UNSOLVED PROBLEMS SECTION

CHARACTERIZATION OF SOME MULTIPLICATIVE FUNCTIONS

Imre Kátai and Bui Minh Phong

(Budapest, Hungary)

Abstract. We state the following problem: Let $a < b$, $a, b \in \mathbb{Z}$ and f be a completely multiplicative function for which

$$f(n^2 + b) - f(n^2 + a) = b - a \quad (n^2 > a)$$

is satisfied. Determine all f for which the above relation holds.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the set of positive integers, integers and complex numbers, respectively. Let $\mathcal{M}^*$ be the class of complex-valued completely multiplicative functions. Let χ_3, χ_4 be the nonprincipal Dirichlet characters $\pmod 3$ and $\pmod 4$, that is

$$\chi_3(n) = \begin{cases} 1, & n \equiv 1 \pmod 3 \\ -1, & n \equiv -1 \pmod 3 \\ 0, & n \equiv 0 \pmod 3 \end{cases}$$

and

$$\chi_4(n) = \begin{cases} 1, & n \equiv 1 \pmod 4 \\ -1, & n \equiv -1 \pmod 4 \\ 0, & n \equiv 0 \pmod 2. \end{cases}$$

Let I be the identity function, that is $I(n) = n$ for every $n \in \mathbb{N}$.

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Theorem 1. *Assume that $f \in \mathcal{M}^*$, $A \in \mathbb{C}$ satisfy the condition*

$$f(n^2) = f(n^2-1) + A \quad \textit{for every} \quad n \in \mathbb{N},\ n \geq 2. \tag{1.1}$$

Then

(a1) *No solution exists if $A \notin \{0,1\}$.*

(b1) *If $A = 1$, then $f \in \{I,\ \chi_3,\ \chi_4\}$*

(c1) *If $A = 0$, then either $f(n) = 1$ for every $n \in \mathbb{N}$ or $f(n) = 0$ for every $n \in \mathbb{N}, n \geq 2$.*

We note that the condition $f \in \mathcal{M}^*$ implies that $f(1) = 1$.

Theorem 2. *Assume that $f \in \mathcal{M}^*$, $B \in \mathbb{C}$ satisfy the condition*

$$f(n^2+1) = f(n^2) + B \quad \textit{for every} \quad n \in \mathbb{N}. \tag{1.2}$$

Then

(a2) *No solution exists if $B \notin \{0,1\}$.*

(b2) *If $B = 1$, then*

$$f(2) = 2, f(p) = p, (p \in \mathcal{P}) \quad \textit{if} \quad p \equiv 1 \pmod 4$$

and

$$f(q) = \pm q\ (q \in \mathcal{P}) \quad \textit{if} \quad q \equiv 3 \pmod 4.$$

(c2) *If $B = 0$, then*

$$f(2) = 1, f(p) = 1, (p \in \mathcal{P}) \quad \textit{if} \quad p \equiv 1 \pmod 4$$

and

$$f(q) = \pm 1\ (q \in \mathcal{P}) \quad \textit{if} \quad q \equiv 3 \pmod 4.$$

If these conditions hold, then (1.2) is satisfied.

Unsolved problem. *Let $a < b$, $a, b \in \mathbb{Z}$ and f be a completely multiplicative function for which*

$$f(n^2+b) - f(n^2+a) = b - a \quad (n^2 > a) \tag{1.3}$$

is satisfied. Determine all f for which (1.3) holds.

2. Proof of Theorem 1

First, by using the fact $f \in \mathcal{M}^*$, we obtain from (1.1) that $f(1) = 1$ and

$$E_n := f^2(n) - f(n-1)f(n+1) - A = 0 \quad \text{for every} \quad n \in \mathbb{N},\ n \geq 2. \tag{2.1}$$

For $n = 2, 3, 4, 5, 9$, we have

$$\begin{cases} E_2 &= f^2(2) - f(3) - A = 0, \\ E_3 &= f^2(3) - f^3(2) - A = 0, \\ E_4 &= f^4(2) - f(3)f(5) - A = 0, \\ E_5 &= f^2(5) - f^3(2)f(3) - A = 0, \\ E_9 &= f^4(3) - f^4(2)f(5) - A = 0. \end{cases}$$

With the help of a Maple program, by solving this system of equations, we obtain that the solutions are:

$$\big(f(2),\ f(3),\ f(5),\ A\big) \in \big\{(0,\ 0,\ 0,\ 0),\ (1,\ 1,\ 1,\ 0),$$
$$(2,\ 3,\ 5,\ 1),\ (-1,\ 0,\ -1,\ 1),\ (0,\ -1,\ 1,\ 1)\big\}.$$

Therefore the assertion (a1) is proved.

First we consider the case $A = 1$. Then

$$\big(f(2),\ f(3),\ f(5),\ A\big) \in \big\{(2,\ 3,\ 5,\ 1),\ (-1,\ 0,\ -1,\ 1),\ (0, -1, 1, 1)\big\}.$$

We get from (1.1) that

$$f^2(N-1) = f\left((N-1)^2\right) = f(N-2)f(N) + A = f(N-2)f(N) + 1. \tag{2.2}$$

If $\big(f(2),\ f(3),\ f(5),\ A\big) = (2,\ 3,\ 5,\ 1)$, then $f(n) = n$ for every $n < N$, where $N \geq 6$ is a prime number. Thus, we infer from (2.2) that

$$(N-1)^2 = f^2(N-1) = f(N-2)f(N) + 1 = (N-2)f(N) + 1,$$

and so $f(N) = N$.

If $\big(f(2),\ f(3),\ f(5),\ A\big) = (-1,\ 0,\ -1,\ 1)$, then $f(n) = \chi_3(n)$ for every $n < N$, where $N \geq 6$ is a prime number. Thus, we infer from (2.2) that

$$\chi_3^2(N-1) = f^2(N-1) = f(N-2)f(N) + 1 = \chi_3(N-2)f(N) + 1,$$

from which we have $f(N) = \chi_3(N)$ for $N \not\equiv 2 \pmod 3$. If $N \equiv 2 \pmod 3$, then $2N - 1 \equiv 0 \pmod 3$ and $\frac{2N-1}{3} < N$. Consequently

$$0 = \chi_3^2(2N-1) = f\left((2N-1)^2\right) = f(2N-2)f(2N) + A =$$

$$= \chi_3^2(2)\chi_3(N-1)f(N) + 1$$

and

$$\chi_3^2(2)\chi_3(N-1)f(N) + 1 = 0, \quad f(N) = -1.$$

Hence

$$f(n) = \chi_3(n) \quad \text{for every} \quad n \in \mathbb{N}.$$

If $\big(f(2), f(3), f(5), A\big) = (0, -1, 1, 1)$, then $f(n) = \chi_4(n)$ for every $n < N$, where $N \geq 6$ is a prime number. Thus, we infer from (2.2) that

$$\chi_4^2(N-1) = f^2(N-1) = f(N-2)f(N) + 1 = \chi_4(N-2)f(N) + 1,$$

which shows that $f(N) = \chi_4(N)$, consequently $f(n) = \chi_4(n)$ for every $n \in \mathbb{N}$.

Now we consider the case $A = 0$. Then

$$\big(f(2), f(3), f(5), A\big) \in \big\{(0, 0, 0, 0),\ (1, 1, 1, 0)\big\}.$$

It is easy to show from (1.1) and from (2.2) that $f(1) = 1$ and

$$f(n) = \begin{cases} 0, & \text{for every} \quad n \in \mathbb{N},\ n \geq 2 \quad \text{if} \quad \big(f(2), f(3), f(5)\big) = (0, 0, 0) \\ 1, & \text{for every} \quad n \in \mathbb{N},\ n \geq 2 \quad \text{if} \quad \big(f(2), f(3), f(5)\big) = (1, 1, 1). \end{cases}$$

Theorem 1 thus is proved. ■

3. Proof of Theorem 2

By using the fact $f \in \mathcal{M}^*$, we obtain from (1.2) that $f(1) = 1$ and

(3.1) $$F_n := f(n^2 + 1) - f^2(n) - B = 0 \quad \text{for every} \quad n \in \mathbb{N},\ n \geq 1.$$

From $F_1 = f(2) - f(1) - B = 0$, we have $f(2) = B + 1$, and so we obtain from $F_2,\ F_4$ that

$$f(5) = B^2 + 3B + 1 \quad \text{and} \quad f(17) = B^4 + 4B^3 + 6B^2 + 5B + 1.$$

Therefore we have

$$F_5 = -(1+B)(B^3+5B^2+6B-f(13)+1) = 0.$$

If $B=-1$, then

$$F_8 = f(13)-1 = 0 \quad \text{and} \quad F_{18} = f(13)+1 = 0,$$

which are impossible. Then

$$B \neq -1 \quad \text{and} \quad f(13) = B^3+5B^2+6B+1,$$

consequently

$$F_8 = -B(B+2)(B-1)(B^3+4B^2+5B+1) = 0,$$

$$F_{13} = B(B-1)(1+B)(B+2)^2(B^2+3B+1) = 0.$$

These imply that

$$B(B-1)(B+2) = 0.$$

Since

$$F_3 = B^3+4B^2-f(3)^2+3B+1 = 0$$

and

$$F_{18} = (1+B)^2(B^5+9B^4-f(3)^4+28B^3+33B^2+9B+1) = 0,$$

we have $B+2 \neq 0$.

Thus we proved that $B \in \{0,1\}$.

Case $B=1$.

It is clear that $f(2) = f(1^2+1) = f(1)^2+1 = 2$, $f(5) = f\left(2^2+1\right) = f(2)^2+1 = 5$, $10 = f(10) = f(9)+1$, thus $f(9)=9$.

Let us assume that $f(p)=p$, $p \equiv 1 \pmod 4$ and $f(q^2) = q^2$ is satisfied if $p<N$, $q<N$, where $N>9$. If N is a critical number, then either $N=P$ ($P \in \mathcal{P}, P \equiv 1 \pmod 4$), or $N=Q^2$ ($Q \in \mathcal{P}, Q \equiv 3 \pmod 4$).

Assume that $N=P$. Let n be the smallest positive integer for which $n^2+1 \equiv 0 \pmod P$. Then $n \leq \frac{P}{2}$, consequently, $n^2+1 = kP$ and $k<N$. Since $f(n^2)=n^2$, we have

$$kf(P) = f(k)f(P) = f(kP) = f(n^2+1) = f(n^2)+1 = n^2+1 = kP,$$

which implies that $f(P)=P$.

In the second case, $N=Q^2$, we obtain that $2|Q^2+1$, and $f\left(\frac{Q^2+1}{2}\right) = \frac{Q^2+1}{2}$, since if $\pi \equiv 3 \pmod 4$, $\pi \in \mathcal{P}$, and $\pi | \frac{Q^2+1}{2}$, then $\pi^\alpha \| \frac{Q^2+1}{2}$ and α is even. The assertion (b2) is proved.

Case $B = 0$.

It is clear that $f(2) = f(1^2+1) = f(1)^2 + B = 1$, $f(5) = f\left(2^2+1\right) = f(2)^2 + B = 1$, $f(9) = f(9) + B = f(9^2+1) = f(2)f(5) = 1$.

Similar as in the proof of the case $B = 1$, we obtain the assertion (c2) of Theorem 2.

Theorem 2 thus is proved. ■

I. Kátai and B. M. Phong
Department of Computer Algebra
Eötvös Loránd University
H-1117 Budapest
Pázmány Péter Sétány 1/C
Hungary
katai@inf.elte.hu
bui@inf.elte.hu