Annales Univ. Sci. Budapest., Sect. Comp. 47 (2018) 211–226

EXPONENTIAL DICHOTOMY AND THE STABILITY OF LINEAR SYSTEMS

Szilvia Császár and Sándor Kovács
(Budapest, Hungary)

Communicated by Ferenc Schipp

(Received March 30, 2018; accepted June 16, 2018)

Abstract. This paper examines the relation of the exponential dichotomy and the stability concepts for systems of linear differential equations. We are going to show some relationship between the studied concepts, more precisely we are presenting how the stability of a linear non-autonomous system is investigated with the help of the exponential dichotomy. Furthermore we are going to show how the stable and unstable subspace of an exponentially dichotomic system can be specified using the definition of the exponential dichotomy.

1. Introduction

The asymptotic behaviour of the solutions and the stability of the equilibrium points is an important element in the investigation of systems of differential equations. As it is well known the equilibrium point ξ of the autonomous system

$$\dot{x} = f \circ x \tag{1.1}$$

is asymptotically stable, provided the Jacobian $f'(\xi)$ is Hurwitz-stable, where $f \in \mathfrak{C}^1(\Omega, \mathbb{R}^n)$ with a domain $\Omega \subset \mathbb{R}^n$ and $\xi \in \Omega$.

Key words and phrases: Exponential dichotomy, stability.
2010 Mathematics Subject Classification: 34D09, 34D20.

Let us consider the non-autonomous system

$$\dot{x} = Ax. \tag{1.2}$$

with coefficient matrix $A \in \mathfrak{C}(\mathbb{R}, \mathbb{R}^{n\times n})$. If A is constant, then the stability of the trivial equilibrium of system (1.2) or the system itself is determined by the spectral properties of the coefficient matrix A (cf. [5]). Nevertheless, there is some non-autonomous system, where we couldn't determine the stability of the system by looking at the eigenvalues of the coefficient matrix. This is the case for example by the classical Markus–Yamabe system (cf. [6]) with coefficient matrix

$$A(t) := \begin{bmatrix} -2+2\cos^2(t) & 1-\sin(2t) \\ -1-\sin(2t) & -2+2\sin^2(t) \end{bmatrix} \qquad (t \in \mathbb{R}).$$

One checks that for all $t \in \mathbb{R}$, $A(t)$ admits -1 as an eigenvalue of algebraic multiplicity 2. Indeed, the characteristic polynomial of $A(t)$ with a fixed $t \in \mathbb{R}$ has the form

$$z^2 - \mathrm{Tr}(A)z + \det(A) = z^2 + 2z + 1 = (z+1)^2 \qquad (z \in \mathbb{C}).$$

It is not difficult to see that (1.2) has the unbounded solution

$$\varphi(t) = \begin{bmatrix} -e^t\cos(t) \\ e^t\sin(t) \end{bmatrix} \qquad (t \in \mathbb{R}). \tag{1.3}$$

This solution and henceforth system (1.2) is not stable, because the stability of a solution of a linear system implies its boundedness (in the positive half line).

Therefore another approach is needed for the investigation of the stability of non-autonomous linear systems. The notion of the exponential dichotomy offers a possibility to the generalization of the concept of the stability and asymptotically stability.

Hereinafter in this paper we will consider the linear system

$$\dot{x} = Ax, \tag{1.4}$$

where $A \in \mathfrak{C}(\mathbb{R}, \mathbb{R}^{n\times n})$ with a fixed dimension $n \in \mathbb{N}$. Let us denote the fundamental matrix of system (1.4) by Φ, i.e. let Φ be a regular matrix solution of (1.4). Thus, the entire solution φ of system (1.4) with an initial condition $x(\tau) = \xi$ has the form

$$\varphi(t) = \Lambda(t,\tau)\xi := \Phi(t)\Phi^{-1}(\tau)\xi \qquad (t \in J)$$

where $\Lambda(\cdot,\tau)$ denotes the Cauchy matrix of (1.4). The paper is organized as follows. In the next section after a brief introduction we define the concept of the exponential dichotomy in the sense of Coppel (cf. [3]), then we shortly review some basic tools from exponential dichotomies. In Section 3 we show what the connection is between the exponential dichotomy and the stability of the linear system (1.4). In the last section of the paper we examine the stable and unstable subspaces of system (1.4) when it possesses an exponential dichotomy.

2. Exponential dichotomy

In order to prepare the concept of the exponential dichotomy we consider a constant matrix $A \in \mathbb{R}^{n\times n}$ and a function $f \in \mathfrak{C}^1(\mathbb{R}, \mathbb{R}^n)$ which form the inhomogeneous and at the same time non-autonomous system

$$\dot{x} = Ax + f \tag{2.1}$$

defined on the whole real line. It can be easily shown that the following three statements are equivalent:

(i) A is hyperbolic, i.e. A has no eigenvalues on the imaginary axis,

(ii) the homogeneous part of (2.1) has no nontrivial bounded solution on $\mathbb{R}$,

(iii) there exists a projection P such that for each bounded $f \in \mathfrak{C}(\mathbb{R}, \mathbb{R}^n)$ the inhomogeneous system (2.1) has a unique solution φ that is bounded on $\mathbb{R}$ and given by
(2.2)
$$\varphi(t) \equiv \int_{-\infty}^{\infty} K(t-s)f(s)\mathrm{d}s \equiv \int_{-\infty}^{t} e^{A(t-s)}Pf(s)\mathrm{d}s - \int_{t}^{\infty} e^{A(t-s)}(I-P)f(s)\mathrm{d}s,$$
where
$$K(t-s) := \begin{cases} \exp((t-s)A)P & (t \geq s), \\ \exp((t-s)A)(I-P) & (t < s). \end{cases} \tag{2.3}$$

Indeed,

- (i) equivalent to (ii), because with the help of the Jordan canonical form of A
$$A = T^{-1}\begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} T =: T^{-1}(A_- + A_+)T$$

we can write

$$\Phi(t) = T^{-1}\exp(tA_-)\exp(tA_+)T \qquad (t \in \mathbb{R}),$$

where J_-, resp. J_+ are matrices having eigenvalues in the negative, resp. in the positive half plane:

$$\sigma(J_-) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \Re(z) < 0\},$$

resp.

$$\sigma(J_+) \subset \mathbb{C}^+ := \{z \in \mathbb{C} \,:\, \Re(z) > 0\}.$$

Hence every solution φ can be written as a sum of two functions,

$$\varphi = \varphi_- + \varphi_+,$$

such that

$$\lim_{t\to+\infty} \|\varphi_+(t)\| = +\infty \qquad \text{and} \qquad \lim_{t\to-\infty} \|\varphi_-(t)\| = +\infty,$$

and these properties imply statement (ii). For the reverse direction, let us assume indirectly that (ii) holds, but there is a $\lambda \in \sigma(A)$ eigenvalue such that $\Re(\lambda) = 0$, i.e. $\lambda_{1,2} := \pm\alpha\imath$ are eigenvalues for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Similarly to the previous part of the proof we can decompose the matrix A into the Jordan form

$$A = T^{-1}\begin{bmatrix} B & 0 \\ 0 & \tilde{A} \end{bmatrix} T,$$

where $B \in \mathbb{R}^{2\times 2}$, $\sigma(B) = \{\pm i\alpha\}$. Every solution φ is a sum of two functions, say $\varphi = \varphi_B + \varphi_{\tilde{A}}$, where φ_B is bounded on $\mathbb{R}$, which contradicts to statement (ii).

- Let us assume that (ii) holds. First, we are going to show that the function φ defined in (2.2) is a solution of system (2.1):

$$\begin{aligned} \dot{\varphi}(t) &= Pf(t) + \int_{-\infty}^{t} Ae^{A(t-s)}Pf(s)\mathrm{d}s + \\ &+ (I-P)f(t) - \int_{t}^{\infty} Ae^{A(t-s)}(I-P)f(s)\mathrm{d}s = A\varphi(t) + f(t). \end{aligned}$$

Let

$$S := \{\lambda \in \sigma(A) \,:\, \Re(\lambda) < 0\}$$

be the set of eigenvalues of matrix A in the open-left-half plane and let P be the Riesz-projection for S, i.e.

$$P := \frac{1}{2\pi\imath} \int\limits_{\gamma} (zI - A)\mathrm{d}z, \tag{2.4}$$

where γ is any rectifiable simple closed curve in the open left half-plane containing in its interior all eigenvalues of A with negative real part. Then due to the hyperbolicity of A we have

$$\sigma(A|_M) = S, \qquad \sigma(A|_L) = \sigma(A) \setminus S (= \{\lambda \in \sigma(A) \; : \; \Re(\lambda) > 0\}),$$

where $M = \mathrm{Im}(P)$ and $L = \mathrm{Ker}(P)$ (cf. [7]), furthermore there exist constants K_1, K_2 and α_1, $\alpha_2 > 0$ such that

$$\begin{aligned} \|\exp{(t-s)AP}\| &\leq K_1 e^{-\alpha_1(t-s)} \qquad (t \geq s), \\ \|\exp((t-s)A)(I-P)\| &\leq K_2 e^{-\alpha_2(s-t)} \qquad (s \geq t). \end{aligned}$$

With the above two inequalities we can show the boundedness of the solution φ. Because for each $t \in \mathbb{R}$ the estimation

$$\begin{aligned} |\varphi(t)| &\leq \left(\left| \int\limits_{-\infty}^{t} K_1 e^{-\alpha_1(t-s)} \mathrm{d}s \right| + \left| \int\limits_{t}^{\infty} K_2 e^{-\alpha_2(s-1)} \mathrm{d}s \right| \right) \cdot \|f\|_\infty = \\ &= \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right) \|f\|_\infty, \end{aligned}$$

holds (cf. (2.2)), we have

$$\|\varphi\|_\infty \leq \left(\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right) \|f\|_\infty.$$

Finally one can prove the uniqueness of φ by the superposition principle from the uniqueness in (ii). Conversely if (iii) holds, we can get with $f \equiv 0$ (or by the superposition principle) that statement (iii) also holds.

The concept of the exponential dichotomy generalizes the third property for non-autonomous systems. In this paper we will use the definition from Coppel's book (cf. [3]).

Definition 2.1. (See [3].) The equation (1.4) is said to possess an exponential dichotomy if there exists a projection P and positive constants K_1, K_2, α_1, α_2

such that

$$
\|\Phi(t)P\Phi^{-1}(s)\| \le K_1 e^{-\alpha_1(t-s)} \qquad (t \ge s),
$$

(2.5)

$$
\|\Phi(t)(I-P)\Phi^{-1}(s)\| \le K_2 e^{-\alpha_2(s-t)} \qquad (s \ge t).
$$

It is said to possess an ordinary dichotomy if the inequalities (2.5) hold with $\alpha_1 = \alpha_2 = 0$.

For the completeness of the treatment we recall a theorem from [3] which will be useful later on.

Proposition 2.1. (See [3], [10].) *Let $A \in \mathbb{R}^{n\times n}$ be a matrix and let us consider the autonomous system*

$$
\dot{x} = Ax. \tag{2.6}
$$

It has an exponential dichotomy on $J := [0, +\infty)$ if and only if no eigenvalue of the constant matrix A has zero real part. It has an ordinary dichotomy, if and only if all eigenvalues of A with zero real part are semisimple (which means that this eigenvalue is a simple root of the minimal polynomial of A). In each case we can take the projection P to be the spectral projection defined by as in (2.4).

3. Stability and exponential dichotomy

In the first claim of this chapter we remind of two necessary and sufficient conditions for stability of linear systems.

Proposition 3.1. (See [2].) *Let us consider the homogeneous linear system* (1.4) *on the interval $J = [\tau, +\infty)$, $\tau \in \mathbb{R}$. The linear system* (1.4) *is*

1. *stable if and only if for each $s > \tau$ there exists a constant $K > 0$ such that*
$$
\|\Phi(t)\Phi^{-1}(s)\| \le K \qquad (t \ge s),
$$
2. *asymptotically stable if and only if for each $s > \tau$ there exist two constants $K,\ \alpha > 0$ such that*

$$
\|\Phi(t)\Phi^{-1}(s)\| \le K e^{-\alpha(t-s)} \qquad (t \ge s).
$$

The above statement is a consequence of the point 1 of Proposition 3.1.

Corollary 3.1. *The linear system* (1.4) *is stable if and only if each of its solutions is bounded on the positive half line, that is for every solution* $\varphi : (\tau, +\infty) \to \mathbb{R}^n$ *there exists a constant* $K \geq 0$ *such that*

$$\|\varphi(t)\| \leq K$$

holds for all $t \in (\tau, +\infty)$.

In the next theorem we are going to show a connection between the notion of the exponential dichotomy and the stability of linear systems, more precisely we are going to show that a linear system (1.4) is stable, resp. asymptotically stable if and only if it has ordinary resp. exponential dichotomy with a suitable projection P, therefore the concept of the exponential dichotomy generalizes the concept of stability.

Theorem 3.2. *The linear system* (1.4) *is*

1. *asymptotically stable if and only if it admits an exponential dichotomy with projection* $P = I$*, furthermore*

2. *stable if and only if it admits an ordinary dichotomy with projection* $P = I$.

Proof. We are going to prove the statement 1 of Theorem 3.2, the verification of the second part is exactly like the first part.

Step 1. Suppose that system (1.4) is asymptotically stable. Let $J = [\tau, +\infty)$ be an interval with some $\tau \in \mathbb{R}$. It follows from the part 2 of Proposition 3.1 that there are positive constants K_1, $\alpha > 0$ such that

$$\|\Phi(t)\Phi^{-1}(\tau)\| \leq K_1 e^{-\alpha(t-\tau)} \qquad (t \geq \tau),$$

hence proceeding from the assumption $\Phi(\tau) = I$ one can get the

$$\|\Phi(t)\| \leq K_1 e^{-\alpha(t-\tau)} \qquad (t \geq \tau) \tag{3.1}$$

estimate. Furthermore, from Corollary 3.1 we get a constant $K_2 \in \mathbb{R}$ such that

$$\|\Phi^{-1}(s)\| \leq K_2 \qquad (s \geq \tau). \tag{3.2}$$

Let $K := K_1 \cdot K_2$, based on the inequalities (3.1) and (3.2) it can be seen that the following estimate is true

$$\|\Phi(t)\Phi^{-1}(s)\| = \|\Phi(t)\| \cdot \|\Phi^{-1}(s)\| \leq K e^{-\alpha(t-\tau)} \leq K e^{-\alpha(t-s)} \qquad (s, t \geq \tau).$$

Finally, if we choose the projection $P := I$, the inequalities (2.5) in the Definition 2.1 hold for all $s, t \in J$

$$\begin{aligned} \|\Phi(t)P\Phi^{-1}(s)\| &= \|\Phi(t)\Phi^{-1}(s)\| \leq K e^{-\alpha(t-s)} && (t \geq s), \\ \|\Phi(t)(I-P)\Phi^{-1}(s)\| &= \|0\| = 0 && (t \leq s), \end{aligned}$$

consequently the system (1.4) has an exponential dichotomy with projection $P = I$.

Step 2. Suppose that system (1.4) has an exponential dichotomy with constants K_1, K_2, α_1, α_2 and projection $P = I$. Then by using the first inequality from equations (2.5) for each $t, s \in J$ we receive the following

$$\|\Phi(t)\Phi^{-1}(s)\| = \|\Phi(t)P\Phi^{-1}(s)\| \leq K_1 e^{-\alpha_1(t-s)} \qquad (t \geq s),$$

hence it follows from Proposition 3.1 that system (1.4) is asymptotically stable.

■

Remark 3.1. Following Theorem 3.2 with compared to Corollary 3.1 we can say that all of the solutions of system (1.4) are bounded on J if and only if the system possesses an ordinary dichotomy with projection $P = I$ on the interval J.

Remark 3.2. It can be easily seen from Proposition 2.1 that if we consider the autonomous system (2.6), then the conditions in Theorem 3.2 for the stability and the asymptotically stability of system (2.6) are equal to the usual conditions about the sign of the real parts of the eigenvalues of the coefficient matrix in system (2.6), because in case of (asymptotic) stability the Riesz projection defined in (2.4) is identical with the identity matrix.

Remark 3.3. In the above theorem the condition with projector $P = I$ is essential, cf. Example 3.3.

Finally we are going to show four examples to illustrate the above theorem.

Example 3.1. Let us consider the autonomous system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x. \tag{3.3}$$

From Proposition 2.1 we know that system (3.3) possesses an ordinary dichotomy with projection $P = I$. On the other hand the eigenvalues of the coefficient matrix are $\lambda_1 = -1$, $\lambda_2 = 0$, hence system (3.3) is stable.

Example 3.2. In the second example let us consider the following system:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x. \tag{3.4}$$

Again from Proposition 2.1 we know that system (3.4) has an exponential dichotomy with the same projection as in the previous example. On the other hand for each eigenvalues λ of the coefficient matrix $\Re(\lambda) < 0$, hence system (3.4) is asymptotically stable.

Example 3.3. In the third example let us consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x. \tag{3.5}$$

The following function is a solution of system (3.5) with initial value $x(0,0) =$ $= (1,0)$

$$\varphi(t) = \Phi(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} \qquad (t \in \mathbb{R}).$$

As it can be seen, φ is not bounded on $\mathbb{R}$, hence system (3.5) is unstable following from Corollary 3.1, so it is not asymptotically stable. Furthermore system (3.5) has got an exponential dichotomy with projection

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is not the identity operator.

The last example is a similar counterexample to the Markus–Yamabe's one mentioned in the Introduction.

Example 3.4. Let us consider the linear system (1.4) with coefficient function $A(t) = U^{-1}(t)A_0U(t)$ for $t \in \mathbb{R}$, where

$$U(t) := \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}, \qquad A_0 := \begin{bmatrix} -1 & 5 \\ 0 & -1 \end{bmatrix}.$$

As it can be seen for each $t \in \mathbb{R}$ the matrices $A(t)$ and A_0 are similar, thus both of the eigenvalues of $A(t)$ are -1. Even so the system corresponding to $A(t)$ is unstable, cf. [3], because the fundamental operator of the system is the following:

$$\Phi(t) = \begin{bmatrix} e^{t}(\cos(t) + \frac{1}{2}\sin(t)) & e^{-3t}(\cos(t) - \frac{1}{2}\sin(t)) \\ e^{t}(\sin(t) - \frac{1}{2}\cos(t)) & e^{-3t}(\sin(t) + \frac{1}{2}\cos(t)) \end{bmatrix} \qquad (t \in \mathbb{R}).$$

We will prove this stability result with Theorem 3.2 by showing that the system possesses an exponential dichotomy with projection $P \neq I$.

The coefficient function $A(t)$ is periodic with period $T = \pi$, so we can ascertain the existence of the exponential dichotomy and calculate the corresponding projection using the characteristic multipliers of the system (cf. [3], [11]). These numbers are the eigenvalues of the monodromy matrix B which is determined as

$$B := \Phi^{-1}(0)\Phi(T) = \Phi^{-1}(0)\Phi(\pi) = \begin{bmatrix} -e^{\pi} & 0 \\ 0 & e^{-3\pi} \end{bmatrix}.$$

Thus the system possesses an exponential dichotomy with projection

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Remark 3.4. In some specific cases of linear systems it is easy to check the existence of the exponential dichotomy and calculate the projection P. For example as we have shown in Proposition 2.1 (cf. [3]), in the case of autonomous systems it is sufficient to examine the sign of the real part of the eigenvalues of the coefficient matrix. In a special non-autonomous case, when the coefficient function of system (1.4) is periodic we can prove the existence of the exponential dichotomy using the Floquet-theory (cf. [3], [11]), as we have seen that in Example 3.4. Finally in more general cases we can use perturbation theorems (cf. [1], [3], [8]) or numerical methods (cf. [4]) to show the existence of the exponential dichotomy.

3.1. On the stable and unstable subspaces

Proposition 3.3. *Let us assume that system* (1.4) *possesses an exponential dichotomy on the interval* $J := [0, +\infty)$ *with projection* P. *For any* $t \in J$ *we are defining the following two sets:*

$$S_-(t) := \{\Phi(t)Px_0 \ : \ x_0 \in \mathbb{R}^n\} = \operatorname{Im}\Phi(t)P,$$

$$S_+(t) := \{\Phi(t)(I-P)x_0 \ : \ x_0 \in \mathbb{R}^n\} = \operatorname{Im}\Phi(t)(I-P).$$

Then for any $t \in J$

$$S_-(t) \oplus S_+(t) = \mathbb{R}^n,$$

the sets $S_-(\cdot)$ *and* $S_+(\cdot)$ *are positively invariants in the following sense: if for a solution* φ *of system* (1.4) $\varphi(0) \in S_-(0)$, *resp.* $\varphi(0) \in S_+(0)$ *then for all* $t \in \mathbb{R}^0_+$ $\varphi(t) \in S_-(t)$, *resp.* $\varphi(t) \in S_+(t)$, *furthermore for each solution of the system* φ *with* $\varphi(0) \in S_-(0)$, *resp.* $\varphi(0) \in S_+(0)$, *two following statements hold:*

$$\lim_{t\to+\infty} \|\varphi(t)\| = 0, \qquad \textit{resp.} \qquad \lim_{t\to+\infty} \|\varphi(t)\| = +\infty. \tag{3.6}$$

Proof. As a first step we have to show that

$$S_-(t) \oplus S_+(t) = \mathbb{R}^n.$$

Let us assume that φ is a solution of system (1.4) and let $\mathbb{R}^n \ni x_0 := \varphi(\tau)$. Then for any $t \geq \tau$ we can rewrite $\varphi(t)$ as

$$\varphi(t) = \Phi(t)x_0 = \Phi(t)Px_0 + \Phi(t)(I-P)x_0,$$

thus for all $x_0 \in \mathbb{R}^n$ φ can be written as the sum of an element from $S_-(t)$ and an other from $S_+(t)$.

Let $t \in \mathbb{R}$ be fixed, $z \in S_-(t) \cap S_+(t)$. Then there exist two constants $z_s,\ z_u \in \mathbb{R}^n$ such that

$$z = \Phi(t)Pz_s, \qquad z = \Phi(t)(I-P)z_u,$$

so

$$\Phi(t)Pz_s = \Phi(t)(I-P)z_u,$$

and so because of the regularity of the matrix $\Phi(t)$ we get $Pz_s = (I-P)z_u$. Then by using the fact that P is a projection, i.e. $P^2 = P$, one can receive the identity

$$Pz_s = P^2 z_s = P(I-P)z_u = Pz_u - P^2 z_u = Pz_u - Pz_u = 0.$$

From this

$$z = \Phi(t)Pz_s = 0,$$

hence $S_-(t) \cap S_+(t) = \{0\}$.

After that we are going to prove the invariance of $S_-(\cdot)$. Let $\varphi(0) \in S_-(0)$ be a fixed element, so with a suitable $x_0 \in \mathbb{R}^n$ vector $\varphi(0) = \Phi(0)Px_0$. In this case for each $t \in \mathbb{R}_+$

$$\varphi(t) = \Lambda(t,0)\varphi(0) = \Lambda(t,0)\Phi(0)Px_0 = \Lambda(t,0)\Lambda(0,0)Px_0 = \Lambda(t,0)Px_0,$$

thus $\varphi(t) \in S_-(t)$. The proof is the same for $S_+(\cdot)$.

In the remaining part we are going to show the asymptotic properties (3.6). Let $\varphi(0) \in S_-(0)$ be again a fixed element such that $\varphi(0) \neq 0$, so $Px_0 \neq 0$. With a fixed $s \in \mathbb{R}_0^+$ for each $t \geq s$ it follows

$$\begin{aligned} \frac{\|\varphi(t)\|}{\|\varphi(s)\|} &= \frac{\|\Phi(t)Px_0\|}{\|\Phi(s)Px_0\|} = \frac{\|\Phi(t)P\Phi^{-1}(s)\Phi(s)Px_0\|}{\|\Phi(s)Px_0\|} \leq \\ &\leq \|\Phi(t)P\Phi^{-1}(s)\| \leq K_1 e^{-\alpha_1(t-s)}, \end{aligned}$$

thus

$$\|\varphi(t)\| \leq K_1 e^{-\alpha_1(t-s)}\|\varphi(s)\|, \tag{3.7}$$

and from the above inequality (3.7) follows the first statement in (3.6), i. e.

$$\lim_{t\to+\infty} \|\varphi(t)\| = 0.$$

The second statement in (3.6) can be proved in a similar way. Let $\varphi(0) \in S_+(0)$ such that $\varphi(0) \neq 0$, so $(I-P)x_0 \neq 0$. Then for all $s \geq t$

$$\frac{\|\varphi(t)\|}{\|\varphi(s)\|} \leq \|\Phi(t)(I-P)\Phi^{-1}(s)\| \leq K_2 e^{-\alpha_2(s-t)},$$

hence the below inequality holds

$$\|\varphi(s)\| \geq \frac{1}{K_2} e^{\alpha_2(s-t)} \|\varphi(t)\|, \tag{3.8}$$

from which follows

$$\lim_{t\to+\infty} \|\varphi(t)\| = +\infty. \qquad \blacksquare$$

Based on the inequalities (3.7) and (3.8) in the above proof we could think that if system (1.4) admits an exponential dichotomy, not only the asymptotic properties (3.6) hold for the solutions of the system, but also the solutions are monotone decreasing or increasing in norm. Nevertheless this monotonicity property is not true, that we are illustrating in the next example.

Example 3.5. Let us consider the following autonomous system:

$$\dot{x} = \begin{bmatrix} -1 & 2 \\ -8 & -1 \end{bmatrix} x. \tag{3.9}$$

The eigenvalues of the coefficient matrix are $\lambda_{1,2} = -1 \pm 4i$, thus the system (3.9) is asymptotically stable, and hence, according to Theorem 3.2 the system (3.9) possesses an exponential dichotomy on the interval $J = \mathbb{R}$ with projection $P = I$. The phase portrait of system (3.9) can be seen in Figure 1.

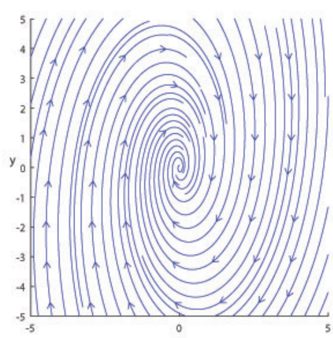

Figure 1. The phase portrait of system (3.9). It can be seen that the origin is asymptotically stable.

Let us consider the solution of system (3.9) with initial condition $x_0 = (1, 1)$:

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} = \begin{bmatrix} \cos(4t)e^{-t} + \dfrac{\sin(4t)e^{-t})}{2} \\ \cos(4t)e^{-t} - 2\sin(4t)e^{-t} \end{bmatrix} \qquad (t \in \mathbb{R}). \tag{3.10}$$

In Figure 3.5 the norm $\|\cdot\|_2$ of the solution φ is shown on the interval $t \in [0, 6]$. The asymptotic properties (3.6) hold for the solution (3.10), but as also it is outlined in Figure 2, the solution is not monotone decreasing.

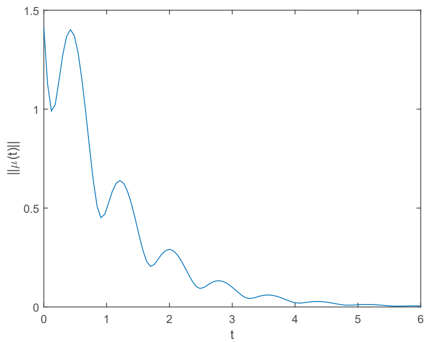

Figure 2. The euclidian norm of the solution (3.10) of system (3.9) with initial value $x_0 = (1, 1)$ on the interval $t \in [0, 6]$.

Based on Proposition 3.3 we have made some remarks on the stable and unstable subspaces and the topologically equivalence of autonomous linear systems.

Proposition 3.4. *Let us consider the autonomous system* (2.6) *with coefficient matrix* $A \in \mathbb{R}^{n\times n}$. *Let us assume that the system has an exponential dichotomy with projection* P. *Let* $H := \{u_1, \ldots, u_n\} \subset \mathbb{R}^n$ *be a Jordan basis for* A. *Then for the sets* $S_-(0)$ *and* $S_+(0)$ *in Proposition* 3.3*:*

$$S_-(0) = E_s(A) \qquad \text{and} \qquad S_+(0) = E_u(A),$$

where $E_s(A)$ *and* $E_u(A)$ *denote the stable and unstable subspaces of the coefficient matrix* A *of the system* (1.4)*, i.e.*

$$E_s(A) = \operatorname{span}\{u_k \in H \ : \ \operatorname{Re}(\lambda) < 0\},$$

$$E_u(A) = \operatorname{span}\{u_k \in H \ : \ \operatorname{Re}(\lambda) > 0\}.$$

Proof. We have to show that $S_-(0) = E_s(A)$. The proof for $S_+(0) = E_u(A)$ is similar. Let $u_k \in H$ be any vector from the Jordan basis such that $u_k \in \operatorname{Im}(P)$

holds. Following from the asymptotic statements (3.6) in Proposition 3.3 we receive the equality

$$\lim_{t\to\infty} \|\Phi(t)u_k\| = \lim_{t\to\infty} \|e^{At}u_k\| = 0 \tag{3.11}$$

and so the inclusion $S_-(0) = \mathrm{Im}(P) \subset E_s(A)$ is true. Let us assume indirectly that $\mathrm{Im}(P) \subset E_s(A)$ doesn't hold, so for the basis vector $u_k \in E_s(A)$, but $u_k \notin \mathrm{Im}(P)$. Then, because of the relation $\mathrm{Im}(P) \oplus \mathrm{Ker}(P) = \mathbb{R}^n$, $u_k \in \mathrm{Ker}(P)$. Thus from property (3.6) in Proposition 3.3 we have

$$\lim_{t\to\infty} \|e^{At}u_k\| = \infty,$$

which contradicts to the assumption $u_k \in E_s(A)$. Thus the other inclusion $E_s(A) \subset \mathrm{Im}(P)$ holds, too. ■

Remark 3.5. Let A, $B \in \mathbb{R}^{n\times n}$ be hyperbolic matrices, and let us consider the autonomous systems of the form (1.2) determined by A and B. From Proposition 2.1 we know that these systems possess an exponential dichotomy. Let us denote by P_A the projection of the exponential dichotomy of the system determined by A, and by P_B the projection of the other system. Due to the Proposition 3.4 the systems are C^0-equivalent and C^0-conjugate if and only if $\mathrm{Im}(P_A) = \mathrm{Im}(P_B)$ holds.

In conclusion we are going to show how we can formulate and extend the three statements (i), (ii) and (iii) with the help of the notion of the exponential dichotomy for the non-autonomous system (1.4), which we have studied at the beginning of Section 2. Let us consider the above three statements:

(i) system (1.4) possesses exponential dichotomy on $\mathbb{R}$,

(ii) system (1.4) has no nontrivial bounded solution on $\mathbb{R}$,

(iii) a projection P exists such that for each bounded $f \in \mathfrak{C}(\mathbb{R}, \mathbb{R}^n)$ function the inhomogeneous system

$$\dot{x} = Ax + f \tag{3.12}$$

has a unique solution φ which is bounded on $\mathbb{R}$, where

$$\begin{aligned}\varphi(t) &= \int_{-\infty}^{\infty} K(t-s)f(s)\mathrm{d}s = \\ &= \int_{-\infty}^{t} \Phi(t)P\Phi^{-1}(s)f(s)\mathrm{d}s - \int_{t}^{\infty} \Phi(t)(I-P)\Phi^{-1}(s)f(s)\mathrm{d}s,\end{aligned} \tag{3.13}$$

where

$$K(t-s) = \begin{cases} \Phi(t)P\Phi^{-1}(s) & (t \geq s), \\ \Phi(t)(I-P)\Phi^{-1}(s) & (t < s). \end{cases} \tag{3.14}$$

As a consequence of Proposition 3.3 it can be seen that (i) implies (ii), and in a similar way as in Section 2 we can prove that (ii) is equivalent to (iii) by the superposition principle, thus (i) implies (iii). Furthermore we know that the homogeneous system (1.4) has an exponential dichotomy if and only if for every bounded and continuous function f the inhomogeneous system (3.12) has at least one bounded solution (cf. [3], [9]). Therefore if the statement (iii) holds, i.e. the inhomogeneous system has at least one bounded solution, we receive that the statement (i) holds, too.

References

[1] **Coppel, W.A.,** Dichotomies and reducibility, *Journal of Differential Equations*, **3** (1967), 500–521.

[2] **Coppel, W.A.,** *Stability and Asymptotic Behaviour of Differential Equations*, D. C. Heath and Company, Boston, 1965.

[3] **Coppel, W.A.,** *Dichotomies in Stability Theory*, Lecture Notes in Mathematics, **629**, Springer-Verlag, Berlin-New York, 1978.

[4] **Dieci, L., C. Elia and E. Vleck,** Detecting exponential dichotomy on the real line: SVD and QR algorithms *BIT Numerical Mathematics*, **51(3)** (2011), 555–579.

[5] **Farkas, M.,** *Periodic Motions*, Berlin, Heidelberg and New York: Springer-Verlag, 1994.

[6] **Johnson, R. and F. Mantellini,** Non-autonomous differential equations, in: J.W. Macki and P. Zecca (Eds.) *Dynamical Systems*, Berlin, Heidelberg: Springer-Verlag, 2000.

[7] **Kato, T.,** *Perturbation Theory for Linear Operators*, Springer, New York, 1980.

[8] **Lin, Z. and X-Y. Lin,** *Linear Systems Exponential Dichotomy and Structure of Sets of Hyprbolic Points*, World Scientific, 2000.

[9] **Naulin, R.,** A remark on exponential dichotomies, *Revista Colombiana de Matematicas*, **33** (1999), 9–13.

[10] **Pötzsche, C.,** Bifurcations in nonautonomous dynamical systems: Results and tools in discrete time, in: *Future Directions in Difference Equations*, June 13–17, Vigo, Spain, 2011, 163–212.

[11] **Sacker, R.J. and G.R. Sell,** Existens for dichotomies and invariant splittings for linear differential systems, *Journal of Differential Equations*, **15** (1974), 429–458.

Sz. Császár and S. Kovács
Department of Numerical Analysis
Eötvös L. University
Pázmány P. sétány 1/C.
H-1117 Budapest
Hungary
csaszar@cs.elte.hu
alex@ludens.elte.hu