$f(x * y) = \frac{f(x) + f(y)}{1 + f(x)f(y)} \tag{2}$$

for all $x, y \in G$*, if and only if there exists a homomorphism* $A : G \longrightarrow \mathbb{R}$ *such that*

$$f(x) = \tanh A(x)\,, \quad x \in G\,.$$

In the case where the groupoid $(G, *)$ *in question is 2*-divisible*, i.e. each element* $x \in G$ *admits an element* $y \in G$ (*not necessarily unique*) *such that* $x = y * y$*, then the range* $f(G)$ *of any solution* $f : G \longrightarrow \mathbb{R}$ *of equation* (2) *is contained in the interval* $[-1, 1]$.

Equation (2) *has no solutions* $f : G \longrightarrow \mathbb{R}$ *with the range* $f(G)$ *contained in the set* $(-\infty, -1) \cup (1, \infty)$.

If $(G, +)$ *yields a group* (*not necessarily commutative*) *and* $f : G \longrightarrow \mathbb{R}$ *is a solution of an alternative functional equation*

$$f(x)f(y) \neq -1 \text{ implies } f(x * y) = \frac{f(x) + f(y)}{1 + f(x)f(y)} \tag{1+}$$

for all $x, y \in G$, *such that* $f(0) = 0$ *and*

$$(*) \qquad 1 \notin f(G) \not\subset (-1, 1)\,,$$

then there exists a subgroup $(\Gamma, +)$ *of index* 2 *in the group* $(G, +)$ *and an additive function* $B : G \longrightarrow \mathbb{R}$ *such that* $\ker B \subset \Gamma$ *and*

$$f(x) \quad = \quad \begin{cases} \tanh \circ B(x) & \textit{for } x \in \Gamma \\ \coth \circ B(x) & \textit{for } x \in G \setminus \Gamma\,. \end{cases}$$

Conversely, any function f *of that form yields a solution to equation* (1+) *and satisfies condition* $(*)$.

In non-2-divisible groupoids the assertion of Proposition 2 is invalid, in general (see [4] for suitable example).

In order to get a complete description of **all** real solutions of equation (1+), we proceed to examine the case where the critical value 1 falls into the range of the unknown function defined on a group.

Theorem (DG). *Let* $(G, *)$ *be a groupoid. The only constant solutions of equation* (1+) *are* $f = -1$, $f = 0$ *and* $f = 1$.

Assume that $(G, +)$ *yields a group* (*not necessarily commutative*) *and* $f : G \longrightarrow \mathbb{R}$ *is a nonconstant solution of equation* (1+) *such that the set* $S := f^{-1}(\{1\})$ *is nonvoid. If* $f(0) \neq 0$, *then* $f(0) \in \{-1, 1\}$ *and setting* $S' := G \setminus S$ *we get*

$$f(x) := \begin{cases} \;\;1 & \textit{for } x \in S \\ -1 & \textit{for } x \in S', \end{cases}$$

where $(S, +)$ *is a subgroup of index* 2 *of the group* $(G, +)$ *provided that* $f(0) = 1$, *whereas* $(S, +)$ *and* $(S', +)$ *are subsemigroups of* $(G, +)$ *whenever* $f(0) = -1$. *Conversely, any function* f *of that form yields a solution to equation* (1+).

If $f(0) = 0$ *then the pair* $(S, +)$ *forms a subsemigroup of* $(G, +)$, $S \cap (-S) = \emptyset \neq G_0 := G \setminus (S \cup (-S))$, $(G_0, +)$ *is a subgroup of* $(G, +)$ *and*

$$(4) \qquad S + G_0 = S \qquad \textit{as well as} \qquad -S + G_0 = -S\,,$$

whereas the function $f_0 := f|_{G_0}$ *yields a solution of equation* (1+) *on the group* $(G_0, +)$ *enjoying the property* $1 \notin f_0(G_0)$.

Conversely, let $(S, +)$ *be an arbitrary subsemigroup of* $(G, +)$ *such that* $S \cap (-S) = \emptyset \neq G_0 := G \setminus (S \cup (-S))$, $(G_0, +)$ *is a subgroup of* $(G, +)$ *and* *relationships* (4) *hold true. Then, for any solution* $f_0 : G_0 \longrightarrow \mathbb{R}$ *of* (1+) *having* 1 *off its range the function* $f : G \longrightarrow \mathbb{R}$ *defined by*

$$f(x) := \begin{cases} 1 & \text{for } x \in S \\ -1 & \text{for } x \in -S \\ f_0(x) & \text{for } x \in G_0 \end{cases}$$

satisfies equation (1+) *on* G.

More generally, given an associative binary operation F defined on a subset (possibly disconnected) of the real plane $\mathbb{R}^2$ one may look for a method of solving Cauchy type functional equation of the form

$$f(x+y) = F(f(x), f(y))$$

(so called addition formulas). We admit "singularities" of the binary operation F, e.g. zeros of the denominator in the case where F is a rational two-place real-valued function.

Our approach has just been visualized by dealing with the case where

$$F(u,v) := \frac{u+v}{1+uv}\,,$$

but the method applied might serve also while solving addition formulas with the corresponding binary operations

$$F(u,v) := \frac{u+v}{1-uv}\,, \quad u,v \in \mathbb{R},\ uv \neq 1\,,$$

$$F(u,v) := \frac{1+uv}{u+v}\,, \quad u,v \in \mathbb{R},\ u+v \neq 0\,,$$

$$F(u,v) := \frac{uv-1}{u+v}\,, \quad u,v \in \mathbb{R},\ u+v \neq 0\,,$$

$$F(u,v) := \frac{u+v+2uv}{1-uv}\,, \quad u,v \in \mathbb{R},\ uv \neq 1\,,$$

and other associative rational mappings.

Leaving the land of addition formulas but staying in the world of alternative functional equations we proceed with further examples of the situation where the alternative character of the equation in question is forced by the nature of the problem. Let us begin with the so called

Inverse additive functions

(cf. a paper of R. Ger and M. Kuczma [7]). To characterize productivity and price indices in mathematical economics the following functional equation (say, for selfmappings of $\mathbb{R}$) has been considered (cf. J. Aczél [1] and J. Aczél [2]):

$$\frac{1}{f(x+y)} = \frac{1}{f(x)} + \frac{1}{f(y)} \tag{5}$$

Applying the corresponding Bessenyei–Szabó prelude rule (*we assume $f(x) \neq 0$ for all $x \in \mathbb{R}$, so equation* (5) *makes sense*), we infer that (5) has no solutions at all; indeed, on setting $a := \frac{1}{f}$ we derive the additivity of a as well as the inequality $a(x) \neq 0$ for all $x \in \mathbb{R}$, contradicting the fact that each additive function vanishes at zero.

Therefore, it seems reasonable to deal with the case where by a solution of equation (5) we understand any function f mapping a groupoid $(G, *)$ into $\mathbb{R}$ (with no restrictions on its range) that satisfies equality (5) for every pair $(x, y) \in G^2$ such that $f(x) \neq 0 \neq f(y)$ and $f(x * y) \neq 0$. In other words - to deal with an alternative functional equation

$$f(x)f(y)f(x*y) \neq 0 \quad \text{implies} \quad \frac{1}{f(x*y)} = \frac{1}{f(x)} + \frac{1}{f(y)}\,. \tag{5+}$$

That approach changes the situation dramatically: instead of the lack of solutions we are faced to the greatly excessive number of them. In fact, in [7] an open subset $U \subset (0, \infty)$ of infinite Lebesgue measure has been defined enjoying the property that

$$x, y \in U \quad \text{implies} \quad x + y \notin U.$$

An immediate verification shows that for an **arbitrary** function $g : U \longrightarrow \mathbb{R}$ the map

$$f(x) := \begin{cases} g(x) & \text{for } x \in U \\ 0 & \text{for } x \in \mathbb{R} \setminus U \end{cases}$$

yields a solution of (5+). Clearly, the shape of the right hand side part of the implication (5+) plays no role at all. Actually, for any nonempty set $U \subset G$ such that

$$x, y \in U \quad \text{implies} \quad x * y \notin U$$

and for any function $g : U \longrightarrow \mathbb{R}$ the map $f : G \longrightarrow \mathbb{R}$ given by the formula

$$f(x) := \begin{cases} g(x) & \text{for } x \in U \\ 0 & \text{for } x \in G \setminus U \end{cases}$$

solves **every** alternative functional equation of the form

$$f(x)f(y)f(x*y) \neq 0 \quad \text{implies} \quad P\left(\frac{1}{f(x*y)}, \frac{1}{f(x)}, \frac{1}{f(y)}\right) = Q(x,y)\,,$$

where $P : \mathbb{R}^3 \longrightarrow H$ and $Q : G^2 \longrightarrow H$ are given arbitrarily; here H stands for any set that is nonvoid.

On the other hand, if the set of zeros of the unknown function f is not too large, then equation (5+) has nontrivial solutions. To visualize it we shall prove the following:

Theorem. *Let* $(G,+)$ *be a Baire topological group* (*not necessarily commutative*) *and let* $f : G \longrightarrow \mathbb{R}$ *be a solution of equation* (5+) *with* $* = +$ *such that* $f^{-1}(\{0\})$ *is of the first Baire category. Then there exists an additive function* $a : G \longrightarrow \mathbb{R}$ *and a set* $Z \subset G$ *such that* $a(x) \neq 0$ *for* $x \in G \setminus Z =: Z'$ *and*

$$f(x) := \begin{cases} 0 & \text{for } x \in Z \\ 1/a(x) & \text{for } x \in Z'. \end{cases}$$

Proof. Let $Z := f^{-1}(\{0\})$. It is not hard to check that the set theoretical union M of the sets $G \times Z,\ Z \times G$ and $\{(x,y) \in G^2 : x+y \in Z\}$ is of the first Baire category in the product group $G^2 := G \times G$ equiped with the product topology. Put

$$F(x) := \begin{cases} 0 & \text{for } x \in Z \\ 1/f(x) & \text{for } x \in Z'. \end{cases}$$

Equation (5+), in the case where $(G,*) = (G,+)$, states now nothing else but the validity of the equality

$$F(x+y) = F(x) + F(y)$$

whenever $(x,y) \in G^2 \setminus M$. An appeal to the main result of author's paper [6] shows that the exists a unique additive function $a : G \longrightarrow \mathbb{R}$ such that the set

$$E := \{x \in G : F(x) \neq a(x)\}$$

yields a first category subset of G. Consequently, so is the set $T := Z \cup E$. Fix arbitrarily an $x \in Z'$. If for each $u \in T' := G \setminus T$ we had $x - u \in T$ then T' would be contained in $x - T$. Consequently, both T and T' would be of the first category, contradicting the fact that $G = T \cup T'$ is a Baire space. Thus there exists a $u \in T'$ such that also $v := x - u \in T'$ and, consequently,

$$u \in Z',\ x-u \in Z',\ (x-u)+u = x \in Z',\ u \notin E \quad \text{and} \quad x-u \notin E.$$

Therefore, the pair $(x-u,u)$ stays off the set M and, finally,

$$F(x) = F((x-u)+u) = F(x-u) + F(u) = a(x-u) + a(u) = a(x),$$

which completes the proof. ■

Remark 2. The intuitive feeling of "largness" may sometimes be misleading. It is well known that the real line $\mathbb{R}$ (which obviously yields a Baire topological group with respect to the usual addition) splits out into two small sets: a first category set Z and a set $W := \mathbb{R} \setminus Z$ of one dimensional Lebesgue measure ℓ_1 equal to zero. In this situation, the theorem just proved may be reformulated as follows:

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $Z = f^{-1}(\{0\})$. Then

$$(**) \qquad x, y \text{ and } x+y \text{ are in } W \quad \text{implies} \quad \frac{1}{f(x+y)} = \frac{1}{f(x)} + \frac{1}{f(y)}\,,$$

if and only if there exists an additive function $a : \mathbb{R} \longrightarrow \mathbb{R}$ such that $a(x) \neq 0$ for $x \in W$ and

$$f(x) := \begin{cases} 0 & \text{for } x \in Z \\ 1/a(x) & \text{for } x \in W. \end{cases}$$

We are faced to a slightly shocking phenomenon: the requirement in the antecendent of the implication $(**)$ seems to be fulfilled very rarely (recall that $\ell_1(W) = 0$) and, a fortiori, equation $(**)$ seems to yield very poor information about the unknown function f; nevertheless, the equation is uniquely solvable.

We proceed with

Cuculière's problem and a conditional d'Alembert's equation

dealing with another interesting alternative functional equation that appears naturally while solving the following problem posed by a French mathematician Roger Cuculière (*Problem 11998*, The American Mathematical Monthly **124** no. 7 (2017)):

Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that satisfy $f(z) \leq 1$ for some nonzero real number z and

$$\text{(C)} \qquad f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

for all real numbers x and y.

Let $(G,+)$ be a group (not necessarily commutative). Cuculière's equation (C) is equivalent to

$$[f(x+y)-f(x)f(y)]^2=\left(1-f(x)^2\right)\left(1-f(y)^2\right),$$

in the class of all real functions f defined on G. Without essential loss of generality we may assume that $f(0)=1$; then f is even and setting

$$Z:=\{x\in G:\ f(x)=1\}$$

we conclude that $(Z,+)$ yields a subgroup of $(G,+)$.

Now, replacing y by $-y$ in (C) and subtracting the resulting equation from (C) side by side we arrive at

$$\text{(CA)} \qquad f(x+y)\neq f(x-y) \quad \text{implies} \quad f(x+y)+f(x-y)=2f(x)f(y)$$

valid for all $x,y\in G$.

So, in that way, we have entered again the world of alternative functional equations like, for instance:

- Mikusiński's equation

$$f(x+y)\neq 0 \quad \text{implies} \quad f(x+y)=f(x)+f(y)$$

- functional equation of Dhombres

$$f(x)+f(y)\neq 0 \quad \text{implies} \quad f(x+y)=f(x)+f(y)$$

- Mikusiński–Pexider functional equation

$$f(x+y)\neq 0 \quad \text{implies} \quad g(x+y)=h(x)+h(y)$$

- numerous others (see e.g. J.G.Dhombres and R. Ger, *Conditional Cauchy equations*, (Glasnik Matematički **13** (33) (1978), 39-62, for more detailed review).

In a paper of mine (not published yet) I have shown that for Lebesgue measurable solutions $f:\mathbb{R}\longrightarrow\mathbb{R}$ of equation (CA) such that $f(0)=1$ and card $f(G)\geq 3$ the antecendent of (CA) is redundant, i.e. f has to satisfy the d'Alembert functional equation

$$\text{(A)} \qquad f(x+y)+f(x-y)=2f(x)f(y)$$

for all real numbers x and y. It is commonly known that the only nonconstant Lebesgue measurable solutions of (A) are $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh bx$, $x \in \mathbb{R}$, with any nonzero real constant b. Consequently, we get the following

Theorem (G). *A Lebesgue measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ yields a solution of equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \tag{C}$$

for all $x, y \in \mathbb{R}$, if and only if either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh bx$, $x \in \mathbb{R}$, where $b \in \mathbb{R}$ is an arbitrary constant.

Corollary (a solution of Cuculière's problem). *A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(z) \leq 1$ for some nonzero real number z yields a solution of equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \tag{C}$$

for all $x, y \in \mathbb{R}$, if and only if either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or $f(x) = \cos bx$, $x \in \mathbb{R}$, where $b \in \mathbb{R}$ is an arbitrary constant.

Proof. Any continuous selfmapping of $\mathbb{R}$ is Lebesgue measurable and it suffices to apply Theorem (G); the cosh function is eliminated by the assumption that $f(z) \leq 1$ for some nonzero real number z yields a solution of equation. ■

Finally, we shall present briefly some examples of conditional equations where the antecendent of the implication in question does not depend on the unknown function. We shall begin with the celebrated

Orthogonal additivity equation

$$(x|y) = 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y)\,, \tag{OA}$$

which occured among others while deriving the so called Maxwell–Boltzmann distribution law for velocities of molecules in an ideal gas. Here $(\cdot|\cdot)$ stands for the standard inner product in $\mathbb{R}^3$ but there is no reason not to consider equation (OA) for real functionals f on an abstract inner product (Hilbert) space $(X, (\cdot|\cdot))$. If that is the case, then

$$f(x) = a\left(\|x\|^2\right) + b(x),\ x \in X,$$

where the functions $a : \mathbb{R} \longrightarrow \mathbb{R}$ and $b : X \longrightarrow \mathbb{R}$ are additive (see e.g. a survey paper of J. Sikorska [8]). The classical orthogonality relation

$x \perp y :\Longleftrightarrow (x|y) = 0$ may be replaced, for instance, by the so called Birkhoff-James orthogonality in a real normed linear space $(X, \|\cdot\|)$ of dimension at least 2 :

$$x \perp y :\Longleftrightarrow \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \in \mathbb{R},$$

to get a similar representation of solutions (see [8]).

For the so called Pythagorean orthogonality

$$x \perp y :\Longleftrightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2$$

the (OA) equation assumes the form

$$\text{(POA)} \qquad \|x + y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{implies} \quad f(x + y) = f(x) + f(y)\,.$$

This equation has been solved by Gy. Szabó [9] in the class of odd functions $f : X \longrightarrow \mathbb{R}$ where $(X, \|\cdot\|)$ is a real normed linear space of dimension at least 3. What provides here an essential difficulty is that Pythagorean orthogonality fails to be homogeneous. Since quarter of a century the problem of finding the general solution of equation (POA) remains open. The more so is, of course, a more general (but very natural, in that context) problem of solving the following alternative equation

$$g(x + y) = g(x) + g(y) \quad \text{implies} \quad f(x + y) = f(x) + f(y)\,,$$

where $g : X \longrightarrow \mathbb{R}$ stands for a reasonably given functional.

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