A NEW CLASS OF UNIQUE RANGE SETS FOR MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we give a new class of unique range sets for meromorphic functions. Note that this class different from Yi's [6], Frank–Reinders's [3] and Fujimoto's [4].

1. Introduction

In this paper, by a meromorphic function we mean a meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the notations in the Nevanlinna theory (see [4], [5] and [8]). Let f be a non-constant meromorphic function on $\mathbb{C}$. For every $a \in \mathbb{C}$, define the function $\nu_f^a : \mathbb{C} \to \mathbb{N}$ by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ m & \text{if } f(z) = a \text{ with multiplicity } m, \end{cases}$$

and set $\nu_f^\infty = \nu_{\frac{1}{f}}^0$. For $f \in \mathcal{M}(\mathbb{C})$ and $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in \mathbb{C}\}.$$

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Two meromorphic functions f, g are said to *share* S, *counting multiplicity*, if $E_f(S) = E_g(S)$. Let a set $S \subset \mathbb{C} \cup \{\infty\}$ and f and g be two non-constant meromorphic (entire) functions. If $E_f(S) = E_g(S)$ implies $f = g$ for any two non-constant meromorphic (entire) functions f, g, then S is called a unique range set for meromorphic(entire) functions or, in brief, $URSM(URSE)$. Gross and Yang [2] showed that the set $S = \{z \in \mathbb{C} |\ z + e^z = 0\}$ is a $URSE$. Since then, $URSE$ and $URSM$ with finitely many elements have been found by Yi [6], Mues and Reinders [1], Frank and Reinders [3], Fujimoto [4]. In fact, examples of unique range sets given by most authors are sets of the form $\{z \in \mathbb{C} |\ z^n + az^m + b = 0\}$ under suitable conditions on the constants a and b and on the positive integers n and m(see[6]). So far, the smallest unique range set for meromorphic functions has 11 elements and was given by Frank and Reinders[3]. They proved the following result.

Theorem A. *The set*

$$\left\{ z \in \mathbb{C} \;\Big|\; \frac{(n-1)(n-2)}{2} z^n + n(n-2) z^{n-1} + \frac{(n-1)n}{2} z^{n-2} + b = 0 \right\},$$

where $n \geq 11$ *and* $b \neq 0, 1$, *is a unique range set for meromorphic functions.*

Fujimoto [4] extended this result to zero sets of more general polynomials $P_F(z)$ satisfying the condition: for any zeros $e_i \neq e_j$ of $P_F'(z)$ we have $P_F(e_i) \neq P_F(e_j)$.

In this paper, we give a new class of unique range sets for meromorphic functions. Note that this class is different from Yi's [6], Frank–Reinders's [3] and Fujimoto's [4] (see Theorem 2.1,Theorem 2.2).

2. A new class of unique range sets for meromorphic functions

We assume that the reader is familiar with the notations in the Nevanlinna theory (see [3], [4] and [8]).

We first need the following Lemmas.

Lemma 2.1. (See [8].) *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r, f) \leq \sum_{i=1}^{q} N_1(r, \frac{1}{f - a_i}) + S(r, f),$$

where $S(r, f) = o(T(r, f))$ for all r, except for a set of finite Lebesgue measure.

Lemma 2.2. (See [7].) *Let $d, n \in \mathbb{N}^*$, $d \geq n^2$, and let $f_1, ..., f_{n+1}$ be entire functions on $\mathbb{C}$, not identically zero and satisfying the condition $f_1^d + f_2^d + ... + f_{n+1}^d = 0$. Then there is a decomposition of indices, $\{1, ..., n+1\} = \cup I_v$, such that*

i. *Every I_v contains at least 2 indices;*

ii. *For $j, i \in I_v$; $f_i = c_{ij} f_j$, where c_{ij} is a non-zero constant.*

Now let us describe main result of the paper.

$$\text{Let } d \in \mathbb{N}^*,\ d \geq 25 \text{ and } a, b, c \in \mathbb{C},\ a, b, c \neq 0,$$

$$(A_1) \qquad \text{with } c \neq \frac{b^d}{a^d},\ a^{2d} \neq 1,\ c \neq a^d b^d,\ c \neq \frac{(-1)^d b^d}{a^{2d}},\ c \neq (-1)^d b^d.$$

Then we consider following polynomial

$$(A_2) \qquad P(z) = z^d + (az+b)^d + c, \text{ and let } P(z) \text{ has only simple zeros.}$$

We need following lemma.

Set $v_1 = (1, 0)$, $v_2 = (0, e)$ with $e^d = c$, $v_3 = (a, b)$. Define the set

$A := \{\alpha = (\alpha_1, \alpha_2)\}$, where α_1, α_2 are 2 distinct numbers of $\{1, 2, 3\}$. For each element $\alpha \in A$, we associate the matrix

$$A_\alpha = \begin{pmatrix} v_{\alpha_1} \\ v_{\alpha_2} \end{pmatrix}.$$

Main result of the paper is following theorem.

Theorem 2.1. *Let S be the set of zeros of the above polynomial $P(z)$. Assume that the conditions $(A_1), (A_2)$ are satisfied. Then S is a $URSM$.*

Proof. Write $f = \frac{f_1}{f_2}$ (resp., $g = \frac{g_1}{g_2}$), where f_1, f_2 (resp., g_1, g_2) are entire functions on $\mathbb{C}$ having no common zeros. Set

$$Q(z_1, z_2) = z_1^d + (az_1 + bz_2)^d + e^d z_2^d, \text{ with } e^d = c$$

We consider following linear forms $L_i(z_1, z_2), i = 1, 2, 3$, on $\mathbb{C}^2$:

$$L_1(z_1, z_2) = z_1,\ L_2(z_1, z_2) = ez_2,\ L_3(z_1, z_2) = az_1 + bz_2.$$

We first prove that if

$$Q(f_1, f_2) = Q(g_1, g_2), \text{ then } g_i = tf_i, i = 1, 2, \text{ where } t \in \mathbb{C}, t \neq 0,$$

and therefore $f = g$. From $Q(f_1, f_2) = Q(g_1, g_2)$ we have

$$(L_1(f_1, f_2))^d + (L_2(f_1, f_2))^d + (L_3(f_1, f_2))^d = (L_1(g_1, g_2))^d + (L_2(g_1, g_2))^d +$$

$$+(L_3(g_1, g_2))^d. \tag{2.1}$$

For simplicity, set $L_i(\tilde{f}) = L_i(f_1, f_2), L_i(\tilde{g}) = L_i(g_1, g_2)$. Then from (2.1) we have

$$(L_1(\tilde{f}))^d + (L_2(\tilde{f}))^d + (L_3(\tilde{f}))^d = (L_1(\tilde{g}))^d + (L_2(\tilde{g}))^d + (L_3(\tilde{g}))^d. \tag{2.2}$$

We shall prove that for each $i = 1, 2, 3$, there exists a non-zero constant c_i such that $L_i(\tilde{f}) = c_i L_i(\tilde{g})$.

By non-constant of the functions f and g we give $L_i(\tilde{f}) \not\equiv 0$, $L_i(\tilde{g}) \not\equiv 0$. Since $d \geq 25$, from Lemma 2.2 it follows that for each $i = 1, 2, 3$, we have one of the following possibilities:

i/ there exists a $i^{'} \in \{1, 2, 3\}$ with $i^{'} \neq i$ such that

$$L_i(\tilde{f}) = b_{ii'} L_{i'}(\tilde{f}), b_{ii'} \neq 0. \tag{2.3}$$

ii/ there exists a $i^{'} \in \{1, 2, 3\}$ such that

$$L_i(\tilde{f}) = c_{ii'} L_{i'}(\tilde{g}), c_{ii'} \neq 0. \tag{2.4}$$

iii/ there exist $i^{'}, i^{''} \in \{1, 2, 3\}, i^{'} \neq i^{''}$ such that

$$L_i(\tilde{f}) = c_{ii'} L_{i'}(\tilde{g}) = c_{ii''} L_{i''}(\tilde{g}), c_{ii'}, c_{ii''} \neq 0,$$

and then

$$L_{i'}(\tilde{g}) = c_{i' i''} L_{i''}(\tilde{g}), c_{i' i''} \neq 0. \tag{2.5}$$

If we have (2.3) or (2.5), we get a contradiction to the hypothesis of non-constant of the functions f and g. Thus, we have only possibility (2.4), i. e., for each $i = 1, 2, 3$, there exists an unique $\sigma(i) \in \{1, 2, 3\}$ with σ is a permutation of $\{1, 2, 3\}$ such that

$$L_i(\tilde{f}) = c_{\sigma(i)} L_{\sigma(i)}(\tilde{g}), \text{ this means that, } L_i(f_1, f_2) = c_{\sigma(i)} L_{\sigma(i)}(g_1, g_2), \tag{2.6}$$

where $c^d_{\sigma(i)} = 1$.

Set $\alpha = (1, 2)$, $\beta = (2, 3)$, and $\alpha' = (\sigma(1), \sigma(2))$, $\beta' = (\sigma(2), \sigma(3))$. Then

$$A_\alpha = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ A_\beta = \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}, \text{ and } \det A_\alpha = e, \ \det A_\beta = -ae. \tag{2.7}$$

Now we consider the following possibilities for (2.6):

Case 1. $\alpha' = (2,1)$, $\beta' = (1,3)$. Then

(2.8) $$A_{\alpha'} = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}, \ A_{\beta'} = \begin{pmatrix} v_1 \\ v_3 \end{pmatrix}, \text{ and } \det A_{\alpha'} = -e, \ \det A_{\beta'} = b.$$

From this and (2.6) we give

$$L_1(f_1, f_2) = c_2 L_2(g_1, g_2), \ L_2(f_1, f_2) = c_1 L_1(g_1, g_2),$$

(2.9) $$L_3(f_1, f_2) = c_3 L_3(g_1, g_2).$$

Then we get by (2.9)

(2.10) $$A_\alpha f^t = B A_{\alpha'} g^t,$$

where

$$B = \begin{pmatrix} c_2 & 0 \\ 0 & c_1 \end{pmatrix},$$

and

(2.11) $$A_\beta f^t = C A_{\beta'} g^t,$$

where

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_3 \end{pmatrix}.$$

From the equations (2.10), (2.11) we get

(2.12) $$f^t = A_\alpha^{-1} B A_{\alpha'} g^t, f^t = A_\beta^{-1} C A_{\beta'} g^t.$$

By deleting f^t from the equations (2.12) we obtain $A_\alpha^{-1} B A_{\alpha'} g^t = A_\beta^{-1} C A_{\beta'} g^t$. By non-constant of g we have $A_\alpha^{-1} B A_{\alpha'} = A_\beta^{-1} C A_{\beta'}$. By $c_i^d = 1, i = 1,2,3,$ and noting that

$$\det A_\alpha \det A_\alpha^{-1} = 1, \det A_\beta \det A_\beta^{-1} = 1,$$

we obtain

$$(\det B)^d = 1, (\det C)^d = 1,$$

$$\left(\frac{\det A_\alpha}{\det A_{\alpha'}}\right)^d = \left(\frac{\det A_\beta}{\det A_{\beta'}}\right)^d, c = \frac{b^d}{a^d}.$$

a contradiction to the hypothesis $c \neq \frac{b^d}{a^d}$.

Case 2. $\alpha' = (3,2),\ \beta' = (2,1)$. From this and (2.6) we give

$$L_1(f_1,f_2) = c_3L_3(g_1,g_2),\ L_2(f_1,f_2) = c_2L_2(g_1,g_2),$$

$$L_3(f_1,f_2) = c_1L_1(g_1,g_2). \tag{2.13}$$

By the similar arguments as in **Case 1** we obtain a contradiction to the hypothesis $a^{2d} \neq 1$.

Case 3. $\alpha' = (3,1),\ \beta' = (1,2)$. From this and (2.6) we give

$$L_1(f_1,f_2) = c_3L_3(g_1,g_2),\ L_2(f_1,f_2) = c_1L_1(g_1,g_2),$$

$$L_3(f_1,f_2) = c_2L_2(g_1,g_2). \tag{2.14}$$

By the similar arguments as in **Case 1** we obtain a contradiction to the hypothesis $c \neq a^d b^d$.

Case 4. $\alpha' = (2,3),\ \beta' = (3,1)$. From this and (2.6) we give

$$L_1(f_1,f_2) = c_2L_2(g_1,g_2),\ L_2(f_1,f_2) = c_3L_3(g_1,g_2),$$

$$L_3(f_1,f_2) = c_1L_1(g_1,g_2). \tag{2.15}$$

By the similar arguments as in **Case 1** we obtain a contradiction to the hypothesis $c \neq \dfrac{(-1)^d b^d}{a^{2d}}$.

Case 5. $\alpha' = (1,3),\ \beta' = (3,2)$. From this and (2.6) we give

$$L_1(f_1,f_2) = c_1L_1(g_1,g_2),\ L_2(f_1,f_2) = c_3L_3(g_1,g_2),$$

$$L_3(f_1,f_2) = c_2L_2(g_1,g_2). \tag{2.16}$$

By the similar arguments as in **Case 1** we obtain a contradiction to the hypothesis $c \neq (-1)^d b^d$.

Case 6. $\alpha' = (1,2),\ \beta' = (2,3)$. From this and (2.6) we give

$$L_1(f_1,f_2) = c_1L_1(g_1,g_2),\ L_2(f_1,f_2) = c_2L_2(g_1,g_2),$$

$$L_3(f_1,f_2) = c_3L_3(g_1,g_2). \tag{2.17}$$

Since L_1, L_2 are linearly independent, L_1, L_2, L_3 are linearly dependent, there exist non-zero constants t_k such that

$$L_3 = \sum_{k=1}^{2} t_k L_k, \quad \text{and} \quad L_3(\tilde{f}) = \sum_{k=1}^{2} t_k L_k(\tilde{f}),\ L_3(\tilde{g}) = \sum_{k=1}^{2} t_k L_k(\tilde{g}),$$

$$L_k(\tilde{f}) = c_k L_k(\tilde{g}), k = 1, 2,\ L_3(\tilde{f}) = c_3 L_3(\tilde{g}).$$

Thus,

$$\sum_{k=1}^{2} (c_3 - c_k) t_k L_k(\tilde{g}) = 0.$$

Since f_1, f_2 are linearly independent, it follows that all the c_i are equal each to other, say $c_i = t$. Then we have $g_i = tf_i$ for $i = 1, 2$. Therefore $f = g$.

Now we are going to complete the proof of Theorem 2.1. By $E_f(S) = E_g(S)$ it is easy to see that there exists an entire function h such that $Q(f_1, f_2) = e^h Q(g_1, g_2)$. Set $l = e^{\frac{h}{d}}$ and $G_1 = lg_1, G_2 = lg_2$. Then $Q(f_1, f_2) = Q(G_1, G_2)$. By the similar arguments as above we have $\frac{f_1}{f_2} = \frac{G_1}{G_2}$. Therefore $f = g$. Theorem 2.1 is proved. ■

A example of new class of unique range sets for meromorphic functions in Theorem 2.1 is following.

Theorem 2.2. *Let $d \in \mathbb{N}^*$, $d \geq 25$ and S be the set of zeros of polynomial $P(z) = z^d + (2z+5)^d + 1$. Then S is a URSM.*

Proof. By $P(z) = z^d + (2z+5)^d + 1$ we have $a = 2$, $b = 5, c = 1$. From this it follows that

$$a, b, c \neq 0, \text{and } c \neq \frac{b^d}{a^d},\ a^{2d} \neq 1,\ c \neq a^d b^d,\ c \neq \frac{(-1)^d b^d}{a^{2d}},\ c \neq (-1)^d b^d.$$

So the condition (A_1) is satisfied. We shall prove that the condition (A_2) is satisfied. Take l is a any zero of $P'(z) = d(z^{d-1} + 2(2z+5)^{d-1})$. Then

$$l^{d-1} + 2(2l+5)^{d-1} = 0,\ (2 + \frac{5}{l})^{d-1} = -\frac{1}{2}. \text{ Set } 2 + \frac{5}{l} = h. \text{ Then } h^{d-1} = -\frac{1}{2},$$

$$l = \frac{5}{h-2},\ (2l+5)^{d-1} = -\frac{1}{2} l^{d-1},\ l^d + (2l+5)^d + 1 = l^d - \frac{1}{2} l^{d-1}(2l+5) + 1$$

$$= -\frac{5}{2} l^{d-1} + 1 = -\frac{5}{2}\frac{5^{d-1}}{(h-2)^{d-1}} + 1 = -\frac{5^d}{2(h-2)^{d-1}} + 1. \tag{2.18}$$

Moreover

$$|h|^{d-1} = \frac{1}{2},\ |h| = (\frac{1}{2})^{\frac{1}{d-1}},\ 0 < |h-2|^{d-1} \leq (|h| + 2)^{d-1},$$

$$0 < |h-2|^{d-1} \leq ((\frac{1}{2})^{\frac{1}{d-1}} + 2)^{d-1} = \frac{(2.2^{\frac{1}{d-1}} + 1)^{d-1}}{2},$$

$$0 < 2.|h-2|^{d-1} \leq (2.2^{\frac{1}{d-1}} + 1)^{d-1},$$

$$\frac{5^d}{2.|h-2|^{d-1}} \geq \frac{5^d}{(2.2^{\frac{1}{d-1}}+1)^{d-1}} > 1. \tag{2.19}$$

Combining (2.18) and (2.19) we get $-\dfrac{5^d}{2(h-2)^{d-1}} + 1 \neq 0$. Thus $P(l) \neq 0$. So the condition (A_2) is satisfied.

Now applying Theorem 2.1 to the set of zeros of polynomial $P(z) = z^d + (2z+5)^d + 1$ we obtain conclusion of Theorem 2.2. ■

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