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ON SYMMETRY OF MAKÓ-PÁLES MEANS

Pál Burai (Debrecen, Hungary)
Justyna Jarczyk (Zielona Góra, Poland)

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Abstract. Given a nonempty real interval I, a continuous strictly monotonic function $\varphi : I \to \mathbb{R}$, and a Borel probability measure μ on $[0,1]$, we characterize all symmetric Makó-Páles means $M_{\varphi,\mu}$ defined on $I \times I$ by

$$M_{\varphi,\mu}(x,y) = \varphi^{-1}\left(\int_0^1 \varphi\left(tx + (1-t)y\right) d\mu(t)\right).$$

1. Introduction

In the whole paper I stands for a nonempty interval of reals. Among two variable means on I, that is functions $M : I \times I \to I$ satisfying the condition

$$\min\{x,y\} \le M(x,y) \le \max\{x,y\}, \qquad x,y \in I,$$

the Makó–Páles means play an important role. Denote by $\mathcal{CM}(I)$ the class of all continuous strictly monotonic functions mapping I into $\mathbb{R}$. Let $\varphi \in \mathcal{CM}(I)$

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and μ be a Borel probability measure on $[0,1]$. In 2008 Makó and Páles [2] introduced the mean $M_{\varphi,\mu}$ on I by the formula

$$M_{\varphi,\mu}(x,y)=\varphi^{-1}\left(\int_0^1\varphi\left(tx+(1-t)y\right)d\mu(t)\right).$$

The class of Makó–Páles means is pretty large. Taking any $p\in(0,1)$ and $\mu=(1-p)\delta_0+p\delta_1$, where δ_t denotes the Dirac measure on $[0,1]$ concentrated at a given fixed $t\in[0,1]$, one can see that $M_{\varphi,\mu}$ is a weighted quasi-arithmetic mean

$$A_p^{\varphi}(x,y)=\varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right),\quad x,y\in I.$$

If μ is the Lebesgue measure on $[0,1]$, then for all $x,y\in I$, substituting $u=tx+$ $+(1-t)y$ and using the change-of-variable theorem, one can see that $M_{\varphi,\mu}=$ $=L_{\varphi}$, where the formula

$$L_{\varphi}(x,y)=\begin{cases}\varphi^{-1}\left(\frac{1}{x-y}\int\limits_x^y\varphi(u)du\right), & \text{if}\quad x,y\in I,\quad x\neq y,\\ x, & \text{if}\quad x,y\in I,\quad x=y,\end{cases}$$

defines the Lagrangean mean. Therefore the class of Makó–Páles means contains two important classes of means. The investigation presented here is, in some sense, a continuation of that given in [1].

On one hand, observe that any Lagrangean mean is symmetric. That is to say, $L_{\varphi}(x,y)=L_{\varphi}(y,x)$ for all $x,y\in I$. On the other hand, the mean A_p^{φ} is symmetric, i.e.

$$A_p^{\varphi}(x,y)=A_p^{\varphi}(y,x),\quad x,y\in I,$$

if and only if $p=1/2$. Thus the following natural problem arises: Describe all the Makó–Páles means that are symmetric, that is, satisfy the condition

$$M_{\varphi,\mu}(x,y)=M_{\varphi,\mu}(y,x).\tag{1.1}$$

The result below provide a complete answer to that question.

Theorem. *Let μ be a Borel probability measure on $[0,1]$ and $\varphi\in\mathcal{CM}(I)$ be a continuous strictly monotonic function. Then the mean $M_{\varphi,\mu}$ is symmetric if and only if at least one of the conditions below holds:*

(i) *the measure μ is symmetric with respect to $1/2$, that is*

$$\mu(1-A)=\mu(A)\tag{1.2}$$

for all Borel subsets A of $[0,1]$;

(ii) φ *is a polynomial of degree at most* $n \in \mathbb{N}$ *and*

$$\int_0^1 t^k d\mu(t) = \int_0^1 (1-t)^k d\mu(t) \quad \textit{for all } k = 1, \ldots, n. \tag{1.3}$$

Condition (1.1) means that

$$\int_0^1 \varphi(tx + (1-t)y) d\mu(t) = \int_0^1 \varphi((1-t)x + ty) d\mu(t), \quad x, y \in I,$$

or equivalently, by putting

$$\mu^*(A) = \mu(1 - A) \tag{1.4}$$

for all Borel sets $A \subset [0, 1]$ and using the change-of-variable theorem,

$$\int_0^1 \varphi(tx + (1-t)y) d\mu(t) = \int_0^1 \varphi(tx + (1-t)y) d\mu^*(t), \quad x, y \in I.$$

Consequently, the original condition (1.1) is equivalent to the requirement

$$M_{\varphi,\mu} = M_{\varphi,\mu^*}, \tag{1.5}$$

which is a particular case of the general equality problem

$$M_{\varphi,\mu} = M_{\psi,\nu} \tag{1.6}$$

solved in [2]. So one can expect that the description of the symmetric Makó–Páles means can be derived from the paper [2]. A thorough reading of it shows that this is almost true. Applying [2, Corollary 3, Theorems 7-9 and 12-15], which solve equation (1.6), to (1.5) being a particular case of (1.6), the reader can obtain the assertion of the Theorem. However, the results of [2] require some regularity of φ and ψ.

In the next section we give a short immediate proof of the Theorem, making use of the main result of the paper [3] by Páles (used also in the proof of [2, Theorem 8]).

2. Proof of the Theorem

In what follows, with any Borel probability measure μ on $[0,1]$ we associate its moments $\hat{\mu}_k$, $k \in \mathbb{N} \cup \{0\}$, given by

$$\hat{\mu}_k = \int_0^1 t^k d\mu(t).$$

Then, by the change-of-variable theorem, condition (1.3) can be reformulated as

$$\hat{\mu}_k = \hat{\mu}_k^* \quad \text{for all } k = 1, \ldots, n, \tag{2.1}$$

where the measure μ^* is defined by (1.4).

Proof of the Theorem. Assume that the mean $M_{\varphi,\mu}$ is symmetric. If $\mu = \mu^*$, that is equality (1.2) holds for all Borel sets $A \subset [0,1]$, then μ is symmetric with respect to $1/2$, i.e. (i) is satisfied. So assume that $\mu \neq \mu^*$. By well-known results of measure theory and Weierstrass approximation theorem there exists an $n \in \mathbb{N} \cup \{0\}$ such that

$$\hat{\mu}_{n+1} \neq \hat{\mu}_{n+1}^*.$$

Let n be the smallest such a number. Now, rewriting (1.5) in the form

$$\int_0^1 \varphi(tx + (1-t)y) d(\mu - \mu^*)(t) = 0, \quad x, y \in I,$$

and applying [3, Theorem 1] by Páles for the continuous function φ and signed Borel measure $\mu - \mu^*$, we see that φ is a polynomial of degree at most n and condition (2.1), i.e. (1.3), holds. However, since φ is strict monotonic, it is not constant and consequently, n cannot be zero. Having $n \geq 1$ we come to (ii).

Now assume (i). Then $\mu = \mu^*$, hence

$$\int_0^1 \varphi(tx + (1-t)y) d\mu(t) =$$

$$= \int_0^1 \varphi((1-s)x + sy) d\mu^*(s) = \int_0^1 \varphi(sy + (1-s)x) d\mu(s)$$

for all $x, y \in I$, which implies the symmetry of $M_{\varphi,\mu}$.

Finally assume condition (ii). Fix any $x, y \in I$. Then also $I \ni t \longmapsto \longmapsto \varphi(tx + (1-t)y)$ is a polynomial of degree at most n, say

$$\varphi(tx + (1-t)y) = \sum_{k=0}^{n} a_k t^k, \qquad t \in [0,1],$$

with some $a_0, \dots, a_n \in \mathbb{R}$. Then, by (1.3),

$$\int_0^1 \varphi(tx + (1-t)y) d\mu(t) = \int_0^1 \sum_{k=0}^{n} a_k t^k d\mu(t) = \int_0^1 \sum_{k=0}^{n} a_k (1-t)^k d\mu(t) =$$
$$= \int_0^1 \varphi((1-t)x + ty) d\mu(t)$$

which completes the proof. ■

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P. Burai
University of Debrecen
Faculty of Informatics
Debrecen
Hungary
burai.pal@inf.unideb.hu

J. Jarczyk
University of Zielona Góra
Faculty of Mathematics
Computer Science and Econometrics
Zielona Góra
Poland
j.jarczyk@wmie.uz.zgora.pl