BALL CONVERGENCE OF AN EFFICIENT FIFTH ORDER ITERATIVE METHOD UNDER WEAK CONDITIONS

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Abstract. The aim of this paper is to expand the applicability of a fast iterative method in a Banach space setting. Moreover, we provide computable radius of convergence, error bounds on the distances involved and a proof of uniqueness of solution based on Lipschitz-type functions not given before. Furthermore, we avoid hypotheses on high order derivatives which limit the applicability of the method. Instead, we only use hypotheses on the first derivative. The convergence order is determined using the computational order of convergence or the approximate order of convergence.

1. Introduction

Let $F : \mathfrak{D} \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ be a continuously Fréchet-differentiable operator between the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ and $\mathfrak{D}$ be a convex set. Let $B(x,h) = \{y \in \mathcal{B}_1 : \|x-y\| < h\}$ for $h > 0$. Denote by $\overline{B}(x,h)$ the closure of $B(x,h)$. Let also $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ stand for the set of bounded linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$.

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In this study, we consider the problem of approximating a solution x^* of nonlinear equation

$$F(x) = 0, \tag{1.1}$$

We can recall that in [16] the authors Sharma and Gupta introduced the following method to increasing the order of convergence of iterative methods and given by

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = x_n + \frac{1}{2}(y_n - x_n), \\ u_n = x_n - F'(z_n)^{-1}F(x_n) \\ x_{n+1} = u_n - [2F'(z_n)^{-1} - F'(x_n)^{-1}]F(u_n), \quad \text{if } n = 0, 1, 2, \dots. \end{cases} \tag{1.2}$$

where $x_0 \in \mathfrak{D}$ is an initial point. They considered the above method for solving system of equations, when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i$ (i a natural number). Using Taylor expansion and the assumptions on derivatives of order up to four on F, they proved the order of convergence of the method (1.2) is five. But one can clearly see that the assumptions on the higher order Fréchet derivatives of the operator F restricts the applicability of method (1.2). For example consider the following:

Example 1.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \mathfrak{D} = \overline{B}(x^*, 1)$ and consider the integral equation of the mixed Hammerstein-type ([1, 2, 6, 7, 8, 9, 12]) defined by

$$x(s) = \int_0^1 G(s,t)\big(x(t)^{3/2} + x(t)^2/2\big)dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where F is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\big(x(t)^{3/2} + x(t)^2/2\big)dt,$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)\Big(\frac{3}{2}x(t)^{1/2} + x(t)\Big)dt,$$

One can see that, higher order derivatives of F at $x^*(s)$ do not exist in this example. Since $F^{'}(x^*(s)) = I$ and $\left\| \int\limits_0^1 G(s,t)dt \right\| \leq \frac{1}{8}$, we have

$$\|F^{'}(x^*)^{-1}(F^{'}(x) - F^{'}(y))\| < \frac{1}{8}\left(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\right).$$

Our goal is to weaken the assumptions in [8] and apply the method for solving equation (1.1) in Banach spaces, so that the applicability of the method (1.2) can be extended. The study of the local convergence is important because it shows the degree of difficulty for choosing initial points. Notice that in the studies using Taylor expansions and high order derivatives the choice of the initial point is a shot in the dark. The technique introduced in this paper can be used on other iterative methods ([1]–[17]).

The rest of the paper is structured as follows. In Section 2, we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and a uniqueness result. Special cases and numerical examples are given in the concluding Section 3.

2. Local convergence analysis

We shall base the local convergence that follows on some scalar functions and parameters. Let $w_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function with $w_0(0) = 0$. Suppose that equation

$$w_0(t) = 1 \tag{2.1}$$

has zeros in $(0, +\infty)$. Denote by ρ_0 the smallest such zero. Let $w : [0, \rho_0) \to$ $\to [0, +\infty)$ be a continuous and nondecreasing function with $w(0) = 0$. Define functions g_1, h_1, g_2 and h_2 on $[0, \rho_0)$ by

$$\begin{aligned}
g_1(t) &= \frac{\int\limits_0^1 w\big((1-\theta)t\big)d\theta}{1 - w_0(t)}, \\
h_1(t) &= g_1(t) - 1, \\
g_2(t) &= \frac{1}{2}\left(1 + \frac{\int\limits_0^1 w\big((1-\theta)t\big)d\theta}{1 - w_0(t)}\right),
\end{aligned}$$

and

$$h_2(t) = g_2(t) - 1.$$

We have that $h_1(0) = -1 < 0$, $h_2(0) = -1/2 < 0$, $h_1(t) \to +\infty$ and $h_2(t) \to +\infty$ as $t \to \rho_0^-$. It then follows from the intermediate value theorem that functions h_1 and h_2 have zeros in $(0, \rho_0)$. Denote by r_1 and r_2 the smallest such zeros, respectively.

Suppose that equation

$$w_0\big(g_2(t)t\big) = 1 \tag{2.2}$$

has zeros in $(0, r_2)$. Denote by ρ_2 the smallest such zero. Let $v : [0, \rho_2) \to \to [0, +\infty)$ be a continuous and nondecreasing function. Define functions g_3 and h_3 on the interval $[0, \rho_2)$ by

$$g_3(t) = g_1(t) + \frac{\Big(w_0\big(g_2(t)t\big) + w_0(t)\Big)\int_0^1 v(\theta t)d\theta}{\big(1 - w_0(t)\big)\Big(1 - w_0\big(g_2(t)t\big)\Big)}$$

and $h_3(t) = g_3(t) - 1$. We get $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \rho_2^-$. Denote by r_3 the smallest zero of function h_3 on $(0, \rho_2)$.

Suppose that equation

$$w_0\big(g_3(t)t\big) = 1 \tag{2.3}$$

has zeros in $(0, r_3)$. Denote by ρ_3 the smallest such zero. Define functions g_4 and h_4 on $[0, \overline{\rho})$ by

$$\begin{aligned} g_4(t) \;=\; & \frac{\int_0^1 w\big((1-\theta)g_3(t)t\big)g_3(t)}{1 - w_0\big(g_3(t)t\big)} + \frac{\Big(w_0(t) + w_0\big(g_2(t)t\big)\Big)\int_0^1 v\big(\theta g_3(t)t\big)d\theta g_3(t)}{\big(1 - w_0(g_2(t)t)\big)\big(1 - w_0(t)\big)} + \\ & + \frac{\Big(w_0\big(g_2(t)t\big) + w_0\big(g_3(t)t\big)\Big)\int_0^1 v\big(\theta g_3(t)t\big)d\theta g_3(t)}{\big(1 - w_0\big(g_2(t)t\big)\big)\big(1 - w_0\big(g_3(t)t\big)\big)} \end{aligned}$$

and $h_4(t) = g_4(t) - 1$, where $\overline{\rho} = \min\{\rho_2, \rho_3\}$. We obtain $h_4(0) = -1 < 0$ and $h_4(t) \to +\infty$ as $t \to \overline{\rho}$. Denote by r_4 the smallest zero of function h_4 on $(0, \overline{\rho})$. Define the radius of convergence r by

$$r = \min\{r_i\}, \quad i = 1, 2, 3, 4. \tag{2.4}$$

Then, we have that for each $t \in [0, r)$

$$0 \le g_i(t) < 1, \tag{2.5}$$

$$0 \le w_0(t) < 1,\ 0 \le w_0\big(g_2(t)t\big) < 1 \tag{2.6}$$

and

$$0 \le w_0\big(g_3(t)t\big) < 1. \tag{2.7}$$

Next, we provide the local convergence analysis of method (1.2) based on the following set of conditions which is known as condition (A):

(a_1) $F : \mathfrak{D} \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ is a continuously Fréchet-differentiable operator.

(a_2) There exists x^* such that $F(x^*) = 0$ and $F^{'}(x^*)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$.

(a_3) $\|F^{'}(x^*)^{-1}\big(F^{'}(x) - F^{'}(x^*)\big)\| \le w_0\big(\|x - x^*\|\big)$ for each $x \in \mathfrak{D}$, where w_0 and ρ_0 are given previously. Set $\mathfrak{D}_0 = \mathfrak{D} \bigcap B(x^*, \rho_0)$.

(a_4) $\|F^{'}(x^*)^{-1}\big(F^{'}(x) - F^{'}(y)\big)\| \le w\big(\|x - y\|\big)$ for each $x, y \in \mathfrak{D}_0$, where w is given previously.

(a_5) $\|F^{'}(x^*)^{-1}F^{'}(x)\big)\| \le v\big(\|x - x^*\|\big)$ for each $x \in \mathfrak{D}_0$, where v is given previously.

(a_6) $\overline{B}(x^*, r) \subseteq \mathfrak{D}$, where r is defined by Eq. (2.4).

Theorem 2.1. *Suppose that the conditions (A) hold. Then, sequence $\{x_n\}$ generated for $x_0 \in B(x^*, r) - \{x^*\}$ by method (1.2) is well defined in $B(x^*, r)$ for each $n = 0, 1, 2, \ldots,$ remains in $B(x^*, r)$ and converges to x^*. Moreover, the following error bounds hold*

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{2.8}$$

$$\|z_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{2.9}$$

$$\|u_n - x^*\| \le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| \tag{2.10}$$

and

$$\|x_{n+1} - x^*\| \le g_4(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| \tag{2.11}$$

where the functions $g_i = 1, 2, 3, 4$ are defined previously. Furthermore, if there exists $R \ge r$, such that

$$\int_0^1 w_0(\theta R)d\theta < 1, \tag{2.12}$$

then the point x^* is the only solution of equation $F(x) = 0$ in $\mathfrak{D}_1 = \mathfrak{D} \bigcap \overline{B}(x^*, R)$.

Proof. We shall show using mathematical induction that sequence $\{x_k\}$ converges to $\{x^*\}$ so that estimates (2.8) – (2.11) hold. By hypothesis $x_0 \in B(x^*, r) - \{x^*\}$, (2.4), (2.5), (a_2) and (a_3), we have that

$$\|F'(x^*)^{-1}\big(F'(x_0) - F'(x^*)\big)\| \leq w_0\big(\|x_0 - x^*\|\big) \leq w_0(r) < 1. \tag{2.13}$$

It follows from (2.13) and the Banach Lemma on invertible operators [2, 6, 12, 15] that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.14}$$

We also have that y_0 and z_0 are well defined by the first and second sub step of method (1.2). Using the first sub step of method (1.2) for $n = 0$ and (a_2), we can write

$$y_0 - x^* = x_0 - x^* - F(x_0)^{-1}F(x_0). \tag{2.15}$$

Then, by (2.4), (2.5) (for $i = 1$), (a_4), (2.14) and (2.15), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}\Big(F'\big(x^* + \theta(x_0 - x^*)\big) - \right. \\
&\quad \left. - F'(x_0)\Big)(x_0 - x^*)d\theta \right\| \leq \\
&\leq \frac{\int_0^1 w\big((1-\theta)\|x_0 - x^*\|\big)\|x_0 - x^*\|d\theta}{1 - w_0(\|x_0 - x^*\|)} = \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.16)
\end{aligned}$$

which shows (2.8) for $n = 0$ and $y_0 \in B(x^*, r)$. By (2.4), (2.5) (for $i = 2$) and (2.16), we have in turn that

$$\begin{aligned}
\|z_0 - x^*\| &\leq \left\| \frac{1}{2}(y_0 - x^*) + \frac{1}{2}(x_0 - x^*) \right\| \leq \\
&\leq \frac{1}{2}\big(1 + g_1(\|x_0 - x^*\|)\big)\|x_0 - x^*\| = \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.17)
\end{aligned}$$

which shows (2.9) for $n = 0$ and $z_0 \in B(x^*, r)$. In view of (2.6), (2.13), (2.14) (for $z_0 = x_0$), we get that $F'(z_0)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\begin{aligned}
\|F'(z_0)^{-1}F'(x^*)\| &\leq \frac{1}{1 - w_0(\|z_0 - x^*\|)} \leq \\
&\leq \frac{1}{1 - w_0\big(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}. \qquad (2.18)
\end{aligned}$$

We also have that u_0 and x_1 are well defined by the third and fourth sub step of method (1.2) for $n = 0$. We can also write by the third sub step of method (1.2) for $n = 0$.

$$\begin{aligned} u_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + \big(F'(x_0)^{-1} - F'(z_0)^{-1}\big)F(x_0) = \\ &= y_0 - x^* + F'(x_0)^{-1}\big(F'(z_0)^{-1} - F'(x_0)^{-1}\big)F'(z_0)^{-1}F(x_0). \end{aligned} \tag{2.19}$$

Moreover, we can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'\big(x^* + \theta(x_0 - x^*)\big)d\theta(x_0 - x^*), \tag{2.20}$$

so by (a_5) and (2.20), we get

$$\|F'(x^*)^{-1}F(x_0)\| \leq \int_0^1 v\big(\theta\|x_0 - x^*\|\big)d\theta\|x_0 - x^*\|. \tag{2.21}$$

Using $(2.4), (2.5)$ (for $i = 3$), $(2.14), (2.16), (2.17), (2.18)$ and (2.21), we obtain in turn that

$$\begin{aligned} \|u_0 - x^*\| &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + \\ &\quad +\|F'(x_0)^{-1}F'(x^*)\|\Big(\|F'(x^*)^{-1}\big(F'(z_0) - F'(x^*)\big)\| + \\ &\quad +\|F'(x^*)^{-1}\big(F'(x_0) - F'(x^*)\big)\|\Big) \times \\ &\quad \times\|F'(z_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \leq \\ &\leq \Bigg[\frac{\Big(w_0\big(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|\big) + w_0(\|x_0 - x^*\|)\Big)}{(1 - w_0(\|x_0 - x^*\|))} \times \\ &\quad \times\frac{\int_0^1 v(\theta\|x_0 - x^*\|)d\theta}{\Big(1 - w_0\big(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|\big)\Big)} + \\ &\quad +g_1(\|x_0 - x^*\|)\Bigg]\|x_0 - x^*\| = \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.22}$$

which shows (2.10) for $n = 0$ and $u_0 \in B(x^*, r)$. Furthermore, from (2.4), (2.5) (for $i = 4$), (2.14) (for $u_0 = x_0$), (2.16), (2.18) and (2.22) and the last sub step of method (1.2) for $n = 0$, we get in turn that

$$\begin{aligned}
\|x_1 - x^*\| &= \|u_0 - x^* - F^{'}(u_0)^{-1}F(u_0) + \big(F^{'}(z_0)^{-1} - F^{'}(x_0)^{-1}\big)F(u_0) + \\
&\quad +\big(F^{'}(z_0)^{-1} - F^{'}(u_0)^{-1}\big)F(u_0)\| \leq \\
&\leq \|u_0 - x^* - F^{'}(u_0)^{-1}F(u_0)\| + \|F^{'}(z_0)^{-1}F^{'}(x^*)\| \times \\
&\quad \times\Big(\|F^{'}(x^*)^{-1}\big(F^{'}(x_0) - F^{'}(x^*)\big)\| + \\
&\quad +\|F^{'}(x^*)^{-1}\big(F^{'}(z_0) - F^{'}(x^*)\big)\|\Big)\|F^{'}(x_0)^{-1}F^{'}(x^*)\| \times \\
&\quad \times\|F^{'}(x^*)^{-1}F(u_0)\| + \\
&\quad +\|F^{'}(z_0)^{-1}F^{'}(x^*)\|\Big(\|F^{'}(x^*)^{-1}\big(F^{'}(z_0) - F^{'}(x^*)\big)\| + \\
&\quad +\|F^{'}(x^*)^{-1}\big(F^{'}(u_0) - F^{'}(x^*)\big)\|\Big)\|F^{'}(u_0)^{-1}F^{'}(x^*)\| \times \\
&\quad \times\|F^{'}(x^*)^{-1}F(u_0)\| \leq \\
&\leq \frac{\int_0^1 w\big((1-\theta)\|u_0 - x^*\|\big)\|u_0 - x^*\|d\theta}{1 - w_0(\|u_0 - x^*\|)} + \\
&\quad +\frac{\Big(w_0(\|x_0 - x^*\|) + w_0\big(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|\big)\Big)}{\Big(1 - w_0\big(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|\big)\Big)\big(1 - w_0(\|x_0 - x^*\|)\big)} \times \\
&\quad \times \int_0^1 v(\theta\|u_0 - x^*\|)d\theta\|u_0 - x^*\| + \\
&\quad +\Big(w_0\big(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|\big) + \\
&\quad +w_0\big(g_3(\|x_0 - x^*\|)\|x_0 - x^*\|\big)\Big) \times \\
&\quad \times \frac{1}{\Big(1 - w_0\big(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|\big)\Big)} \times \\
&\quad \times \frac{1}{\Big(1 - w_0\big(g_3(\|x_0 - x^*\|)\|x_0 - x^*\|\big)\Big)} \times \\
&\quad \times \int_0^1 v(\theta\|u_0 - x^*\|)d\theta\|u_0 - x^*\| \leq \\
&\leq g_4(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r,
\end{aligned}$$

which shows (2.11) for $n = 0$ and $x_1 \in B(x^*, r)$. The induction for (2.8)–(2.11) is completed by simply replacing x_0, y_0, z_0, u_0 and x_1 by x_k, y_k, z_k, u_k and x_{k+1}

in the preceding estimates, respectively. Then, from the estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r, \tag{2.23}$$

where $c = g_4(\|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim\limits_{k\to+\infty} x_k = x^*$ and $x_{k+1} \in$ $\in B(x^*, r)$. Finally, let $y^* \in D_1$ with $F(y^*) = 0$ and define $Q = \int\limits_0^1 F'\big(x^* +$ $+\theta(y^* - x^*)\big)d\theta$. Using (a_3) and (2.12), we get that

$$\|F'(x^*)^{-1}\big(Q - F'(x^*)\big)\| \leq \int\limits_0^1 w_0\big(\theta\|y^* - x^*\|\big)d\theta \leq \int\limits_0^1 w_0(\theta R)d\theta < 1, \tag{2.24}$$

so, $Q^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity

$$Q(y^* - x^*) = F(y^*) - F(x^*) = 0, \tag{2.25}$$

we conclude that $x^* = y^*$. ■

Remark 2.1. (a) In the case when $w_0(t) = L_0 t, w(t) = Lt$ and $\mathfrak{D}_0 = \mathfrak{D}$, the radius $r_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [2] as the convergence radius for Newton's method under condition (a_1)-(a_4) and (a_6). Notice that the convergence radius for Newton's method given independently by Rheinboldt [15] and Traub [17] is given by

$$r_{TR} = \frac{2}{3L} < r_A \tag{2.26}$$

As an example, let us consider the function $F(x) = e^x - 1$. Then $x^* = 0$. Set $\mathfrak{D} = \mathcal{B}(0, 1)$. Then, we have that $L_0 = e - 1 < L = e$, so $r_{TR} = 0.24252961 <$ $< r_A = 0.324947231$.

Moreover, the new error bounds [2] are

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [5, 7]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, the radius of convergence of method (1.2) given by r cannot be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2]–[6].

(c) The results can be also used to solve equations where the operator $F^{'}$ satisfies the autonomous differential equation [2]–[6]:

$$F^{'}(x) = P\big(F(x)\big),$$

where $P : \mathcal{B}_2 \to \mathcal{B}_2$ is a known continuous operator. Since $F^{'}(x^*) = P\big(F(x^*)\big) = P(0)$, we can apply the results without actually knowing the solution x^*. As an example let $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) is not changing if we use the new instead of the old conditions [16]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{\ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{\ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } n = 0, 1, 2, \ldots$$

(e) In view of (a_3) and the estimate

$$\begin{aligned} \|F^{'}(x^*)^{-1}F^{'}(x)\| &= \|F^{'}(x^*)^{-1}\big(F^{'}(x) - F^{'}(x^*)\big) + I\| \\ &\leq 1 + \|F^{'}(x^*)^{-1}\big(F^{'}(x) - F^{'}(x^*)\big)\| \leq 1 + w_0(\|x - x^*\|) \end{aligned}$$

condition (a_5) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t)$$

or

$$v(t) = 1 + w_0(\rho_0),$$

since $t \in [0, \rho_0)$.

3. Numerical examples

We present two numerical examples in this section.

Example 3.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \overline{U}(0,1), s^* = (0,0,0)^T$. Define function F on D for $s = (s_1, s_2, s_3)^T$ by

$$F(s) = \left(e^{s_1} - 1, \frac{e-1}{2}s_2^2 + s_2, s_3\right)^T.$$

Then, the Fréchet-derivative is given by

$$F^{'}(s) = \begin{bmatrix} e^{s_1} & 0 & 0 \\ 0 & (e-1)s_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we can choose $w_0(t) = L_0 t, w(t) = e^{\frac{1}{L_0}} t, v(t) = e^{\frac{1}{L_0}}, L_0 = e - 1$. Then, $r_1 = 0.3827$, $r_2 = 0.3827$, $r_3 = 0.1234$, $r_4 = 0.1029$ and hence $r = 0.1029$ is concluded from (2.4). Also we can check that conditions given in (2.5)-(2.7) are satisfied. Hence the radius of convergence r is given by $r = 0.1029$.

Example 3.2. Again coming back to the motivational Example 1.1, we can choose (see also Remark 2.1(e) for function v) $w_0(t) = w(t) = \frac{1}{8}\left(\frac{3}{2}\sqrt{t} + t\right)$ and $v(t) = 1 + w_0(r_0), r_0 \backsimeq 4.7354$. Then, $r_1 = 2.6303$, $r_2 = 2.6303$, $r_3 = 0.4491$, $r_4 = 0.3152$ and hence $r = 0.3152$ is concluded from (2.4). Also we can check that conditions given in (2.5)–(2.7) are satisfied. Hence the radius of convergence r is given by $r = 0.3152$.

4. Conclusions

In this paper we have studied the local convergence of an efficient fifth order method by assuming only conditions on the first derivative of the operator. We also provided computable radius of convergence, error bounds on the distances involved and a uniqueness of the solution result based on Lipschitz-type functions not given before. Numerical examples are computed to give computable radius of convergence.

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