SEGRE’S UPPER BOUND FOR THE REGULARITY INDEX OF 2n + 2 NON-DEGENERATE DOUBLE POINTS IN \( \mathbb{P}^n \)

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Abstract. We prove the Segre’s upper bound for the regularity index of 2n + 2 non-degenerate double points that do not exist n+1 points lying on a (n − 2)-plane in \( \mathbb{P}^n \).

1. Introduction

Let \( P_1, \ldots, P_s \) be a set of distinct points in a projective space with n-dimension \( \mathbb{P}^n := \mathbb{P}^n_k \), with \( k \) as an algebraically closed field. Let \( \wp_1, \ldots, \wp_s \) be the homogeneous prime ideals of the polynomial ring \( R := k[x_0, \ldots, x_n] \) corresponding to the points \( P_1, \ldots, P_s \). Let \( m_1, \ldots, m_s \) be positive integers and \( I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s} \). Denote \( Z = m_1P_1 + \cdots + m_sP_s \) the zero-scheme defined by \( I \), and we call \( Z \) a set of \( s \) fat points in \( \mathbb{P}^n \).

The homogeneous coordinate ring of \( Z \) is

\[ A = R/(\wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}). \]

The ring \( A = \oplus_{t \geq 0} A_t \) is a one-dimension Cohen-Macaulay \( k \)-graded algebra whose multiplicity is \( e(A) = \sum_{i=1}^s \binom{m_i+n-1}{n} \). The Hilbert function \( H_A(t) = Key words and phrases: Fat points, regularity index, zero-scheme.

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In 1961, Segre (see [10]) showed the upper bound for regularity index of generic fat points \( Z = m_1 P_1 + \cdots + m_s P_s \) in \( \mathbb{P}^2 \):

\[
\text{reg}(Z) \leq \max\left\{ m_1 + m_2 - 1, \frac{m_1 + \cdots + m_s}{2} \right\}
\]

with \( m_1 \geq \cdots \geq m_s \).

For arbitrary fat points \( Z = m_1 P_1 + \cdots + m_s P_s \) in \( \mathbb{P}^2 \), in 1969 Fulton (see [9]) gave the following upper bound:

\[
\text{reg}(Z) \leq m_1 + \cdots + m_s - 1.
\]

This bound was later extended to arbitrary fat points in \( \mathbb{P}^n \) by Davis and Geramita (see [6]). They also showed that this bound is attained if and only if points \( P_1, \ldots, P_s \) lie on a line in \( \mathbb{P}^n \).

A set of fat points \( Z = m_1 P_1 + \cdots + m_s P_s \) in \( \mathbb{P}^n \) is said to be in general position if no \( j + 2 \) of the points \( P_1, \ldots, P_s \) are on any \( j \)-plane for \( j < n \). A set of fat points \( Z = m_1 P_1 + \cdots + m_s P_s \) of \( \mathbb{P}^n \) is said to be non-degenerate if all points \( P_1, \ldots, P_s \) do not lie on a hyperplane of \( \mathbb{P}^n \). In 1991, Catalisano (see [3], [4]) extended Segre’s result to fat points in general position in \( \mathbb{P}^2 \), and later Catalisano, Trung and Valla (see [5]) extended the result to fat points in general position in \( \mathbb{P}^n \), they proved:

\[
\text{reg}(Z) \leq \max\left\{ m_1 + m_2 - 1, \frac{m_1 + \cdots + m_s + n - 2}{n} \right\}.
\]

In 1996, N.V. Trung gave the following conjecture: Let \( Z = m_1 P_1 + \cdots + m_s P_s \) be arbitrary fat points in \( \mathbb{P}^n \). Then

\[
\text{reg}(Z) \leq \max \left\{ T_j \mid j = 1, \ldots, n \right\},
\]

where

\[
T_j = \max \left\{ \left\lfloor \frac{\sum_{i=1}^{q} m_i + j - 2}{j} \right\rfloor \mid P_i, \ldots, P_q \text{ lie on a } j\text{-plane} \right\}.
\]

This upper bound nowadays is called the Segre’s upper bound.

The Segre’s upper bound is proved right in projective spaces with \( n = 2 \), \( n = 3 \) (see [12], [13]), for the case of double points \( Z = 2P_1 + \cdots + 2P_s \) in \( \mathbb{P}^n \) with \( n = 4 \) (see [14]) by Thien; also for case \( n = 2, n = 3 \), independently by Fatabbi and Lorenzini (see [7], [8]).
In 2012, Benedetti, Fatabbi and Lorenzini proved the Segre’s bound for any set of \( n+2 \) non-degenerate fat points \( Z = m_1P_1 + \cdots + m_{n+2}P_{n+2} \) of \( \mathbb{P}^n \) (see [1]), and independently Thien also proved the Segre’s bound for a set of \( s+2 \) fat points which is not on a \((s-1)\)-space in \( \mathbb{P}^n, s \leq n \) (see [15]).

Recently, Ballico, Dumitrescu and Postinghen proved the Segre’s upper bound for the case \( n+3 \) non-degenerate fat points \( Z = m_1P_1 + \cdots + m_{n+3}P_{n+3} \) in \( \mathbb{P}^n \) (see [2]) and Sinh proved the Segre’s upper bound for the regularity index of \( 2n+1 \) double points \( Z = 2P_1 + \cdots + 2P_{2n+1} \) that do not exist \( n+1 \) points lying on a \((n-2)\)-plane in \( \mathbb{P}^n \) (see [11]). Up to now, there have not been any other result of Trung’s conjecture published yet.

In this article, we prove the Segre’s upper bound in the case \( 2n+2 \) non-degenerate double points \( Z = 2P_1 + \cdots + 2P_{2n+2} \) that do not exist \( n+1 \) points lying on a \((n-2)\)-plane in \( \mathbb{P}^n \).

2. Preliminaries

We will use the following lemmas which have been proved. The first lemma allows us to compute the regularity index by induction.

**Lemma 2.1.** [5, Lemma 1]. Let \( P_1, ..., P_r, P \) be distinct points in \( \mathbb{P}^n \), and let \( \wp \) be the defining ideal of \( P \). If \( m_1, ..., m_r \) and \( a \) are positive integers, \( J = \wp_{m_1} \cap \cdots \cap \wp_{m_r} \), and \( I = J \cap \wp^a \), then

\[
\text{reg}(R/I) = \max \left\{ a - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp^a)) \right\}.
\]

To compute \( \text{reg}(R/(J + \wp^a)) \), we need the following lemma.

**Lemma 2.2.** [5, Lemma 3]. Let \( P_1, ..., P_r \) be distinct points in \( \mathbb{P}^n \) and \( a, m_1, ..., m_r \) positive integers. Put \( J = \wp_{m_1} \cap \cdots \cap \wp_{m_r} \) and \( \wp = (x_1, ..., x_n) \). Then

\[
\text{reg}(R/(J + \wp^a)) \leq b
\]

if and only if \( x_0^{b-1}M \in J + \wp^{i+1} \) for every monomial \( M \) of degree \( i \) in \( x_1, ..., x_n \), \( i = 0, ..., a - 1 \).

To find such a number \( b \), we will find \( t \) hyperplanes \( L_1, ..., L_t \) avoiding \( P \) such that \( L_1 \cdots L_t M \in J \). For \( j = 1, ..., t \), since we can write \( L_j = x_0 + G_j \) for some linear form \( G_j \in \wp \), we get \( x_0^i M \in J + \wp^{i+1} \). Therefore, if we put

\[
\delta = \max \left\{ t + i | M \text{ is a monomial of degree } i, 0 \leq i \leq a - 1 \right\}
\]
then
\[
\text{reg}(R/(J + \phi^n)) \leq \delta.
\]

The hyperplanes \(L_1, \ldots, L_t\) will be constructed by the help of the following lemma.

**Lemma 2.3.** [5, Lemma 4]. Let \(P_1, \ldots, P_r, P\) be distinct points in general position in \(\mathbb{P}^n\), let \(m_1 \geq \cdots \geq m_r\) be positive integers, and let \(J = \phi_1^{m_1} \cap \cdots \cap \phi_r^{m_r}\).
If \(t\) is an integer such that \(nt \geq \sum_{i=1}^{r} m_i\) and \(t \geq m_1\), we can find \(t\) hyperplanes, say \(L_1, \ldots, L_t\) avoiding \(P\) such that for every \(P_{l}, l = 1, \ldots, r\), there exist \(m_l\) hyperplanes of \(\{L_1, \ldots, L_t\}\) passing through \(P_l\).

The two following lemmas are used to prove main results by induction.

**Lemma 2.4.** [11, Proposition 2.1]. Let \(X = \{P_1, \ldots, P_{2n+1}\}\) be a set of \(2n + 1\) distinct points that do not exist \(n + 1\) points of \(X\) lying on an \((n-2)\)-plane in \(\mathbb{P}^n\). Let \(\varphi_i\) be the homogeneous prime ideal corresponding \(P_i, i = 1, \ldots, 2n + 1\).
Let
\[
Z = 2P_1 + \cdots + 2P_{2n+1}.
\]

Put
\[
T_j = \max\{\left\lfloor \frac{1}{j}(2q + j - 2) \right\rfloor | P_1, \ldots, P_q \text{ lie on a } j\text{-plane}\},
\]
\[
T_Z = \max\{T_j | j = 1, \ldots, n\}.
\]
Then, there exists a point \(P_{i_0} \in X\) such that
\[
\text{reg}(R/(J + \varphi_{i_0}^2)) \leq T_Z,
\]
where
\[
J = \bigcap_{k \neq i_0} \varphi_k^2.
\]

**Lemma 2.5.** [11, Proposition 2.2]. Let \(X = \{P_1, \ldots, P_{2n+1}\}\) be a set of \(2n + 1\) distinct points which do not exist \(n + 1\) points of \(X\) lying on a \((n-2)\)-plane in \(\mathbb{P}^n\). Let \(Y = \{P_1, \ldots, P_s\}, 2 \leq s \leq 2n,\) be a subset of \(X\). Let \(\varphi_i\) be the homogeneous prime ideal corresponding \(P_i, i = 1, \ldots, 2n + 1\).
Let
\[
Z = 2P_1 + \cdots + 2P_{2n+1}.
\]

Put
\[
T_j = \max\{\left\lfloor \frac{1}{j}(2q + j - 2) \right\rfloor | P_1, \ldots, P_q \text{ lie on a } j\text{-plane}\},
\]
\[
T_Z = \max\{T_j | j = 1, \ldots, n\}.
\]
Then, there exists a point \( P_{i_0} \in Y \) such that
\[
\text{reg}(R/(J + \wp_{i_0}^2)) \leq T_Z,
\]
where
\[
J = \bigcap_{P_k \in Y \setminus \{P_{i_0}\}} \wp_k^2.
\]

3. Segre’s upper bound for the regularity index of \( 2n + 2 \) non-degenerate double points in \( \mathbb{P}^n \)

From now on, we consider a hyperplane and its identical defining linear form. These following propositions are important for proving of Segre’ upper bound.

**Proposition 3.1.** Let \( X = \{P_1, ..., P_{2n+2}\} \) be a non-degenerate set of \( 2n + 2 \) distinct points that do not exist \( n + 1 \) points of \( X \) lying on a \((n - 2)\)-plane in \( \mathbb{P}^n \). Let \( \wp_i \) be the homogeneous prime ideal corresponding \( P_i, i = 1, ..., 2n + 2 \), and
\[
Z = 2P_1 + \cdots + 2P_{2n+2}.
\]
Put
\[
T_j = \max \left\{ \left\lfloor \frac{1}{2}(2q + j - 2) \right\rfloor \mid P_{i_1}, ..., P_{i_q} \text{ lie on a } j\text{-plane} \right\},
\]
\[
T_Z = \max \{T_j \mid j = 1, ..., n\}.
\]
Then, there exists a point \( P_{i_0} \in X \) such that
\[
\text{reg}(R/(J + \wp_{i_0}^2)) \leq T_Z,
\]
where
\[
J = \bigcap_{k \neq i_0} \wp_k^2.
\]

**Proof.** We denote \( |H| \) by the number points of \( X \) lying on a \( j\)-plane \( H \). The proposition was proved in projective spaces with \( n \leq 4 \) (see [7], [8], [12]–[14]). Thus, we will prove the case with \( n \geq 5 \).

We can see that there are \((n - 1)\)-planes \( H_1, ..., H_d \) in \( \mathbb{P}^n \) with \( d \) as the least integer such that the two following conditions satisfied:

(i) \( X \subset \cup_{i=1}^d H_i \),
(ii) \( |H_d \cap (X) \setminus \bigcup_{j=1}^{d-1} H_j| = \max\{|H \cap (X \setminus \bigcup_{j=1}^{d-1} H_j)| \mid H \text{ is an } (n - 1)\text{-plane}\} \).
Since $X$ non-degenerate and $n + 1$ points do not lie on a $(n - 2)$-plane, $2 \leq d \leq 3$. We consider the following cases:

**Case 1.** $d = 3$. Since a hyperplane always passes through at least $n$ points of $X$ and $d = 3$, we have the two following cases:

(i) $|H_1| = n$, $|H_2| = n$, $|H_3| = 2$.
(ii) $|H_1| = n + 1| = |H_2 \setminus H_1| = n$, $|H_3| = 1$.

**Case 1.1.** $|H_1| = n$, $|H_2| = n$, $|H_3| = 2$. Since $|H_1| = n$, there do not exist $n + 1$ points of $X$ lying on a hyperplane. Therefore, $X$ is general position. By Lemma 2.3 and Lemma 2.2 we have

$$\text{reg}(R/(J + \varphi_{t_0}^2)) \leq T_Z.$$ 

**Case 1.2.** $|H_1| = n + 1$, $|H_2| = n$, $|H_3| = 1$. We may assume that $P_1 \in H_3$. Choose $P_i = P_i = (1, 0, \ldots, 0)$, then $\varphi_{t_0} = (x_1, \ldots, x_n)$. Clearly, $H_1, H_2$ avoiding $P_i$. We have $H_1 H_1 H_2 H_2 \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i$, $i = 0, 1$. By Lemma 2.2 we have

$$\text{reg}(R/(J + \varphi_{t_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

**Case 2.** $d = 2$. We have $X \subset H_1 \cup H_2$. Therefore, $|H_1| \geq n + 1$ and $H_1 \geq |H_2|$. We call $q$ the number points of $X$ lying on $H_2 \setminus H_1$, we have $1 \leq q \leq n + 1$, without loss of generality, we assume $P_1, \ldots, P_q \in H_2 \setminus H_1$. Put $Y = \{P_1, \ldots, P_q\}$. Since $n + 1$ points of $X$ do not lie on a $(n - 2)$-plane, $Y$ does not lie on a $(q - 3)$-plane. We consider the following cases:

**Case 2.1.** $Y$ lies on a $(q - 1)$-plane and $Y$ does not lie on a $(q - 2)$-plane. Choose $P_q = P_i = (1, 0, \ldots, 0)$, $P_1 = (0, 1, \ldots, 0), \ldots, P_{q-1} = (0, \ldots, 1, \ldots, 0)$, then $\varphi_{t_0} = (x_1, \ldots, x_n)$. Since we always have a $(q - 2)$-plane, say $K$, passing through $P_1, \ldots, P_{q-1}$ and avoiding $P_i$: therefore, we always have a hyperplane, say $L$, containing $K$ and avoiding $P_i$. We have $H_1 H_1 LL M \in J$. Thus $H_1 H_1 L L M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\text{reg}(R/(J + \varphi_{t_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

**Case 2.2.** $Y$ lies on a $(q - 2)$-plane $\alpha, q \geq 3$. We consider the following cases of $Y$:

**Case 2.2.1.** There are $q - 1$ points of $Y$ lying on a $(q - 3)$-plane. Assume that $P_1, \ldots, P_{q-1}$ lying on a $(q - 3)$-plane, say $K$ and $P_q \notin K$. Choose $P_q = P_i = (1, 0, \ldots, 0)$, then $\varphi_{t_0} = (x_1, \ldots, x_n)$. Since $q \leq n + 1$, we have $q - 3 \leq n - 2$ and $P_i \notin K$, we always have a hyperplane $L$ containing $K$ and avoiding
We have $H_1 H_1 LL \in J$, thus $H_1 H_1 LLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have
\[
\text{reg}(R/(J + \varphi_{1_0}^2)) \leq 4 + i \leq T_Z.
\]

Case 2.2.2. There are not $q - 1$ points of $Y$ lying on a $(q - 3)$-plane. We consider the three following cases of $q$:

Case 2.2.2.1. $q \geq 5$. Since any $(q - 3)$-planes only pass through $q - 2$ points of $Y$. Choose $P_q = P_{i_0} = (1, 0, \ldots, 0), P_1 = (0, 1, 0, \ldots, 0), \ldots, P_{q-2} = (0, \ldots, 0, 1, 0, \ldots, 0)$. Put $m_t = 2 - i + c_l, l = 1, \ldots, q - 2, m_{q-1} = 2$ and
\[
t = \max \left\{ 2, \left[ \sum_{i=1}^{q-1} m_t + (q - 2) - 1 \right]/(q - 2) \right\}.
\]
We have
\[
t + i = \max \left\{ 2, \left[ \sum_{i=1}^{q-1} m_t + q - 3 \right]/(q - 2) \right\} + i \leq \max \left\{ 2 + i, \left[ \sum_{i=1}^{q-1} m_t + (q - 2)i + q - 3 \right]/(q - 2) \right\} \leq \max \left\{ 2 + i, \left[ (3q - 4)q - 2) \right] \right\} \leq 3.
\]
Therefore,
\[
t \leq 3 - i.
\]

By Lemma 2.2, we can find $t$ $(q - 3)$-planes, say $G_1, \ldots, G_t$ avoiding $P_{i_0}$ such that for every point $P_l, l = 1, \ldots, q - 1$, there are $m_t$ $(q - 3)$-planes of $G_1, \ldots, G_t$ passing through $P_l$. With $j = 1, \ldots, t$ we find a hyperplane $L_j$ containing $G_j$ and avoiding $P_{i_0}$. Therefore
\[
L_1 \cdots L_t \in \varphi_1^{m_1} \cap \cdots \cap \varphi_{q-2}^{m_{q-2}} \cap \varphi_{q-1}^2.
\]
Moreover, since $H_1 H_1 \in \varphi_{q+1}^2 \cap \cdots \cap \varphi_{2n+2}^2$ and $M \in \varphi_1^{-c_1} \cap \cdots \cap \varphi_{q-2}^{-c_{q-2}}$, then
\[
H_1 H_1 L_1 \cdots L_t M \in J.
\]

By Lemma 2.2 we have
\[
\text{reg}(R/(J + \varphi_{1_0}^2)) \leq 2 + (3 - i) + i \leq T_Z.
\]

Case 2.2.2.2. $q = 4$. We have $P_1, P_2, P_3, P_4 \notin H_1$. Choose $P_1 = P_{i_0} = (1, 0, \ldots, 0), P_3 = (0, 1, 0, \ldots, 0), P_2 = (0, 0, 1, 0, \ldots, 0), \ldots, P_{n+1} = (0, \ldots, 0, 1, 0, \ldots, 0), P_{n+2} = (0, \ldots, 0, 1, 0, \ldots, 0)$, therefore $\varphi_{1_0} = (x_1, \ldots, x_n)$. We
call $l_1$ a line passing through $P_2, P_3$; $l_2$ a line passing through $P_3, P_4$; $l_3$ a line passing through $P_2, P_4$. We consider the two following cases of $i$:

a) $i = 0$. With $j = 1, 2, 3$, since $P_{i0} \notin l_j$, then we always have a hyperplane $L_j$ containing $l_j$ and avoiding $P_{i0}$. We have $H_1H_1L_1L_2L_3 \in J$, thus $H_1H_1L_1L_2L_3M \in J$. By Lemma 2.2 we have

$$\text{reg}(R/(J + \varphi_{i0}^2)) \leq 5 \leq T_Z.$$ 

b) $i = 1$. Since $c_1 + \cdots + c_n = 1$, then there exists $j \in \{1, \ldots, n\}$ such that $c_j = 1, c_k = 0, k \in \{1, \ldots, n\}\{j\}$.

- If $j \in \{1, 2\}$, assume that $c_1 = 1$ then $M \in \varphi_4 \cap \varphi_5 \cap \cdots \cap \varphi_{n+2}$.

We have a $(n - 2)$-plane, say $K_1$ passing through $P_{n+3}, \ldots, P_{2n-1}$ and $l_1$, a $(n - 2)$-plane, say $K_2$ passing through $P_{2n}, P_{2n+1}$ and $l_1$, a $(n - 2)$-plane, say $K_3$ passing through $P_4, P_{2n+2}$ avoiding $P_{i0}$. With $i = 1, 2, 3$, we always have hyperplanes $L_i$ containing $K_i$ and avoiding $P_{i0}$. We have

$$H_1L_1L_2L_3 \in \varphi_2^2 \cap \varphi_3^2 \cap \varphi_4 \cap \varphi_5 \cap \cdots \cap \varphi_{n+2} \cap \varphi_{n+3} \cap \cdots \cap \varphi_{2n+2}^2.$$ 

Therefore

$$H_1L_1L_2L_3M \in J.$$ 

By Lemma 2.2 we have

$$\text{reg}(R/(J + \varphi_{i0}^2)) \leq 4 + i \leq T_Z.$$ 

- If $j \in \{3, \ldots, n\}$, assume that $c_3 = 1$ then $M \in \varphi_3 \cap \varphi_4 \cap \cdots \cap \varphi_6 \cap \cdots \cap \varphi_{n+2}$.

We call $l_1$ a line passing through $P_2, P_3$ and $l_2$ a line passing through $P_2, P_4$. With $i = 1, 2$, since $P_{i0} \notin l_i$, then we always have hyperplanes $L_i$ containing $l_i$ and avoiding $P_{i0}$. We have

$$L_1L_2 \in \varphi_2^2 \cap \varphi_3 \cap \varphi_4$$

Since $H_1H_1 \in \varphi_3^2 \cap \cdots \cap \varphi_{2n+2}^2$ then

$$H_1H_1L_1L_2M \in J.$$ 

By Lemma 2.2 we have

$$\text{reg}(R/(J + \varphi_{i0}^2)) \leq 4 + i \leq T_Z.$$
Case 2.2.2.3. $q = 3$. We have $P_1, P_2, P_3 \notin H_1$. We call $l$ a line passing through $P_1, P_2, P_3$ and $W = \{P_4, \ldots, P_{2n+2}\}$ are the points of $X$ lying on $H_1 \cap X$, then there are $(n-2)$-planes $Q_1, \ldots, Q_r$ in $\mathbb{P}^n$ such that the two following conditions satisfied:

(i) $W \subset \cup_{i=1}^r Q_i$,

(ii) $|Q_i \cap (W \setminus \bigcup_{j=1}^{i-1} Q_j)| = \max \{|Q \cap (W \setminus \bigcup_{j=1}^{i-1} Q_j)| \mid Q \text{ is a } (n-2) \text{-plane} \}$.

Since $n + 1$ of $X$ do not lie on a $(n-2)$-plane, then we consider the two following cases of $Q_1$:

a) $|Q_1| = n$. We have $r = 2$ and $|Q_2| = n - 1$. Put $U = \{P_3, \ldots, P_{n+2}\}$ to be $n - 1$ points lying on $Q_2 \vee T = \{P_1, \ldots, P_{n+2}\}$. We consider the two following cases of $T$:

a.1) $T$ does not lie on a $(n-1)$-plane. Since $P_1, P_2, P_3$ lie on a line $l$, then we always have a hyperplane containing $l$ and passing through $n - 2$ points of $U$. Assume that $L$ to be a hyperplane containing $l$ and passing through points $P_4, \ldots, P_{n+1}$. Clearly, the hyperplane $L$ avoiding $P_{n+2}$ (if not, then $T$ lies on a $(n-1)$-plane). Choose $P_{n+2} = P_{i_0} = (1, 0, \ldots, 0)$, then $\varphi_{i_0} = (x_1, \ldots, x_n)$. Since $P_{i_0} \notin Q_1$, therefore we always have a hyperplane $L_1$ containing $Q_1$ and avoiding $P_{i_0}$. We have $LLL_1L_1 \in J$ then $LLL_1L_1M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \varphi_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

a.2) $T$ lies on a $(n-1)$-plane, say $H$. Assume that $|Q_1 \cap H \cap X| = s$. When hyperplane $H$ passing through $n + 2 + s$ points of $X$. Consider $n - s$ points lying on $Q_1 \setminus H$, say $P_{i_1}, \ldots, P_{i_{n-s}} \in Q_1 \setminus H$.

a.2.1) Case $P_{i_1}, \ldots, P_{i_{n-s}}$ lie on a $(n-s-1)$-plane and they do not lie on a $(n-s-2)$-plane. Choose $P_{i_1} = P_{i_0} = (1, 0, \ldots, 0)$, then $\varphi_{i_0} = (x_1, \ldots, x_n)$. Since we always have a $(n-s-2)$-plane, say $\beta$ passing through $P_{i_2}, \ldots, P_{i_{n-s-1}}$. Moreover, since $n - s - 2 \leq n - 2$ then we always have a hyperplane $L$ containing $\beta$ and avoiding $P_{i_0}$. We have $HHLL \in J$ then $HHLLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \varphi_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

a.2.2) Case $P_{i_1}, \ldots, P_{i_{n-s}}$ lie on a $(n-s-2)$-plane. Since $P_1, P_2, P_3$ lie on a line, then $P_1, P_2, P_3, P_{i_1}, \ldots, P_{i_{n-s}}$ lie on a $(n-s)$-plane. So, $n - 1 \leq n - s \leq n$ or $0 \leq s \leq 1$.

- If $\{P_{i_1}, \ldots, P_{i_{n-s}}\}$ has $n - s - 1$ points lying on a $(n-s-3)$-plane, say $\gamma$. Assume that $P_{i_1} \notin \gamma$, then choose $P_{i_1} = P_{i_0} = (1, 0, \ldots, 0)$, then $\varphi_{i_0} = (x_1, \ldots, x_n)$. Since $P_{i_0} \notin \gamma$ therefore we always have a hyperplane $L$ containing $\gamma$ and avoiding $P_{i_0}$. We have $LLHH \in J$ then $LLHHM \in J$ for every
monomial \( M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1 \). By Lemma 2.2 we have
\[
\text{reg}(R/(J + \varphi_i^2)) \leq 4 + i \leq 5 \leq T_Z.
\]

- If \( \{P_{i_1}, ..., P_{i_{n-s}}\} \) without \( n-s-1 \) points lying on a \((n-s-3)\)-plane, then any \((n-s-3)\)-plane only pass through \( n-s-2 \) points of \( \{P_{i_1}, ..., P_{i_{n-s}}\} \). Choose

  \[
P_{i_1} = P_{i_0} = (1, 0, ..., 0), P_{i_2} = (0, 1, 0, ..., 0), ..., P_{i_{n-s-1}} = (0, ..., 0, 1, 0, ..., 0), \]

  \( i_0 = (x_1, ..., x_n) \). Put \( m_l = 2 - i + c_l, l = 2, ..., n - s - 1, m_{n-s} = 2 \) and

  \[
t = \max \left\{ 2, \left[ \sum_{i=1}^{n-s-1} m_l + (n - s - 2) - 1 \right] / (n - s - 2) \right\}.
\]

  We have

  \[
t + i = \max \left\{ 2, \left[ \sum_{i=1}^{n-s-1} m_l + n - s - 3 \right] / (n - s - 2) \right\} + i \leq \max \left\{ 2 + i, \left[ \sum_{i=1}^{n-s-1} m_l + (n - s - 2) + n - s - 3 \right] / (n - s - 2) \right\} \leq \max \left\{ 2 + i, \left[ 3(n - s - 2) + 2 \right] / (n - s - 2) \right\}.
\]

  \( s = 0 \) or \( n \geq 6 \), we have

  \( t \leq 3 - i \).

By Lemma 2.3 we can find \( t \) \((q - 3)\)-planes, say \( G_1, ..., G_t \) avoiding \( P_{i_0} \) such that for every point \( P_l, l = 1, ..., q - 1 \), there are \( m_l \) \((q - 3)\)-planes of \( G_1, ..., G_t \) passing through. With \( j = 1, ..., t \) we find a hyperplane \( L_j \) containing \( G_j \) and avoiding \( P_{i_0} \). Therefore

\[
L_1 \cdots L_t \in \varphi_{i_2}^{m_2} \cap \cdots \cap \varphi_{i_{n-s-1}}^{m_{n-s-1}} \cap \varphi_{i_{n-s}}^{2}.
\]

So, \( HHL_1 \cdots L_t M \in J \) for every monomial \( M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1 \). By Lemma 2.2 we have

\[
\text{reg}(R/(J + \varphi_i^2)) \leq 4 + i \leq 5 \leq T_Z.
\]

\( s = 1 \) and \( n = 5 \). Then hyperplane \( H \) pass through eight points of \( X \) and there are four points \( P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4} \) lying on a \( 2\)-plane, say \( \gamma_{1} \setminus H \). According to Case 2.2.2.2 we have proved it.

b) If \( |Q_1| = n - 1 \), then \( W = \{P_4, ..., P_{2n+2}\} \) lie on the general position in \( H_1 \). We call \( H \) a hyperplane containing \( l \) and passing through \( n - 3 + u \) points of \( W \cap H_1 \). We have \( u \geq 1 \).

- If \( u = 1 \), then consider \( n + 1 \) points of \( H_1 \setminus H \). Without loss of generality, assume that \( P_{n+2}, ..., P_{2n+2} \in H \). Put \( V = \{P_{n+2}, ..., P_{2n+2}\} \). Since there do not exist \( n \) points of \( V \) lying on a \((n - 2)\)-plane. Choose \( P_{n+2} = P_u = \)
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We have

\[ t = \max \left\{ 2, \left[ \sum_{i=n+3}^{2n+2} m_i + (n-1) - 1 \right]/(n-1) \right\}. \]

We have

\[ t + i = \max\{2, \left[ \sum_{i=n+3}^{2n+2} m_i + n - 2 \right]/(n-1) \} + i \leq \max\{2 + i, \left[ \sum_{i=n+3}^{2n+2} m_i + (n-1)i + n - 2 \right]/(n-1) \} \leq \max\{2 + i, (3n-1)/(n-1) \} \leq 3. \]

Therefore

\[ t \leq 3 - i. \]

By Lemma 2.3 we can find \( t(n-2) \)-planes \( G_1, \ldots, G_t \) avoiding \( P_{i_0} \) such that for every \( P_i, i = n + 3, \ldots, 2n + 2 \), there are \( m_l \) \((n-2)\)-planes of \( G_1, \ldots, G_t \) passing through. With \( j = 1, \ldots, t \) we find a hyperplane \( L_j \) containing \( G_j \) and avoiding \( P_{i_0} \). Therefore

\[ L_1 \cdots L_t \in \varphi_{n+3}^{m_{n+3}} \cap \cdots \cap \varphi_{2n+1}^{m_{2n+1}} \cap \varphi_2^{2}. \]

Moreover, since \( HH \in \varphi_2^{2} \cap \cdots \cap \varphi_2^{2} \) and \( M \in \varphi_{n+3}^{i-c_n} \cap \cdots \cap \varphi_{2n+1}^{i-c_n-1} \) then

\[ H_1H_1L_1 \cdots L_tM \in J. \]

By Lemma 2.2 we have

\[ \text{reg}(R/(J + \varphi_{i_0}^{2})) \leq 2 + (3 - i) + i \leq T_Z. \]

- If \( u \geq 2 \), then there are \( n+2-u \) points, assume that \( P_1, \ldots, P_{n+2-u} \in H_1 \setminus H \). Since \( u \geq 2 \) then \( n+2-u \leq n \). Moreover, since \( P_1, \ldots, P_{n+2-u} \) lie on the general position in \( H_1 \), then we have a \((n-u)\)-plane, say \( \pi \), passing through \( n+1-u \) points \( P_{i_1}, \ldots, P_{n+2-u} \) and avoiding \( P_1 \). Choose \( P_i = P_{i_0} = (1, 0, \ldots, 0) \), then \( \varphi_{i_0} = (x_1, \ldots, x_n) \). Since \( P_{i_0} \notin \pi \), we always have a hyperplane, say \( L \), containing \( \pi \) and avoiding \( P_{i_0} \). We have \( HHLL \in J \), therefore \( HHLLM \in J \) for every monomial \( M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1 \). By Lemma 2.2 we have

\[ \text{reg}(R/(J + \varphi_{i_0}^{2})) \leq 4 + i \leq 5 \leq T_Z. \]

The proof of proposition 3.1 is completed.

From Lemma 2.4, Lemma 2.5 and Proposition 3.1, we get the following remark.
Remark 3.1. Let $X = \{P_1, \ldots, P_{2n+2}\}$ be a non-degenerate set of $2n + 2$ distinct points that do not exist $n + 1$ points of $X$ lying on a $(n-2)$-plane in $\mathbb{P}^n$. Let $Y = \{P_1, \ldots, P_s\}, 2 \leq s \leq 2n + 1$, be a subset of $X$. Let $\wp_i$ be the homogeneous prime ideal corresponding $P_i, i = 1, \ldots, 2n + 1$, and

\[ Z = 2P_1 + \cdots + 2P_{2n+2}. \]

Put

\[ T_j = \max \left\{ \left[ \frac{1}{j}(2q + j - 2) \right] \mid P_{1s}, \ldots, P_{s} \text{ lie on a } j\text{-plane} \right\}, \]

\[ T_Z = \max \{ T_j \mid j = 1, \ldots, n \}. \]

Then, there exists a point $P_{i_0} \in Y$ such that

\[ \text{reg}(R/(J + \wp_{i_0}^2)) \leq T_Z, \]

where

\[ J = \bigcap_{P_i \in Y \setminus \{P_{i_0}\}} \wp_i^2. \]

The theorem below is the main result of this paper.

Theorem 3.2. Let $X = \{P_1, \ldots, P_{2n+2}\}$ be a non-degenerate set of $2n + 2$ distinct points that do not exist $n + 1$ points of $X$ lying on a $(n-2)$-plane in $\mathbb{P}^n$. Let

\[ Z = 2P_1 + \cdots + 2P_{2n+2}. \]

Then

\[ \text{reg}(Z) \leq \max \{ T_j \mid j = 1, \ldots, n \} = T_Z, \]

where

\[ T_j = \left\{ \left[ \frac{2q + j - 2}{j} \right] \mid P_{1s}, \ldots, P_{s} \text{ lie on a } j\text{-plane} \right\}. \]

Proof. Firstly, we have the following claim:

Let $X = \{P_1, \ldots, P_{2n+2}\}$ in $\mathbb{P}^n, Y = \{P_1, \ldots, P_s\}$ be a subset of $X$, $1 \leq s \leq 2n + 1$. Then

\[ \text{reg}(R/J_s) \leq T_Z, \]

where

\[ J_s = \bigcap_{P_i \in Y} \wp_i^2. \]

We will prove this claim by induction on number points of $Y$. If $s = 1$. Let $\wp_1$ be the defining homogeneous prime ideal of $P_1$. Put $J_1 = \wp_1^2, A = R/J_1$. Then,

\[ \text{reg}(R/J_1) = 1 \leq T_Z. \]
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Assume that the claim is right for all subsets $Y$ of $X$, whose number points are smaller or equal $s - 1$. Let $Y = \{P_{i1}, ..., P_{is}\}$. By Remark 3.1, there exists a point $P_{i0} \in Y$ such that

(1) $\text{reg}(R/(J_{s-1} + \varphi^2_{i0})) \leq T_Z,$

where $J_{s-1} = \bigcap_{P_i \in Y \setminus \{P_{i0}\}} \varphi^2_i$. Note that, $J_{s-1}$ is the intersection of ideals containing $s - 1$ double points of $Y$. By conjecture of induction, we have

(2) $\text{reg}(R/J_{s-1}) \leq T_Z.$

By Lemma 2.1 we have

(3) $\text{reg}(R/J_s) = \left\{1, \text{reg}(R/(J_{s-1})), \text{reg}(R/(J_{s-1} + \varphi^2_{i0}))\right\}.$

From (1), (2) and (3) we have

$$\text{reg}(R/J_s) \leq T_Z.$$  

The proof of the above claim is completed.

Now, we prove Theorem 3.2. Let $X = \{P_1, ..., P_{2n+2}\}$ in $\mathbb{P}^n$, by Proposition 3.1, there exists a point $P_{i0} \in X$ such that

(4) $\text{reg}(R/(J + \varphi^2_{i0})) \leq T_Z.$

where $J = \bigcap_{P_i \in X \setminus \{P_{i0}\}} \varphi^2_i$. Note that, $J$ is the intersection of ideals containing $2n + 1$ double points of $X$. Therefore, by the above claim with $s = 2n + 1$, we have

(5) $\text{reg}(R/J) \leq T_Z.$

By Lemma 2.1 we have

(6) $\text{reg } R/I = \left\{1, \text{reg}(R/J), \text{reg}(R/(J + \varphi^2_{i0}))\right\}$

where $I = J \cap \varphi^2_{i0}$.

From (4), (5) and (6) we have

$$\text{reg}(Z) \leq T_Z.$$  

The proof of Theorem 3.2 is completed.
References


[16] Thien, P.V. and T.N. Sinh, On the regularity index of $s$ fat points not on a $(r − 1)$-space, $s \leq r + 3$, Comm. Algebra, 45 (2017), 4123–4138.

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