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CONTINUATION OF THE LAUDATION TO Professor Karl-Heinz Indlekofer on his 75-th birthday

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What has happened with him on the last five years?

He continued his activity as an organist and leader of the church choir in the beautiful village Dörenhagen, where they live during more than thirty years.

Thomas, his son and Katharina built a modern, beautiful house close to their house. He helped in the organization of the construction.

Grand children Lucas and Elias born.

To keep his fitness he runs or has a long walk every day.

He continued his research activity in number theory and in probability.

I. His work in the group of Professor Wolfgang Schwarz

From 1970 during some years he worked in the group of Professor Wolfgang Schwarz in Frankfurt/Main. In [150] he describes the main topics discussed in the group of Wolfgang Schwarz: Equivalent power series, spaces of arithmetical functions, characterization of multiplicative functions, integration via Gelfand theory. Whereas Indlekofer's work on equivalent power series and characterizations of multiplicative functions have been described in Laudation for his sixtieth birthday, we focus here on topic "integration".

W. Schwarz and J. Spilker in their book: *Arithmetical Functions* (Cambridge University Press, 1994) describe the integration theory for algebra $\mathcal{B}^u$. Here $\mathcal{B}$ denote the space of all even functions and is equal to the space of all linear combinations of Ramanujan sum c_r, where

$$c_r(n) = \sum_{\substack{1\le a\le r \\ gcd(a,r)=1}} \exp\left(2\pi \frac{a}{r} n\right)$$

i.e.

$$\mathcal{B} = Lin_{\mathbb{C}}\{c_r : r = 1, 2, \dots\}.$$

Taking the closure of $\mathcal{B}$ with respect to the supremum norm $\|.\|_u$, defines by

$$\|f\|_u := \sup_{n\in\mathbb{N}} |f(u)| \quad \text{for} \quad f \in \ell^\infty,$$

we arrive at the space $\mathcal{B}^u$ of uniformly almost even functions.

$\mathcal{B}^u$ is a B^*-algebra. Therefore the Gelfand Representation Theorem is applicable and implies that $\mathcal{B}^u$ is isomorphic and isometric to the algebra $C(X)$ of all continuous complex-valued functions on a certain compact Hausdorff space X – its maximal ideal space. The mean-value $M(f)$ for $f \in \mathcal{B}^u$ is a nonnegative linear functional on $\mathcal{B}^u$ and can extended to a bounded nonnegative linear functional on $C(X)$. Thus, by Riesz's Theorem the mean-value $M(f)$ can be written as an integral over a continuous function on X. Similar conclusions can be done for the B^*-algebras $\mathcal{D}^u$ and A^u of uniformly limit -periodic and uniformly almost periodic functions, respectively. It should be alluded the special role played by the asymptotic density in the investigation of the space $\mathcal{B}^u$, $\mathcal{D}^u$ and $\mathcal{A}^u$. Further, despite of the ad hoc constructions of the compactifications, the "size" of these spaces is "restricted"; the Möbius μ function, for example, is not an element of any of these spaces. To avoid these limitations Indlekofer described in 1990s an "integration theory" which is based on the Stone–Cech compactification of natural numbers.

The main difficulties concerning the immediate application of probabilistic tools to the investigation of additive and multiplicative functions in number theory arise from the fact that the asymptotic density defines only a finitely additive measure on the family of all subsets of Γ_e having an asymptotic density. To overcome these difficulties Indlekofer proceeded as follows: $\mathbb{N}$, endowed with the discrete topology, will be embedded in a compact space $\beta\mathbb{N}$, the Stone–Cech compactification of $\mathbb{N}$, and then any algebra $\mathcal{A}$ in $\mathbb{N}$ with an arbitrary finitely additive set function, a content or pseudomeasure on $\mathbb{N}$, can be extended to an algebra $\overline{\mathcal{A}}$ in $\beta\mathbb{N}$ together with an extension of this pseudomeasure. The integration theory is based on the following result:

Theorem 1. *Let $\mathcal{A}$ be an algebra in $\mathbb{N}$ and $\delta : \mathcal{A} \to [0,\infty)$ be a content on $\mathcal{A}$. Then the map*

$$\bar{\delta} : \overline{\mathcal{A}} \to [0,\infty), \quad \bar{\delta}(\overline{\mathcal{A}}) = \delta(\mathcal{A})$$

is σ-additive on $\overline{\mathcal{A}}$ and can uniquely extended to a measure on the minimal σ-algebra over $\overline{\mathcal{A}}$.

Then, if $\mathcal{E}$ denote the family of simple functions on $\mathbb{N}$, the set

$$\mathcal{E}(\mathcal{A}) := \left\{ s \in \mathcal{E} : s = \sum_{j=1}^{m} \alpha_j 1_{A_j},\ m \in \mathbb{N},\ \alpha_j \in \mathbb{C},\ A_j \in \mathcal{A} \quad (j = 1, \ldots, m) \right\}$$

of simple functions on $\mathcal{A}$ is a vector space, especially an algebra, such that $\mathcal{E}(\mathcal{A})^u$, the $\|.\|_u$-closure of $\mathcal{E}(\mathcal{A})$, is a B^*-algebra.

The connection with Gelfand's theory is described in the paper [146] with Wagner and reads as follows:

Theorem 2. *Let $\mathcal{F}$ be a B^*-algebra of complex-valued functions from ℓ^∞ and let L be a positive linear functional on $\mathcal{F}$ with $L(1_{\mathbb{N}}) = 1$. Then there exists an algebra $\mathcal{A}$ of subsets of $\mathbb{N}$ and a content δ on $\mathcal{A}$ such that*

(i) *each $f \in \mathcal{F}$ belongs to the $\|.\|_u$-closure of $\mathcal{E}(\mathcal{A})$*

and

(ii) *for each $f \in \mathcal{F}$ the relation*

$$L(f) = \int\limits_{\beta\mathbb{N}} \overline{f} d\overline{\delta}$$

holds

II. On the book [151]

The book [151] written together with Prof. O. K. Klesov, Prof. J. G. Steinebach and the late Prof. V. V. Buldygin is a very important work in the renewal theory and its various important applications. It is developed from the collaboration in several DFG-projects (see Laudation on his seventieth anniversary) and its related to a recent common DFG-project "Ein einheitlicher Zugang zu Grenzwertsätzen für duale Objekte in Wahrscheinlichkeits- und Zahlentheorie" (2016-2019). Here, for example the asymptotic behaviour of means $\sum_{n\le x} f(n) \quad (x \to \infty)$ for (positive) uniformly summable multiplicative functions f and the asymptotic behaviour of the distributions

$$\sharp\left\{n \in \mathbb{N} : \frac{n}{f(n)} \le y\right\} \quad \text{as} \quad y \to \infty$$

if $\frac{n}{f(n)} \nearrow \infty$.

III. Survey article [153]

In the survey article *Equivalent power series* Indlekofer describes the result and applications which have been initiated and motivated by Turán's work on equivalent power series (for details see Laudation on his sixtieth anniversary). Further, he extends subject to functions f which are holomorphic in a region containing $\overline{D} \setminus \{1\}$.

IV. Tauberian Theorem for exp-log functions

Indlekofer [144] introduced the class $\mathcal{F}$ of exp-log functions. For this let

$$(1) \qquad Z(y)=\sum_{n=0}^{\infty}\gamma(n)y^n=\exp\Big(\sum_{m=0}^{\infty}\frac{\lambda(m)}{m}y^m\Big)$$

be holomorphic for $|y|<1$, where

$$(2) \qquad 0\le\lambda(m)=O(1), m\in\mathbb{N},$$

and

$$(3) \qquad |Z(y)|\ll Z(|y|)\Big|\frac{1-|y|}{1-y}\Big|^{\epsilon}, \quad (|y|<1)$$

for some $\epsilon>0$. Further, putting

$$B(n)=\exp\Big(\sum_{m\le n}\frac{\lambda(m)}{m}\Big),$$

we assume that

$$(4) \qquad n\gamma(n)\asymp B(n)$$

and

$$(5) \qquad B(m)=o(B(n)) \quad \text{if} \quad m=o(n), n\to\infty.$$

Then we say that the function Z given in (1) belongs to exp-log class $\mathcal{F}$ in case (2), (3), (4) and (5) hold.

Now, if the function

$$(6) \qquad F(y)=\sum_{n=0}^{\infty}f(n)y^n=\exp\Big(\sum_{m=1}^{\infty}\frac{\lambda_f(m)}{m}y^m\Big)$$

($|y|<1$) is given then the following Tauberian theorem holds:

Theorem 3. *Let Z be an element of the exp-log class $\mathcal{F}$ and let $F(y)$ in* (6) *satisfy $|\lambda_f(m)|\le\lambda(m)$ for all $m\in\mathbb{N}$. Then the following two assertions hold:*

(i) *Let*

$$(7) \qquad \sum_{m=1}^{\infty}\frac{\lambda(m)-Re\lambda_f(m)e^{ima}}{m}$$

converge for some $a \in \mathbb{R}$. *Put*

$$A_n := \exp\Big(-ima + \sum_{m \le n} \frac{\lambda_f(m)e^{ima} - \lambda(m)}{m} \Big).$$

Then

$$f(n) = A_n \gamma(n) + o(\gamma(n)) \quad as \quad n \to \infty.$$

(ii) *Let* (7) *diverge for all* $a \in \mathbb{R}$. *Then*

$$f(n) = o(\gamma(n)) \quad as \quad n \to \infty.$$

Indlekofer proved in [144]:

Theorem 4. *If* $Z(y)$ *defined in* (1), *satisfies* (2) *and has the form*

$$Z(y) = \sum_{n=0}^{\infty} \gamma(m) y^n = \frac{H(y)}{(1-y)^\delta},$$

where $\delta > 0$, $H(y) = O(1)$ *for* $y \in D$ *and* $\lim_{y \to 1^-} H(y) = A > 0$, *then* $Z \in \mathcal{F}$ *and*

$$\gamma(n) \sim A \frac{n^{\delta - 1}}{\Gamma(\delta)} \quad as \quad n \to \infty.$$

V. Uniformly summable multiplicative functions on additive arithmetical semigroups

Let (G, ∂) be an additive arithmetical semigroup. By definition G is a free commutative semigroup with identity element 1_G, generated by a countable subset $\mathcal{P}$ of primes and admitting an integer valued degree mapping $\partial : G \to$ $\to \mathbb{N} \cup \{0\}$, which satisfies

(i) $\partial(1_G) = 0$ and $\partial(p) > 0$ for all $p \in \mathcal{P}$,

(ii) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,

(iii) the total number $G(n)$ of elements $a \in G$ of degree $\partial(a) = n$ is finite for each $n \ge 0$.

Obviously, $G(0) = 1$ and G is countable.

Let

$$\pi(n) := \#\{p \in \mathcal{P} : \partial(p) = n\}$$

denote the total number of primes of degree n in G. We obtain the identity, at least in the formal sense,

$$\hat{Z}(z) := \sum_{n=0}^{\infty} G(n) z^n = \exp\left(\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m} z^m \right) = \prod_{n=1}^{\infty} (1 - z^n)^{-\pi(n)}.$$

$\hat{Z}$ can be considered as the zeta-function associated with the semigroup (G, ∂).

We assume that $\Lambda(n) = O(q^n)$, and the generating function of (G, ∂) has the form

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n) z^n = \frac{\hat{H}(z)}{(1-qz)^\delta} \text{ and converges for } |z| < q^{-1}, \tag{8}$$

where

$$\hat{H}(z) = O(1) \quad \text{for } |z| < q^{-1}, \text{ and } \lim_{z \to q^{-1}} \hat{H}(z) \quad \text{exists and is positive,} \tag{9}$$

and $\delta > 0$. By a paper of K.-H. Indlekofer (see [144]), the formal power series $\hat{H}(z)$ is convergent for $z = q^{-1}$ and equals $\lim_{z \to q^{-1}} \hat{H}(z)$, and

$$G(n) \sim \frac{\hat{H}(q^{-1})}{\Gamma(\delta)} q^n n^{\delta - 1}$$

holds.

For each arithmetical function $\tilde{f}$ on G, $\tilde{f} : G \to \mathbb{C}$, we associate a power series $\hat{F}$, the *generating function* $\hat{F}$ of $\tilde{f}$, which is defined by

$$\hat{F}(z) = \sum_{a \in G} \tilde{f}(a) z^{\partial(a)} = \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a) = n}} \tilde{f}(a) \right) z^n = \sum_{n=0}^{\infty} f(n) z^n.$$

Further, we introduce the *means*

$$M(n, \tilde{f}) := \begin{cases} \frac{1}{G(n)} f(n), & \text{if } G(n) \neq 0, \\ 0, & \text{if } G(n) = 0, \end{cases}$$

and say that the function $\tilde{f}$ possesses an (arithmetical) *mean-value* $M(\tilde{f})$, if the limit

$$M(\tilde{f}) := \lim_{n \to \infty} M(n, \tilde{f})$$

exists.

For $1 \leq \alpha < \infty$, define

$$\|\tilde{f}\|_\alpha := \left(\limsup_{n \to \infty} M(n, |\tilde{f}|^\alpha) \right)^{1/\alpha},$$

and let

$$L^\alpha := \{ \tilde{f} : G \to \mathbb{C}, \|\tilde{f}\|_\alpha < \infty \}$$

denote the linear space of functions on G with bounded seminorm $\|\cdot\|_\alpha$. If

$$\ell^\infty := \{\tilde{f} : G \to \mathbb{C}, \sup_{g\in G}|\tilde{f}(g)| < \infty\}$$

is the space of bounded functions on G, we introduce the space $L^*(G)$ of *uniformly summable functions* on G as the $\|\cdot\|_1$-closure of $\ell^\infty(G)$.

Obviously, $\tilde{f} \in L^*$ if and only if

$$\lim_{K\to\infty} \sup_{n\geq 1} M\left(n, |\tilde{f}_K|\right) = 0,$$

where

$$\tilde{f}_K(a) = \begin{cases} \tilde{f}(a), & \text{if } |\tilde{f}(a)| \geq K, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that, if $1 < \alpha < \infty$,

$$\ell^\infty(G) \subsetneqq L^\alpha \subsetneqq L^* \subsetneqq L^1.$$

The class of uniformly summable functions has been defined by Indlekofer (see [21]) for functions defined on $\mathbb{N}$, and he has given a complete characterization of uniformly summable *multiplicative* functions (see Indlekofer [30]).

Here, as in the classical case, an arithmetical function $\tilde{f} : G \to \mathbb{R}$ is called *multiplicative* if $\tilde{f}(ab) = \tilde{f}(a)\tilde{f}(b)$ whenever $a, b \in G$ are coprime.

If $\tilde{f}$ is a multiplicative function on G, then $\sum\limits_{\substack{a\in G\\ \partial(a)=0}} \tilde{f}(a) = 1 \ (\neq 0)$, and we assume that its generating function $\hat{F}$ converges in some neighborhood of $z = 0$ and satisfies

$$\hat{F}(z) = \sum_{n=0}^{\infty}\Big(\sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a)\Big) z^n = \prod_p \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)}\right) =:$$

$$=: \exp\left(\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m} z^m\right).$$

In the paper [145] Barát and Indlekofer characterize multiplicative functions $f \in L^*$ the means of which satisfy $M(n, f) \asymp 1$ for $n \geq n_1$. The proofs use ideas and results from [21], [133] and [144]. As an example we mention

Theorem 5. *Let (G, ∂) be an additive arithmetical semigroup satisfying $\Lambda(n) = O(q^n)$, (8), and (9) with $\delta > 0$. Let $\tilde{f}$ be a multiplicative function,* *and $\alpha \geq 1$. If $\tilde{f} \in L^* \cap L^\alpha$, and if $M(n, \tilde{f}) \asymp 1$ for $n \geq n_1$, then the following assertions hold:*

$$\sum_{\substack{p\in P,\partial(p)\leq n\\ |\tilde{f}(p)|\leq\frac{3}{2}}} \frac{Re\,\tilde{f}(p)-1}{q^{\partial(p)}} = O(1), \qquad \sum_{\substack{p\in P,\partial(p)\leq n\\ |\tilde{f}(p)|\leq\frac{3}{2}}} \frac{|\tilde{f}(p)|-1}{q^{\partial(p)}} = O(1),$$

$$\sum_{\substack{p\in P\\ |\tilde{f}(p)|\leq 3/2}} \frac{|\tilde{f}(p)-1|^2}{q^{\partial(p)}} \quad \textit{converges},$$

$$\sum_{p\in P;n\geq 2} \frac{|\tilde{f}(p^n)|^\lambda}{(q^{\partial(p)})^n} \quad \textit{converges},$$

$$\sum_{\substack{p\in P\\ ||\tilde{f}(p)|-1|>1/2}} \frac{|\tilde{f}(p)|^\lambda}{q^{\partial(p)}} \quad \textit{converges for } 1\leq\lambda\leq\alpha,$$

and for each prime p

$$\sum_{n=1}^{\infty} \frac{\tilde{f}(p^n)}{q^{n\partial(p)}} + 1 \neq 0.$$

VI. Multiplicative functions with small increment

O. Klurman proved an old conjecture of I. Kátai, namely that if f is a completely multiplicative function, $|f(n)| = 1$ for every $n \in \mathbb{N}$ and

$$\frac{1}{\log x} \sum_{n\leq x} \frac{1}{n}|f(n+1) - f(n)| \to 0 \quad x \to \infty,$$

then $f(n) = n^{i\tau} \quad (\tau \in \mathbb{R})$.

By using this theorem in [148] it is proved: If f is completely multiplicative and

$$\sum_{n\leq x} \frac{|f(n+1) - f(n)|}{n} = O(\log x),$$

then either

$$\sum_{n\leq x} \frac{|f(n)|}{n} = O(\log x),$$

or

$$f(n) = n^{\sigma+it}, \quad 0 < \sigma \leq 1,\ t \in \mathbb{R}.$$

In [149] it was proved: If f is multiplicative,

$$\overline{\lim} \sum_{n \le x} \frac{|f(n)|}{n} = \infty$$

and

$$\overline{\lim} \frac{1}{\log x} \sum_{n \le x} \frac{|f(n+K) - f(n)|}{n} < \infty,$$

then there are real numbers σ, t such that $0 < \sigma \le 1$, and a Dirichlet character $\chi \pmod K$ such that $f(n) = n^{\sigma+it}\chi(n)$.