Annales Univ. Sci. Budapest., Sect. Comp. 47 (2018) 227–237

GENERALIZED CONVOLUTIONS WITH WEIGHT-FUNCTION FOR DISCRETE-TIME FOURIER COSINE AND SINE TRANSFORMS

Nguyen Anh Dai and Nguyen Xuan Thao

(Ha Noi, Viet Nam)

Communicated by Ferenc Schipp

(Received February 28, 2018; accepted June 10, 2018)

Abstract. In this article we study generalized convolutions for discrete-time Fourier cosine and Fourier sine transforms, operator properties and for the application in volving infinite systems of linear algebraic equations.

1. Introduction

Since the end of 19^{th} century, Fourier transform has been studied along with Fourier cosine, Fourier sine transforms and their convolutions. Fourier transform of a function $x(t)$ is defined in [6], [10]. Fourier cosine transform of a function $x(t)$ is defined in [1], [5].

In recent years, there are many interesting results relates to Fourier, Fourier cosine and Fourier sine transforms have been published [2], [3], [4], [7], [11].

Discrete-time Fourier transform is given (see [8], [9])

$$X(\omega) \equiv F_{DT}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n} \tag{1.1}$$

$$\big(x = (x(n)) \in l_1,\ \omega \in [0, 2\pi]\big)$$

Key words and phrases: Discrete-time Fourier cosine transform, discrete-time Fourier sine transform, convolution, generalized convolution.

2010 Mathematics Subject Classification: 42A38, 65T50.

and

$$x(n) \equiv F_{DT}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})e^{i\omega n}d\omega \qquad \big(n \in \mathbb{N}\big). \tag{1.2}$$

In this article we study generalized convolutions with weighted-function for discrete-time Fourier cosine and Fourier sine transforms, operator properties and applications.

2. Discrete-time Fourier cosine and Fourier sine transforms

2.1. Definitions. The *discrete-time Fourier cosine transform* of $x(n)$ sequence has the form

$$X_c(\omega) \equiv \mathcal{F}_{cDT}\{x(n)\}(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos(n\omega), \quad \omega \in [0,\pi] \tag{2.1}$$

and *inverse transform*

$$x(n) = \mathcal{F}_{cDT}^{-1}\{X_c(\omega)\}(\omega) = \frac{1}{\pi}\int_{0}^{\pi} X_c(\omega)\cos(n\omega)d\omega \quad (n \in \mathbb{N}). \tag{2.2}$$

The *Discrete-time Fourier sine transform* of $x(n)$ sequence has the form

$$X_s(\omega) \equiv \mathcal{F}_{sDT}\{x(n)\}(\omega) = 2\sum_{n=0}^{\infty} x(n)\sin(n\omega), \omega \in [0,\pi] \tag{2.3}$$

and *inverse transform*

$$x(n) = \mathcal{F}_{sDT}^{-1}\{X_s(\omega)\}(\omega) = \frac{1}{\pi}\int_{0}^{\pi} X_s(\omega)\cos(n\omega)d\omega \quad (n \in \mathbb{N}). \tag{2.4}$$

Here $X_c(\omega)$, $X_s(\omega)$ are determined, bounded function in $[0,\pi]$. These imply that all the spectral information contained in the fundamental interval is necessary for the complete description with the signal.

2.2. Properties. Let $x(n), y(n) \in l_1$ denote sequences, where

$$l_1 = \Big\{x(n) : \sum_{n=0}^{\infty} |x(n)| < \infty\Big\}.$$

(a) *Linearity:*

$$\mathcal{F}_{cDT}\{x(n) + y(n)\}(\omega) = \mathcal{F}_{cDT}\{x(n)\}(\omega) + \mathcal{F}_{cDT}\{y(n)\}(\omega) \tag{2.5}$$

(b) *Time shifting:*

$$\mathcal{F}_{cDT}\{x(n - n_0)\}(\omega) = \mathcal{F}_{cDT}\{x(n)\}(\omega). \tag{2.6}$$

(c) *Modulation:*

$$\begin{aligned} &\mathcal{F}_{cDT}\{x(n)\cos n\,\omega_0\}(\omega) = \\ &= \frac{1}{2}\big[\mathcal{F}_{cDT}\{x(n)\}(\omega + \omega_0) + \mathcal{F}_{cDT}\{x(n)\}(\omega - \omega_0)\big], \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\mathcal{F}_{cDT}\{x(n)\sin n\,\omega_0\}(\omega) = \\ &= \frac{1}{2}\big[\mathcal{F}_{sDT}\{x(n)\}(\omega + \omega_0) - \mathcal{F}_{cDT}\{x(n)\}(\omega - \omega_0)\big]. \end{aligned} \tag{2.8}$$

(d) *Differentation in the frequency domain:* If $nx(n) \in l_1$, then we have:

$$\frac{dX_c(\omega)}{d\omega} = -\mathcal{F}_{sDT}\{nx(n)\}(\omega) \tag{2.9}$$

Remark. For the discrete-time Fourier sine we also have similar results.

3. Generalized convolutions

3.1. Generalized convolution with weighted function for discrete-time Fourier cosine transform

Definition 3.1. A generalized convolution for the Fourier cosine transforms with weighted function $\gamma(\omega) = \sin\omega$ is defined by:

$$\begin{aligned} (x \underset{\mathcal{F}_{cDT}}{\overset{\gamma}{*}} y)(n) = 2\sum_{m=0}^{\infty} x(m)\big[&y(n+m-1) + y(|n-m+1|) - \\ &- y(|n+m+1|) - y(|n-m-1|)\big]. \end{aligned} \tag{3.1}$$

Theorem 3.1. *If $x(n),\ y(n) \in l_1$, then $(x \underset{\mathcal{F}_{cDT}}{\overset{\gamma}{*}} y)(n) \in l_1$ and the following factorization equality holds*

$$\mathcal{F}_{cDT}(x \underset{\mathcal{F}_{cDT}}{\overset{\gamma}{*}} y)(\omega) = \sin\omega(\mathcal{F}_{sDT}x)(\omega)(\omega)(\mathcal{F}_{cDT}y)(\omega), \quad \omega \in [0,\pi]. \tag{3.2}$$

Proof. First, we prove that $(x \underset{\mathcal{F}_{cDT}}{\overset{\gamma}{*}} y)(n) \in l_1$:

$$\big\|x \underset{\mathcal{F}_{cDT}}{\overset{\gamma}{*}} y\big\|_1 \le \sum_{m=0}^{\infty} |x(m)| \sum_{r=m+1}^{\infty} |y(r)| + \sum_{m=0}^{\infty} |x(m)| \sum_{r=0}^{m+1} |y(r)| +$$

$$+ \sum_{m=0}^{\infty} |x(m)| \sum_{r=0}^{\infty} |y(r)| + \sum_{m=0}^{\infty} |x(m)| \sum_{r=m-1}^{\infty} |y(r)| + \sum_{m=0}^{\infty} |x(m)| \sum_{r=0}^{m-1} |y(r)| +$$

$$+ \sum_{m=0}^{\infty} |x(m)| \sum_{r=0}^{\infty} |y(r)| = 4 \sum_{m=0}^{\infty} |x(m)| \sum_{r=0}^{\infty} |y(r)| < \infty.$$

Then, we have $\|x \underset{\mathcal{F}_{cDT}}{\overset{\gamma}{*}} y\|_1 \le 4\|x\|_1\|y\|_1$. So $(x \underset{\mathcal{F}_{cDT}}{\overset{\gamma}{*}} y)(n)$ belongs to l_1.

Now we prove factorization equality (3.2). Since

$$\sin\omega \cdot (\mathcal{F}_{sDT}x)(\omega) \cdot (\mathcal{F}_{cDT}y)(\omega) =$$

$$= 4 \sum_{m=0}^{\infty} x(m) \sum_{n=0}^{\infty} y(n) \sin\omega \cdot \sin(m\,\omega) \cdot \cos(n\,\omega),$$

and

$$\sin\omega \cdot \sin(n\,\omega) \cdot \cos(m\,\omega) = \frac{1}{4}\big[\cos\omega\,(n+m-1) + \cos\omega\,(m-n-1) -$$

$$- \cos\omega\,(n+m+1) - \cos\omega\,(m-n+1)\big],$$

we have

$$\sin\omega \cdot (\mathcal{F}_{sDT}x)(\omega) \cdot (\mathcal{F}_{cDT}y)(\omega) =$$

$$= \sum_{m=0}^{\infty} x(m) \sum_{n=0}^{\infty} y(n)[\cos\omega(m+n-1) + \cos\omega(m-n-1) - \tag{3.3}$$

$$- \cos\omega(m+n+1) - \cos\omega(m-n+1)].$$

- With change of variable $m+n+1=t$, we obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty} x(m) \sum_{n=0}^{\infty} y(n) \cos\omega(m+n+1) = \\
&= \sum_{m=0}^{\infty} x(m) \sum_{t=m+1}^{\infty} y(t-m-1)\cos(\omega t) = \\
&= \sum_{m=0}^{\infty} x(m) \sum_{t=0}^{\infty} y(|t-m-1|)\cos(\omega t) - \\
&- \sum_{m=0}^{\infty} x(m) \sum_{t=0}^{m+1} y(|m+1-t|)\cos(\omega t).
\end{aligned} \tag{3.4}$$

- With $\sum\limits_{m=0}^{\infty} x(m) \sum\limits_{n=0}^{\infty} y(n)\cos\omega(m-n+1)$ and $m-n+1=-t$ we have

(3.5)

$$\begin{aligned}
&\sum_{m=0}^{\infty} x(m) \sum_{n=0}^{\infty} y(n) \cos\omega(m+1-n) = \\
&= \sum_{m=0}^{\infty} x(m) \sum_{t=-m-1}^{\infty} y(t+m+1)\cos(\omega t) = \\
&= \sum_{m=0}^{\infty} x(m) \sum_{t=0}^{\infty} y(t+m+1)\cos\omega t + \sum_{m=0}^{\infty} x(m) \sum_{t=-m-1}^{0} y(t+m+1)\cos(\omega t).
\end{aligned}$$

Moreover,

(3.6)

$$\sum_{m=0}^{\infty} x(m) \sum_{t=-m-1}^{0} y(t+m+1)\cos\omega t = \sum_{m=0}^{\infty} x(m) \sum_{t=0}^{m+1} y(|m+1-t|)\cos(\omega t).$$

Form (3.4), (3.5) and (3.6) we have:

$$\begin{aligned}
&\sum_{m=0}^{\infty} x(m) \sum_{n=0}^{\infty} y(n)\big[\cos\omega(m-n+1) + \cos\omega(m+n+1)\big] = \\
&= \sum_{m=0}^{\infty} x(m) \sum_{t=0}^{\infty} \big[y(t+m+1) + y(|t-m-1|)\big]\cos(\omega t).
\end{aligned} \tag{3.7}$$

• Similarly, with change of variable we obtain

$$\sum_{m=0}^{\infty} x(m) \sum_{n=0}^{\infty} y(n)\big[\cos\omega(m-1+n) + \cos\omega(m-1-n)\big] = \tag{3.8}$$
$$= \sum_{m=0}^{\infty} x(m) \sum_{t=0}^{\infty} \big[y(|t-m+1|) + y(|t+m-1|)\big] \cos(\omega t).$$

Finally, from (3.1), (3.7) and (3.8) we have:

$$\sin\omega \cdot (\mathcal{F}_{sDT}x)(\omega)(\mathcal{F}_{cDT}y)(\omega) = \sum_{m=0}^{\infty} x(m) \sum_{t=0}^{\infty} \big[y(t+m-1) + y(|t-m+1|) -$$
$$- y(|t+m+1|) - y(|t-m-1|)\big] \cos(\omega t). \qquad \blacksquare$$

Remark. Formula (3.2) shows that the convolution is non commutative.

3.2. Discrete-time Fourier sine generalized convolution with weighted function

Definition 3.2. The generalized convolution for the discrete-time Fourier sine with $\gamma(\omega) = \sin(\omega)$ weighted function is defined by

$$(x \underset{\mathcal{F}_{sDT}}{\overset{\gamma}{*}} y)(n) = 2 \sum_{m=0}^{\infty} x(m)\big[y(|m+n-1|) + \tag{3.9}$$
$$+ y(|n-m-1|) - y(n+m+1) - y(|n-m+1|)\big].$$

Similar Theorem 3.1, we get the following theorem:

Theorem 3.2. *If $x(n), y(n) \in l_1$, then $(x \underset{\mathcal{F}_{sDT}}{\overset{\gamma}{*}} y)(n) \in l_1$ and factorization equality*

$$\mathcal{F}_{sDT}(x \underset{\mathcal{F}_{sDT}}{\overset{\gamma}{*}} y)(\omega) = \sin\omega \cdot (\mathcal{F}_{cDT}x)(\omega)(\mathcal{F}_{cDT}y)(\omega). \tag{3.10}$$

Remark. Formula (3.10) shows that the convolution is commutative.

In order to construct a of Titchmarch's type theorem, we introduce the weighted space $l_1(e^n)$ as follow: $l_1(e^n) = \{x = \{x(n)\} : \sum\limits_{n=0}^{\infty} |x(n)e^n| < \infty\}$.

Theorem 3.3. (Theorem of Titchmarch's type.) *Let x, y be given sequences in weighted space $l_1(e^n)$. Then $(x \underset{F_{cDT}}{\overset{\gamma}{*}} y)(n) \equiv 0$ or $(x \underset{F_{sDT}}{\overset{\gamma}{*}} y)(n) \equiv 0$ if and only if $x(n) \equiv 0$ or $y(n) \equiv 0$ for all $n \geq 0$.*

4. Applications

In this section, we need to use the following propositions

Proposition 4.1. *Let $x(n), y(n)$ be two sequences. If $x(n), y(n) \in l_1$ then $(x \underset{F_{sDT}}{*} y)(n) \in l_1$ and factorization equality*

$$\mathcal{F}_{sDT}\{(x \underset{F_{sDT}}{*} y)(n)\}(\omega) = \mathcal{F}_{sDT}\{x(n)\}(\omega)\mathcal{F}_{cDT}\{y(n)\}(\omega), \ \forall\omega \in [0, \pi], \tag{4.1}$$

where $(x \underset{F_{sDT}}{*} y)(n)$ is defined by

$$(x \underset{F_{sDT}}{*} y)(n) = 2\sum_{k=0}^{\infty} x(k)\big[y(|n-k|) - y(n+k)\big].$$

Proposition 4.2. *Let $x(n), y(n)$ be two sequences. If $x(n), y(n) \in l_1$ then $(x \underset{F_{cDT}}{*} y)(n) \in l_1$ and factorization equality*

$$\mathcal{F}_{cDT}\{(x \underset{F_{cDT}}{*} y)(n)\}(\omega) = \mathcal{F}_{sDT}\{x(n)\}(\omega)\mathcal{F}_{sDT}\{y(n)\}(\omega), \ \forall\omega \in [0, \pi], \tag{4.2}$$

where $(x \underset{F_{cDT}}{*} y)(n)$ is defined by

$$(x \underset{F_{cDT}}{*} y)(n) = 2\sum_{k=0}^{\infty} x(k)\big[y(k+n) + y(|k-n|)\,\mathrm{sign}\,(k-n)\big].$$

Proposition 4.3. *Let $x(n), y(n)$ be two sequences. If $x(n), y(n) \in l_1$ then $(x * y)(n) \in l_1$ and factorization equality*

$$\mathcal{F}_{cDT}\{(x * y)(n)\}(\omega) = \mathcal{F}_{cDT}\{x(n)\}(\omega)\mathcal{F}_{cDT}\{y(n)\}(\omega), \ \forall\omega \in [0, \pi], \tag{4.3}$$

where $(x * y)(n)$ is defined by

$$(x * y)(n) = 2\sum_{k=0}^{\infty} x(k)\big[y(n+k) + y(|n-k|)\big] + x(0)\,y(n).$$

Proposition 4.4. (Theorem of Wiener–Levy type.)

(i) *Suppose that $x(n) \in l_1$ and $\Phi(z) \in L_\infty(0, \pi)$ is an analytic function. Then, exist $y(n) \in l_1$ so that*

$$\mathcal{F}_{cDT}\{y(n)\}(\omega) = \Phi(\mathcal{F}_{cDT}\{x(n)\}(\omega)).$$

(ii) *In particular, if $\mathcal{F}_{cDT}\{x(n)\}(\omega) \neq 0$, $x(n) \in l_1$, then unique existence $y(n) \in l_1$ such that*

$$\mathcal{F}_{cDT}\{y(n)\}(\omega) = \frac{1}{\mathcal{F}_{cDT}\{x(n)\}(\omega)}.$$

4.1. Infinite systems of linear algebraic equations of the first type

Consider the system of equations in the following form

$$\begin{aligned} x(n) + 2\sum_{m=0}^{\infty}\big(y \underset{F_{sDT}}{*} z\big)(m)\big[x(n+m-1) + x(|n-m+1|) - \\ -x(|n+m+1|) - x(|n-m-1|)\big] = h(n) \end{aligned} \tag{4.4}$$

where $(y \underset{F_{sDT}}{*} z)(n)$ is defined by (4.1); $y(n), z(n), h(n) \in l_1$ are given sequences and $x(n)$ is the unknown sequence.

Theorem 4.5. *Let $y(n), z(n), h(n) \in l_1$ and satisfy*

$$1 + \sin(\omega) \cdot \mathcal{F}_{sDT}\{y(n)\}(\omega)\mathcal{F}_{cDT}\{z(n)\}(\omega) \neq 0, \ \forall \omega \in [0, \pi]$$

Then the system of equations (4.4) *has unique solution $x \in l_1$*

$$x(n) = h(n) - (h * u)(n) \in l_1,$$

where $u(n) \in l_1$ is defined by

$$\mathcal{F}_{cDT}\{u(n)\}(\omega) = \frac{\sin(\omega)\mathcal{F}_{sDT}\{y(n)\}(\omega)\mathcal{F}_{cDT}\{z(n)\}(\omega)}{1 + \sin(\omega)\mathcal{F}_{sDT}\{y(n)\}(\omega)\mathcal{F}_{cDT}\{z(n)\}(\omega)}.$$

4.2. Infinite systems of linear algebraic equations of the second type

Consider the system of equations in the following form

$$\begin{cases} x(n) + 2\displaystyle\sum_{m=0}^{\infty} y(m)\big[u(n+m-1) + u(|n-m+1|) - \\ \qquad -u(|m+n+1|) - u(|n-m-1|)\big] = z(n), \\ 2\displaystyle\sum_{k=0}^{\infty} v(k)[x(|n-k|) - x(n+k)] + y(n) = w(n), \end{cases} \tag{4.5}$$

where, $u(n), z(n), v(n)$ and $w(n)$ are given sequences, $x(n)$ and $y(n)$ is unknow.

Theorem 4.6. *If $u(n), v(n), z(n), w(n) \in l_1$ and satisfy*

$$1 - \sin(\omega)\mathcal{F}_{sDT}\{v(n)\}(\omega)\mathcal{F}_{cDT}\{u(n)\}(\omega) \neq 0, \ \forall \omega \in [0, \pi],$$

then the system of equations (4.5) *has unique solution* $(x(n), y(n)) \in (l_1, l_1)$

$$\begin{cases} x(n) = & z(n) - (w \underset{F_{cDT}}{\overset{\gamma}{*}} u)(n) + (h * z)(n) - \big(h * (w \underset{F_{cDT}}{\overset{\gamma}{*}} u)\big)(n) \in l_1, \\ y(n) = & w(n) - (v \underset{F_{sDT}}{*} z)(n) + (w \underset{F_{sDT}}{*} h)(n) - \big((v \underset{F_{sDT}}{*} z) \underset{F_{sDT}}{*} h\big)(n) \in l_1. \end{cases}$$

where $h(n) \in l_1$, *is defined by*

$$\mathcal{F}_{cDT}\{h(n)\}(\omega) = \frac{\sin(\omega) \cdot \mathcal{F}_{sDT}\{v(n)\}(\omega)\mathcal{F}_{cDT}\{u(n)\}(\omega)}{1 - \sin(\omega) \cdot \mathcal{F}_{sDT}\{v(n)\}(\omega)\mathcal{F}_{cDT}\{u(n)\}(\omega)}.$$

Proof. Applying the discrete-time Fourier cosine transform to both sides of the first equation and discrete-time Fourier sine transform to both sides of the second equation of system (4.5), we obtain

$$\begin{cases} X_c(\omega) + \sin\omega Y_s(\omega)U_c(\omega) & = Z_c(\omega) \\ V_s(\omega)X_c(\omega) + Y_s(\omega) & = W_s(\omega). \end{cases}$$

We have

$$\Delta = 1 - \sin\omega V_s(\omega)U_c(\omega) \neq 0, \tag{4.6}$$

$$\Delta_1 = Z_c(\omega) - \sin\omega W_s(\omega)U_c(\omega) \text{ and } \Delta_2 = W_s(\omega) - V_s(\omega)Z_c(\omega).$$

By the Proposition 4.4 (Theorem of Wiener-Levy type), exist a sequence $h(n) \in$ $\in l_1$ so that

$$H_c(\omega) := \mathcal{F}_{cDT}\{h(n)\}(\omega) = \frac{\sin(\omega) \cdot V_s(\omega)U_c(\omega)}{1 - \sin(\omega) \cdot V_s(\omega)U_c(\omega)}. \tag{4.7}$$

From (4.5), (4.6), (4.7) and convolution theorems, we have:

$$\begin{aligned} X_c(\omega) = \frac{\Delta_1}{\Delta} &= Z_c(\omega) - \sin(\omega) \cdot W_s(\omega)U_c(\omega) + Z_c(\omega)H_c(\omega) - \\ &\quad - \sin(\omega) \cdot W_s(\omega)U_c(\omega)H_c(\omega), \\ Y_s(\omega) = \frac{\Delta_2}{\Delta} &= W_s(\omega) - V_s(\omega)Z_c(\omega) + W_s(\omega)H_c(\omega) - V_s(\omega)Z_c(\omega)H_c(\omega). \end{aligned}$$

From uniaxial of the discrete-time Fourier cosine and discrete-time Fourier sine transforms, we have

$$\begin{cases} x(n) = z(n) - (w \underset{F_{cDT}}{\overset{\gamma}{*}} u)(n) + (h * z)(n) - \big(h * (w \underset{F_{cDT}}{\overset{\gamma}{*}} u)\big)(n) \in l_1, \\ y(n) = w(n) - (v \underset{F_{sDT}}{*} z)(n) + (w \underset{F_{sDT}}{*} h)(n) - \big((v \underset{F_{sDT}}{*} z) \underset{F_{sDT}}{*} h\big)(n) \in l_1. \end{cases} \blacksquare$$

4.3. Infinite systems of linear algebraic equations of the third type

Consider the system of equations

$$\tag{4.8} \begin{cases} x(n) + 2\sum_{m=0}^{\infty} u(m)[y(|n+m-1|) + y(|n-m-1|) - \\ \qquad\qquad -y(n+m+1) - y(|n-m+1|) = z(n) \\ 2\sum_{k=0}^{\infty} v(k)[x(k+n) + x(|k-n|)\,\mathrm{sign}\,(k-n)] + y(n) = w(n), \end{cases}$$

where $u(n), z(n), v(n)$ and $w(n)$ are given sequences; $x(n)$ and $y(n)$ are unknown.

Theorem 4.7. *If $u(n), z(n), v(n), w(n) \in l_1$, and satisfy*

$$1 - \sin(\omega)\mathcal{F}_{cDT}\{u(n)\}(\omega)\mathcal{F}_{sDT}\{v(n)\}(\omega) \neq 0, \ \forall\omega \in [0, \pi],$$

then the system of equations (4.8) has unique solution $(x(n), y(n)) \in l_1 \times l_1$:

$$\begin{cases} x(n) = z(n) - (y \underset{\mathcal{F}_{sDT}}{\overset{\gamma}{*}} w)(n) + (z \underset{F_{sDT}}{*} h)(n) - ((y \underset{\mathcal{F}_{sDT}}{\overset{\gamma}{*}} w) \underset{F_{sDT}}{*} h)(n), \\ y(n) = w(n) - (z \underset{F_{cDT}}{*} v)(n) + (w * h)(n) - (h * (z \underset{F_{cDT}}{*} v))(n). \end{cases}$$

Here, $h(n) \in l_1$ is defined by

$$\mathcal{F}_{cDT}\{h(n)\}(\omega) = \frac{\sin(\omega)\cdot\mathcal{F}_{cDT}\{u(n)\}(\omega)\mathcal{F}_{sDT}\{v(n)\}(\omega)}{1 - \sin(\omega)\cdot\mathcal{F}_{cDT}\{u(n)\}(\omega)\mathcal{F}_{sDT}\{v(n)\}(\omega)}.$$

References

[1] **Anh, H.T.V. and N.X. Thao,** Hartley–Fourier cosine generalized convolusion inequalities, *Math. Inequal. Appl.*, **18** (2015), 1393–1408.

[2] **Strang, G.,** The discrete cosine transform, *SIAM*, **41** (1999), 135–147.

[3] **Pei, S.C. and J.J. Ding,** Fractional cosine, sine and Hartley transforms, *IEEE Trans. Signal Process.*, **50** (2002), 1661–1680.

[4] **Soo-Chang Pei and Min-Hung Yeh,** The discrete fractional cosine and sine transforms, *IEEE Trans. Signal Process.*, **49** (2001), 1198–1207.

[5] **Thao, N.X., V.K. Tuan and N.T. Hong,** A Fourier generalized convolution transform and applications to integral equations, *Fract. Calc. and Appl. Anal.* , **15** (2012), 493–508.

[6] **Thao, N.X., V.K. Tuan and H.T.V. Anh,** On the Toeplitz plus Hankel integral equation II, *Integral Transforms Spec. Funct.*, **25**, (2014), 75–84.

[7] **Bochner, S. and K. Chandrasekharan,** *Fourier Transforms*, Princeton Univ. Press., 1949.

[8] **Oppenheim, A.V. and R.W. Schafer,** *Discrete-Time Signal Processing*, Pretice Hall, Englewood Cliffs, NJ., 1989.

[9] **Poularikas, A.D.,** *The Transforms and Applications*, handbook CLC Press LLC, 2000.

[10] **Sneddon, I.N.,** *The Use of Integral Transforms*, Mc Gray-Hill, New York, 1972.

[11] **Zygmund, A.,** *Trigonometric Series*, Vol. I and Vol. II, Cambridge University Press, Cambridge, 1959.

N. A. Dai and N. X. Thao
Hanoi University of Science and Technology
Ha Noi
Viet Nam
daina@utehy.edu.vn
thaonxbmai@yahoo.com