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THE PELL EQUATION $x^2 - (k^2 - 2)y^2 = 1$ AND THE CORRESPONDING INTEGER SEQUENCE

Arzu Akın and Ahmet Tekcan

(Bursa, Turkiye)

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Abstract. In [11], the second author considered the positive integer solutions of the Pell equation $x^2 - Dy^2 = 1$ for some specific values of D including $D = k^2 - 2$ for an integer $k \geq 2$. In this paper, we are able to give the n^{th} integer solution (x_n, y_n) of $x^2 - (k^2 - 2)y^2 = 1$ by a different method and then we set an integer sequence $W_n = pW_{n-1} - qW_{n-2}$ with parameters $p = k^2 - 2$ and $q = 1$ and derive some algebraic relations on it.

1. Preliminaries

Let p and q be non–zero integers such that $d = p^2 - 4q \neq 0$ (to exclude a degenerate case). We set the sequences U_n and V_n to be

$$
\begin{aligned}
U_n &= U_n(p, q) = pU_{n-1} - qU_{n-2},\\
V_n &= V_n(p, q) = pV_{n-1} - qV_{n-2}
\end{aligned} \tag{1.1}
$$

for $n \geq 2$ with $U_0 = 0, U_1 = 1, V_0 = 2$ and $V_1 = p$. The characteristic equation of them is $x^2 - px + q = 0$ and hence the roots of it are $\alpha = \frac{p+\sqrt{d}}{2}$ and $\beta = \frac{p-\sqrt{d}}{2}$.

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Their Binet formulas are $U_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $V_n = \alpha^n + \beta^n$. For the companion matrix $M = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix}$, one has

$$\begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} p \\ 2 \end{bmatrix}$$

for $n \geq 1$. It is easy to verify the following formal power series developments for any p and q,

$$\sum_{n=0}^{\infty} U_n x^n = \frac{x}{1-px+qx^2} \quad \text{and} \quad \sum_{n=0}^{\infty} V_n x^n = \frac{2-px}{1-px+qx^2}.$$

In (1.1), we note that

$$\begin{aligned} U_n(1,-1) &= F_n \ \text{ Fibonacci numbers (A000045 in OEIS)}, \\ V_n(1,-1) &= L_n \ \text{ Lucas numbers (A000032 in OEIS)}, \\ U_n(2,-1) &= P_n \ \text{ Pell numbers (A000129 in OEIS)}, \\ V_n(2,-1) &= Q_n \ \text{ Pell–Lucas numbers (A002203 in OEIS)}. \end{aligned}$$

(For further details see [2, 4, 6, 8, 9, 10, 12]).

2. The Pell equation $x^2 - (k^2 - 2)y^2 = 1$

In [11], the second author considered the integer solutions of the Pell equation $x^2 - Dy^2 = 1$ (for further details on Pell equations see [1, 5, 7]) for some specific values of D including $D = k^2 - 2$ for some positive integer $k \geq 2$ and proved the following theorem.

Theorem 2.1. ([11, Theorem 2.4]) *Let $k \geq 2$ be any integer and $D = k^2 - 2$.*

1. *The continued fraction expansion of $\sqrt{D}$ is*

$$\sqrt{D} = \begin{cases} [1, \overline{2}], & \text{if } k = 2 \\ [k-1; \overline{1, k-2, 1, 2k-2}], & \text{if } k > 2. \end{cases}$$

2. *$(x_1, y_1) = (k^2 - 1, k)$ is the fundamental solution. Set $\{(x_n, y_n)\}$, where*

$$\frac{x_n}{y_n} = \left[k-1; \underbrace{1, k-2, 1, 2k-2, \cdots, 1, k-2, 1, 2k-2}_{n-1 \text{ times}}, 1, k-1 \right]$$

for $n \geq 2$. Then (x_n, y_n) is a solution of $x^2 - (k^2 - 2)y^2 = 1$.

3. *The consecutive solutions* (x_n, y_n) *and* (x_{n+1}, y_{n+1}) *satisfy*

$$x_{n+1} = (k^2-1)x_n + (k^3-2k)y_n \quad \text{and} \quad y_{n+1} = kx_n + (k^2-1)y_n$$

for $n \geq 1$.

4. *The solutions* (x_n, y_n) *satisfy the following recurrence relations*

$$\begin{aligned} x_n &= (2k^2-3)(x_{n-1}+x_{n-2}) - x_{n-3}, \\ y_n &= (2k^2-3)(y_{n-1}+y_{n-2}) - y_{n-3} \end{aligned}$$

for $n \geq 4$.

In this section, we aim to give a different method (based on binomial expansion) for finding the integer solutions of $x^2-(k^2-2)y^2=1$. But we first give the following theorem which we need it.

Theorem 2.2. *Let* $k \geq 2$ *be any integer. Set* $H = \begin{bmatrix} k^2-1 & k^3-2k \\ k & k^2-1 \end{bmatrix}$. *Then the* n^{th} *power of* H *is* $H^n = \begin{bmatrix} H_{11}^n & H_{12}^n \\ H_{21}^n & H_{22}^n \end{bmatrix}$, *where*

$$\begin{aligned} H_{11}^n &= \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} (k^2-1)^{n-2i} k^i (k^3-2k)^i = H_{22}^n, \\ H_{12}^n &= \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} (k^2-1)^{n-1-2i} k^i (k^3-2k)^{i+1}, \\ H_{21}^n &= \sum_{i=0}^{\frac{n-2}{2}} \binom{n}{2i+1} (k^2-1)^{n-1-2i} k^{i+1} (k^3-2k)^i \end{aligned}$$

for even $n \geq 2$ *or*

$$\begin{aligned} H_{11}^n &= \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} (k^2-1)^{n-2i} k^i (k^3-2k)^i = H_{22}^n, \\ H_{12}^n &= \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} (k^2-1)^{n-1-2i} k^i (k^3-2k)^{i+1}, \\ H_{21}^n &= \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} (k^2-1)^{n-1-2i} k^{i+1} (k^3-2k)^i \end{aligned}$$

for odd $n \geq 1$.

Proof. We prove by induction. Let $n=2$. Then since $H_{11}^2=2k^4-4k^2+1$, $H_{12}^2=2k^5-6k^3+4k, H_{21}^2=2k^3-2k$ and $H_{22}^2=2k^4-4k^2+1$, it is true for $n=2$. Let us assume that it is satisfied for $n-2$. Then

$$\begin{aligned}
H^2\cdot H^{n-2} &= \begin{bmatrix} (k^2-1)^2+k(k^3-2k) & 2(k^2-1)(k^3-2k) \\ 2k(k^2-1) & (k^2-1)^2+k(k^3-2k) \end{bmatrix}\times \\
&\times \begin{bmatrix} H_{11}^{n-2} & H_{12}^{n-2} \\ H_{21}^{n-2} & H_{22}^{n-2} \end{bmatrix}.
\end{aligned} \tag{2.1}$$

Applying (2.1), we deduce that

$$\begin{aligned}
&[(k^2-1)^2+k(k^3-2k)]H_{11}^{n-2}+[2k(k^2-1)]H_{12}^{n-2}= \\
&=[(k^2-1)^2+k(k^3-2k)]\times \\
&\times\left[\begin{array}{c} (k^2-1)^{n-2}+\binom{n-2}{2}(k^2-1)^{n-4}k(k^3-2k)+\cdots \\ +\binom{n-2}{n-4}(k^2-1)^2k^{\frac{n-4}{2}}(k^3-2k)^{\frac{n-4}{2}} \\ +k^{\frac{n-2}{2}}(k^3-2k)^{\frac{n-2}{2}} \end{array}\right]+ \\
&+[2k(k^2-1)]\left[\begin{array}{c} \binom{n-2}{1}(k^2-1)^{n-3}(k^3-2k) \\ +\binom{n-2}{3}(k^2-1)^{n-5}k(k^3-2k)^2+\cdots \\ +\binom{n-2}{n-5}(k^2-1)^3k^{\frac{n-6}{2}}(k^3-2k)^{\frac{n-4}{2}} \\ +\binom{n-2}{n-3}(k^2-1)k^{\frac{n-4}{2}}(k^3-2k)^{\frac{n-2}{2}} \end{array}\right]= \\
&=(k^2-1)^n+\binom{n}{2}(k^2-1)^{n-2}k(k^3-2k)+\binom{n}{4}(k^2-1)^{n-4}k^2(k^3-2k)^2+ \\
&+\cdots+\binom{n}{n-4}(k^2-1)^4k^{\frac{n-4}{2}}(k^3-2k)^{\frac{n-4}{2}}+ \\
&+\binom{n}{n-2}(k^2-1)^2k^{\frac{n-2}{2}}(k^3-2k)^{\frac{n-2}{2}}+k^{\frac{n}{2}}(k^3-2k)^{\frac{n}{2}}= \\
&=\sum_{i=0}^{\frac{n}{2}}\binom{n}{2i}(k^2-1)^{n-2i}k^i(k^3-2k)^i= \\
&=H_{11}^n.
\end{aligned}$$

Similarly it can be shown that $[2(k^2-1)(k^3-2k)]H_{11}^{n-2}+[(k^2-1)^2+k(k^3-2k)]$ $H_{12}^{n-2}=H_{12}^n$, $[(k^2-1)^2+k(k^3-2k)]H_{21}^{n-2}+[2k(k^2-1)]H_{22}^{n-2}=H_{21}^n$ and $[2(k^2-1)(k^3-2k)]H_{21}^{n-2}+[(k^2-1)^2+k(k^3-2k)]H_{22}^{n-2}=H_{22}^n$ as we wanted.

The other case can be proved similarly. ■

From Theorem 2.2, we can give the following main theorem.

Theorem 2.3. *Let $k \geq 2$ be any integer. Then the set of all positive integer solutions of $x^2-(k^2-2)y^2=1$ is $\{(x_n,y_n)\}$, where*

$$x_n=\begin{cases}\sum_{i=0}^{\frac{n}{2}}\binom{n}{2i}(k^2-1)^{n-2i}k^i(k^3-2k)^i & \text{for even } n\geq 2\\[2ex] \sum_{i=0}^{\frac{n-1}{2}}\binom{n}{2i}(k^2-1)^{n-2i}k^i(k^3-2k)^i & \text{for odd } n\geq 1\end{cases}$$

and

$$y_n=\begin{cases}\sum_{i=0}^{\frac{n-2}{2}}\binom{n}{2i+1}(k^2-1)^{n-1-2i}k^{i+1}(k^3-2k)^i & \text{for even } n\geq 2\\[2ex] \sum_{i=0}^{\frac{n-1}{2}}\binom{n}{2i+1}(k^2-1)^{n-1-2i}k^{i+1}(k^3-2k)^i & \text{for odd } n\geq 1.\end{cases}$$

Proof. It can be proved by induction on n as in Theorem 2.2. ■

3. The integer sequence W_n

Even if, in the previous chapter, we had considered the solutions of the Pell equation $x^2-Dy^2=1$, where $D=k^2-2$ for an integer $k\geq 2$, we do not have such a restriction on k in this chapter. Therefore, the properties of the integer sequence can be investigated for all integers k.

Now we set the integer sequence $W=W_n(k)$ as $W_0=0, W_1=1$ and

$$W_n=pW_{n-1}-qW_{n-2} \tag{3.1}$$

for $n\geq 2$, where $p=k^2-2$ and $q=1$. Here one can easily notice the followings:

1. If $k=0$, then $W_n=-2W_{n-1}-W_{n-2}$ and hence $W_n=(-1)^{n+1}n$.
2. If $k=\pm 1$, then $W_n=-W_{n-1}-W_{n-2}$ and so $W_n=1$ for $n\equiv 1 \pmod 3$; -1 for $n\equiv 2 \pmod 3$ or 0 for $n\equiv 0 \pmod 3$.
3. If $k=\pm 2$, then $W_n=2W_{n-1}-W_{n-2}$ and hence $W_n=n$.

The characteristic equation of (3.1) is $x^2-(k^2-2)x+1=0$ and hence the roots of it are $\alpha=\frac{k^2-2+\sqrt{\Delta}}{2}$ and $\beta=\frac{k^2-2-\sqrt{\Delta}}{2}$, where $\Delta=k^4-4k^2$. Hence the Binet formula for W_n is $W_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ for $k\neq 0,\pm 2$ (Note that, if $k=0,\pm 2$, then $\alpha=\beta$ and so W_n is undefined, that is, this formula can not be used).

3.1. Sums

Theorem 3.1. *For the sums of first $n-$terms of W_n, we have*

1. *if $k=0$, then $\sum\limits_{i=1}^{n} W_i = \frac{-n}{2}$ for even $n \geq 2$ or $\sum\limits_{i=1}^{n} W_i = \frac{n+1}{2}$ for odd $n \geq 1$;*
2. *if $k=\pm 1$, then $\sum\limits_{i=1}^{n} W_i = 0$ for $n \equiv 0, 2(mod\ 3)$ or $\sum\limits_{i=1}^{n} W_i = 1$ for $n \equiv 1$ $(mod\ 3)$;*
3. *if $k=\pm 2$, then $\sum\limits_{i=1}^{n} W_i = \frac{n^2+n}{2}$;*
4. *if $|k|>2$, then*
$$\sum_{i=1}^{n} W_i = \frac{W_{n+1}-W_n-1}{k^2-4}.$$

Proof. 1. Let $k=0$. Then as we said above $W_n=(-1)^{n+1}n$. So clearly, the sum of first $n-$terms of W_n is $\frac{-n}{2}$ if n is even or $\frac{n+1}{2}$ if n is odd.

2. Let $k=\pm 1$. Then $W_n=-W_{n-1}-W_{n-2}$, that is, $W_n=1$ for $n \equiv 1$ (mod 3); -1 for $n \equiv 2$(mod 3) or 0 for $n \equiv 0$(mod 3). So if $n \equiv 0, 2$(mod 3), then the sum of first $n-$terms of W_n is 0 and if $n \equiv 1$(mod 3), then the sum of first $n-$terms of W_n is 1.

3. Let $k=\pm 2$. Then $W_n=2W_{n-1}-W_{n-2}$, that is, $W_n=n$. So the sum of first $n-$terms of W_n is $\frac{n(n+1)}{2}=\frac{n^2+n}{2}$.

4. Let $|k|>2$. Notice that $W_{n+2}=(k^2-3)W_{n+1}+W_{n+1}-W_n$ and hence

$$W_{n+2}-W_{n+1}=(k^2-3)W_{n+1}-W_n. \tag{3.2}$$

Applying (3.2), we deduce that

$$\begin{aligned}
W_2-W_1 &= (k^2-3)W_1-W_0,\\
W_3-W_2 &= (k^2-3)W_2-W_1,\\
W_4-W_3 &= (k^2-3)W_3-W_2,\\
&\vdots\\
W_{n+1}-W_n &= (k^2-3)W_n-W_{n-1},\\
W_{n+2}-W_{n+1} &= (k^2-3)W_{n+1}-W_n.
\end{aligned} \tag{3.3}$$

If we sum both sides of (3.3), then we obtain

$$W_{n+2}-W_1=(k^2-4)(W_1+W_2+\cdots+W_n)-W_0+(k^2-3)W_{n+1}. \tag{3.4}$$

Since $W_0 = 0$ and $W_1 = 1$, (3.4) becomes $W_{n+2} - 1 = (k^2-4)(W_1 + W_2 + \cdots + W_n) + (k^2-3)W_{n+1}$. Taking $W_{n+2} \mapsto (k^2-2)W_{n+1} - W_n$, we conclude that $W_1 + W_2 + \cdots + W_n = \frac{W_{n+1} - W_n - 1}{k^2-4}$ as we wanted. ■

In 1876, the French mathematician François Edouard Anatole Lucas discovered an explicit formula for the Fibonacci numbers, namely,

$$F_n = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i},$$

and for the Lucas numbers,

$$L_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \left[\binom{n-i}{i} + \binom{n-1-i}{i-1} \right].$$

Similarly we can give the following theorem which can be proved as in Theorem 2.2.

Theorem 3.2. *Let W_n denote the n^{th} number. Then*

$$W_n = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \binom{n-1-i}{i} (k^2-2)^{n-1-2i}$$

for $n \geq 1$.

Also we can give the following result which can be proved similarly.

Theorem 3.3. *Let W_n denote the n^{th} number. Then for every k*

1. *the sum of $(2i-1)^{st}$ W_n numbers from 1 to n is a perfect square and is*

$$\sum_{i=1}^{n} W_{2i-1} = W_n^2,$$

2. *also*

$$\sum_{i=1}^{2n} W_i = W_n(W_n + W_{n+1}),$$

$$\sum_{i=1}^{2n+1} W_i = W_{n+1}(W_n + W_{n+1}),$$

$$\sum_{i=1}^{n} W_{2i} = W_n W_{n+1},$$

$$\sum_{i=1}^{2n} (W_i + W_{i+1}) = W_{n+1}(W_{n+1} + 2W_n + W_{n-1}),$$

$$\sum_{i=1}^{2n+1}(W_i+W_{i+1})=(W_n+W_{n+1})(W_{n+1}+W_{n+2}),$$
$$\sum_{i=0}^{2n}(W_{2i+1}+W_{2i+2})=W_{2n+1}(W_{2n+1}+W_{2n+2}).$$

3.2. Relations

Theorem 3.4. *Let W_n denote the n^{th} number.*

1. *If $k=0,\pm 2$, then $W_{2n}=2W_{2n-2}-W_{2n-4}$ and $W_{2n+1}=2W_{2n-1}-W_{2n-3}$ for $n\geq 2$.*
2. *If $k=\pm 1$, then $W_{2n}=-W_{2n-2}-W_{2n-4}$ and $W_{2n+1}=-W_{2n-1}-W_{2n-3}$ for $n\geq 2$.*
3. *If $|k|>2$, then $W_{2n}=(k^4-4k^2+2)W_{2n-2}-W_{2n-4}$ and $W_{2n+1}= =(k^4-4k^2+2)W_{2n-1}-W_{2n-3}$ for $n\geq 2$.*

Proof. We only prove 3. The others can be proved similarly. Let $|k|>2$. Then $W_{2n}=(k^2-2)W_{2n-2}-W_{2n-4}$ and hence

$$\begin{aligned}
W_{2n}&=(k^2-2)\left[(k^2-2)W_{2n-2}-W_{2n-3}\right]-W_{2n-2}=\\
&=W_{2n-2}(k^4-4k^2+3)-(k^2-2)\left[(k^2-2)W_{2n-4}-W_{2n-5}\right]=\\
&=W_{2n-2}(k^4-4k^2+3)-(k^2-2)^2W_{2n-4}+(k^2-2)W_{2n-5}=\\
&=W_{2n-2}(k^4-4k^2+2)+(k^2-2)[(k^2-2)W_{2n-4}-W_{2n-5}]+\\
&\quad+W_{2n-4}[-1-(k^2-2)^2]+(k^2-2)W_{2n-5}=\\
&=W_{2n-2}(k^4-4k^2+2)+(k^2-2)^2W_{2n-4}-(k^2-2)W_{2n-5}+\\
&\quad+W_{2n-4}[-1-(k^2-2)^2]+(k^2-2)W_{2n-5}=\\
&=(k^4-4k^2+2)W_{2n-2}-W_{2n-4}.
\end{aligned}$$

The other assertions can be proved similarly. ■

Further we can give the following result.

Theorem 3.5. *Let W_n denote the n^{th} number. Then for every k*

1. *$(W_{n+1}+W_n)(W_{n+1}-W_n)=W_{2n+1}$ for every integer $n\geq 1$,*
2. *$W_nW_{m+1}-W_{n-1}W_m=W_{n+m}$ for every positive integers n,m,*
3. *$(W_n+W_m)(W_n-W_m)=W_{n+m}W_{n-m}$ for every integers $n\geq m\geq 1$.*

4. *The product of $(n+1)^{st}$ and $(n-1)^{st}$ terms of W_n numbers and adding* 1 *is a perfect square and is*

$$\sqrt{W_{n+1}W_{n-1}+1}=W_n$$

for $n\geq 1$. *In fact,*

$$W_n=\sqrt{1+\sum_{i=1}^{n-1}W_{2i+1}}.$$

Proof. 1. Let $k\neq 0,\pm 2$. Since $W_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, we easily deduce that

$$\begin{aligned}(3.5)\qquad (W_{n+1}+W_n)(W_{n+1}-W_n) &= W_{n+1}^2-W_n^2=\\ &=\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)^2-\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2=\\ &=\frac{\alpha^{2n}(\alpha^2-1)+\beta^{2n}(\beta^2-1)}{k^4-4k^2}.\end{aligned}$$

Note that $\frac{\alpha^2-1}{k^4-4k^2}=\frac{\alpha}{\sqrt{k^4-4k^2}}$ and $\frac{\beta^2-1}{k^4-4k^2}=\frac{-\beta}{\sqrt{k^4-4k^2}}$. So (3.5) becomes

$$\begin{aligned}(W_{n+1}+W_n)(W_{n+1}-W_n) &= \frac{\alpha^{2n}(\alpha^2-1)+\beta^{2n}(\beta^2-1)}{k^4-4k^2}=\\ &=\frac{\alpha^{2n+1}-\beta^{2n+1}}{\sqrt{k^4-4k^2}}=\\ &=W_{2n+1}.\end{aligned}$$

Let $k=0$. Then $W_n=(-1)^{n+1}n$. So

$$(W_{n+1}+W_n)(W_{n+1}-W_n)=(-1)^{2n+2}(2n+1)=W_{2n+1}.$$

Similarly let $k=\pm 2$. Then since $W_n=n$, we get

$$(W_{n+1}+W_n)(W_{n+1}-W_n)=2n+1=W_{2n+1}.$$

The others can be proved similarly. ■

Theorem 3.6. *Let W_n denote the n^{th} number.*

1. (a) *If $k=0$, then $\alpha^n+\beta^n=(-1)^n2$ for $n\geq 0$.*
   (b) *If $k=\pm 1$, then $\alpha^n+\beta^n=2$ when $n\equiv 0(mod\ 3)$ or -1 otherwise.*
   (c) *If $k=\pm 2$, then $\alpha^n+\beta^n=2$ for $n\geq 0$.*

(d) *If* $|k| > 2$, *then*

$$\alpha^n + \beta^n = \begin{cases} W_{n+1} - W_{n-1} & \text{for } n \geq 1 \\ 2W_{n+1} - (k^2 - 2)W_n & \text{for } n \geq 0 \\ (k^2 - 2)W_n - 2W_{n-1} & \text{for } n \geq 1. \end{cases}$$

2. (a) *If* $k = 0$, *then* $W_{n+1} - W_{n-1} = (-1)^n 2$ *for* $n \geq 1$.

(b) *If* $k = \pm 1$, *then* $W_{n+1} - W_{n-1} = 2$ *for* $n \equiv 0(mod\ 3)$ *or* $W_{n+1} - W_{n-1} =$ $= -1$ *otherwise.*

(c) *If* $k = \pm 2$, *then* $W_{n+1} - W_{n-1} = 2$ *for* $n \geq 1$.

(d) *If* $|k| > 2$, *then*

$$W_{n+1} - W_{n-1} = \frac{\sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} (k^2 - 2)^{n-2i} (k^4 - 4k^2)^i}{2^{n-1}}$$

for even $n \geq 2$; *or*

$$W_{n+1} - W_{n-1} = \frac{\sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i} (k^2 - 2)^{n-2i} (k^4 - 4k^2)^i}{2^{n-1}}$$

for odd $n \geq 1$.

Proof. 1. (a) Let $k = 0$. Then $\alpha = \beta = -1$. So $\alpha^n + \beta^n = (-1)^n 2$ for $n \geq 0$.

1. (b) Let $k = \pm 1$. Then $\alpha = \frac{-1+i\sqrt{3}}{2}$ and $\beta = \frac{-1-i\sqrt{3}}{2}$. Hence clearly $\alpha^n + \beta^n = 2$ when $n \equiv 0(\text{mod } 3)$ or -1 otherwise.

1. (c) Let $k = \pm 2$. Then $\alpha = \beta = 1$. So $\alpha^n + \beta^n = 2$ for $n \geq 0$.

1. (d) Let $|k| > 2$. Since $W_{n+1} = (k^2 - 2)W_n - W_{n-1}$, we easily get

$$\begin{aligned} W_{n+1} - W_{n-1} &= (k^2 - 2)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - 2\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) = \\ &= \frac{(k^2 - 2)}{\sqrt{\Delta}}(\alpha^n - \beta^n) - \frac{2}{\sqrt{\Delta}}(\beta\alpha^n - \alpha\beta^n) = \\ &= \alpha^n \left(\frac{k^2 - 2 - 2\beta}{\sqrt{\Delta}}\right) + \beta^n \left(\frac{2 - k^2 + 2\alpha}{\sqrt{\Delta}}\right) = \\ &= \alpha^n + \beta^n \end{aligned}$$

since $k^2 - 2 - 2\beta = 2 - k^2 + 2\alpha = \sqrt{\Delta}$.

2. (a) Let $k = 0$. If n is even, then $W_{n+1} - W_{n-1} = n + 1 - (n - 1) = 2$ and if n is odd, then $W_{n+1} - W_{n-1} = -(n + 1) - (1 - n) = -2$. So in both cases, $W_{n+1} - W_{n-1} = (-1)^n 2$ for $n \geq 1$.

2. (b) Let $k = \pm 1$. Then $W_{n+1} = -1$ if $n \equiv 1(\text{mod } 3)$, $W_{n+1} = 1$ if $n \equiv 0(\text{mod } 3)$, $W_{n+1} = 0$ if $n \equiv 2(\text{mod } 3)$ and $W_{n-1} = -1$ if $n \equiv 0(\text{mod } 3)$, $W_{n-1} = 1$ if $n \equiv 2(\text{mod } 3)$, $W_{n-1} = 0$ if $n \equiv 1(\text{mod } 3)$. So $W_{n+1} - W_{n-1} = 2$ if $n \equiv 0(\text{mod } 3)$ or -1 otherwise.

2. (c) Let $k = \pm 2$. Then $W_{n+1} = n + 1$ and $W_{n-1} = n - 1$. So $W_{n+1} - W_{n-1} = n + 1 - (n - 1) = 2$ for $n \geq 1$.

2. (d) Let $|k| > 2$. Then by binomial series expansion, we easily get the desired result. ■

3.3. Greatest common divisor

Theorem 3.7. *Let W_n denote the n^{th} number. Then*

1. *Any two consecutive W_n numbers are relatively prime, that is,*

$$(W_n, W_{n-1}) = 1$$

for every k.

(a) *If $k = 0$, then*

$$(W_n, W_m) = \begin{cases} W_{(n,m)} & \textit{for odd } m \geq 1 \\ (-1)^{n+1} W_{(n,m)} & \textit{for even } m \geq 2. \end{cases}$$

(b) *Let $k = \pm 1$. If $m \equiv 1, 2(\textit{mod } 3)$, then*

$$(W_n, W_m) = 1$$

and if $m \equiv 0(\textit{mod } 3)$, then

$$(W_n, W_m) = \begin{cases} 0 & n \equiv 0(\textit{mod } 3) \\ 1 & \textit{otherwise.} \end{cases}$$

(c) *If $|k| \geq 2$, then*

$$(W_n, W_m) = W_{(n,m)}$$

for every integer $m \geq 1$.

2. *If $m \geq 1$ is odd, then*

$$(W_{nm-1}, W_{nm+1}) = \begin{cases} 1 & \text{for even } n \geq 2 \\ |k^2-2| & \text{for odd } n \geq 1 \end{cases}$$

and if $m \geq 2$ is even, then

$$(W_{nm-1}, W_{nm+1}) = 1$$

for every k.

3. (a) *If $k = \pm 1$, then*

$$(W_n, W_{n+p}) = \begin{cases} 0 & \text{if } n = 3t \\ 1 & \text{otherwise} \end{cases}$$

for prime $p = 3$ or

$$(W_n, W_{n+p}) = |W_p|$$

for other primes $p \geq 5$.

(b) *For other values of k,*

$$(W_n, W_{n+p}) = \begin{cases} W_p & \text{if } n = pt \\ 1 & \text{otherwise} \end{cases}$$

for every primes $p \geq 3$ and every integers $t \geq 1$.

4. (a) *If $k = 0$, then*

$$\frac{W_{mn}}{W_n} = m$$

for odd $m \geq 1$, or

$$\frac{W_{mn}}{W_n} = (-1)^n m$$

for even $m \geq 2$.

(b) *Let $k = \pm 1$. If $m \equiv 1 (mod\ 3)$, then*

$$\frac{W_{mn}}{W_n} = \begin{cases} 1 & \text{for } n \equiv 1, 2 (mod\ 3) \\ \text{undefined} & \text{for } n \equiv 0 (mod\ 3), \end{cases}$$

if $m \equiv 2 (mod\ 3)$, then

$$\frac{W_{mn}}{W_n} = \begin{cases} -1 & \text{for } n \equiv 1, 2 (mod\ 3) \\ \text{undefined} & \text{for } n \equiv 0 (mod\ 3), \end{cases}$$

and if $m \equiv 0 (mod\ 3)$, then

$$\frac{W_{mn}}{W_n} = \text{undefined}$$

for every $n \geq 1$.

(c) *If* $k = \pm 2$, *then*

$$\frac{W_{mn}}{W_n} = m$$

for every integer $m \geq 1$.

(d) *If* $|k| > 2$, *then*

$$W_n | W_{mn}$$

for every positive integer $m \geq 1$.

5. (a) *If* $k = 0$, *then*

$$\frac{W_{mn}}{W_m W_n} = 1$$

for odd $m \geq 1$, *or*

$$\frac{W_{mn}}{W_m W_n} = (-1)^{n+1}$$

for even $m \geq 2$.

(b) *Let* $k = \pm 1$. *If* $m \equiv 1, 2(mod\ 3)$, *then*

$$\frac{W_{mn}}{W_m W_n} = \begin{cases} 1 & \text{for } n \equiv 1, 2(mod\ 3) \\ \text{undefined} & \text{for } n \equiv 0(mod\ 3) \end{cases}$$

and if $m \equiv 0(mod\ 3)$, *then*

$$\frac{W_{mn}}{W_m W_n} = \text{undefined}$$

for every $n \geq 1$.

(c) *If* $k = \pm 2$, *then*

$$\frac{W_{mn}}{W_m W_n} = 1$$

for every integer $m \geq 1$.

(d) *If* $|k| > 2$, *then*

$$W_m W_n \mid W_{mn} W_{(m,n)}$$

for every integer $m \geq 1$.

Proof. 1. Applying the Euclidean algorithm and the relation $W_n = (k^2 - -2)W_{n-1} - W_{n-2}$, we get

$$\begin{aligned} W_n &= (k^2 - 2)W_{n-1} + (-W_{n-1} + W_{n-1}) - W_{n-2}, \\ W_{n-1} &= (W_{n-1} - W_{n-2}) \times 1 + W_{n-2}, \\ (W_{n-1} - W_{n-2}) &= (k^2 - 4)W_{n-2} + (W_{n-2} - W_{n-3}), \\ W_{n-2} &= (W_{n-2} - W_{n-3}) \times 1 + W_{n-3}, \end{aligned}$$

$$(W_{n-2}-W_{n-3})=(k^2-2)W_{n-3}+(W_{n-3}-W_{n-4}),$$
$$\vdots$$
$$W_2=(k^2-3)W_1+(W_1-W_0),$$
$$(k^2-3)=1\times(k^2-3)+0.$$

Since $W_1=1$ and $W_0=0$ we conclude that $(W_n,W_{n-1})=W_1=1$.

The others can be proved similarly. ■

3.4. Matrices

Theorem 3.8. *Let*

$$M=\begin{bmatrix} k^2-2 & -1 \\ 1 & 0 \end{bmatrix}, N=\begin{bmatrix} k^2-2 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } S=\begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(M is the companion matrix for W_n). Then for every k we have

1. $M^n=\begin{bmatrix} W_{n+1} & -W_n \\ W_n & -W_{n-1} \end{bmatrix}$ *for* $n\geq 1$,
2. $W_{n+1}=SM^nS^t$ *for* $n\geq 0$,
3. $W_n=SM^{n-2}NS^t$ *for* $n\geq 2$,
4. $M^{n-1}N=\begin{bmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{bmatrix}$ *and* $\det(M^{n-1}N)=-1$ *for* $n\geq 1$.

Proof. 1. We prove it by induction on n. Let $n=1$. Then

$$M^1=\begin{bmatrix} W_2 & -W_1 \\ W_1 & -W_0 \end{bmatrix}=\begin{bmatrix} k^2-2 & -1 \\ 1 & 0 \end{bmatrix}.$$

So it is true for $n=1$. Let us assume that this relation is satisfied for $n-1$. Since $M^n=M^{n-1}\cdot M$, we get

$$\begin{aligned}\begin{bmatrix} W_n & -W_{n-1} \\ W_{n-1} & -W_{n-2} \end{bmatrix}\begin{bmatrix} k^2-2 & -1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} (k^2-2)W_n-W_{n-1} & -W_n \\ (k^2-2)W_{n-1}-W_{n-2} & -W_{n-1} \end{bmatrix}= \\ &= \begin{bmatrix} W_{n+1} & -W_n \\ W_n & -W_{n-1} \end{bmatrix}= \\ &= M^n.\end{aligned}$$

2. It is easily seen that

$$SM^nS^t=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} W_{n+1} & -W_n \\ W_n & -W_{n-1} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} W_{n+1} \\ W_n \end{bmatrix}=W_{n+1}.$$

3.

$$
\begin{aligned}
SM^{n-2}NS^{t} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} W_{n-1} & -W_{n-2} \\ W_{n-2} & -W_{n-3} \end{bmatrix} \begin{bmatrix} k^2-2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} W_{n-1} & -W_{n-2} \\ W_{n-2} & -W_{n-3} \end{bmatrix} \begin{bmatrix} k^2-2 \\ 1 \end{bmatrix} = \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} (k^2-2)W_{n-1} - W_{n-2} \\ (k^2-2)W_{n-2} - W_{n-3} \end{bmatrix} = \\
&= (k^2-2)W_{n-1} - W_{n-2} = \\
&= W_n.
\end{aligned}
$$

4. Finally,

$$
\begin{aligned}
M^{n-1}N &= \begin{bmatrix} W_n & -W_{n-1} \\ W_{n-1} & -W_{n-2} \end{bmatrix} \begin{bmatrix} k^2-2 & 1 \\ 1 & 0 \end{bmatrix} = \\
&= \begin{bmatrix} (k^2-2)W_n - W_{n-1} & W_n \\ (k^2-2)W_{n-1} - W_{n-2} & W_{n-1} \end{bmatrix} = \\
&= \begin{bmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{bmatrix},
\end{aligned}
$$

and $\det(M^{n-1}N) = \det(M^{n-1})\det(N) = (1)^{n-1}(-1) = -1$. This completes the proof. ■

3.5. Continued fraction expansion

Theorem 3.9. *Let W_n denote the n^{th} number.*

1. *If $k = 0$, then*

$$\frac{W_{n+1}}{W_n} = [-2; 1, n-1] \quad \text{for } n \geq 3,$$

$$\frac{W_{2n+1}}{W_{2n-1}} = [1; n-1, 2] \text{ for } n \geq 2,$$

$$\frac{W_{2n+2}}{W_{2n}} = [1; n] \quad \text{for } n \geq 2.$$

2. *If* $k=\pm 1$, *then*

$$\frac{W_{n+1}}{W_n}=\frac{W_{2n+2}}{W_{2n}}=\begin{cases} [-1] & \text{if } n\equiv 1 (\text{mod } 3)\\ [0] & \text{if } n\equiv 2 (\text{mod } 3)\\ \text{undefined} & \text{if } n\equiv 0 (\text{mod } 3)\end{cases},$$

$$\frac{W_{2n+1}}{W_{2n-1}}=\begin{cases} [0] & \text{if } n\equiv 1 (\text{mod } 3)\\ \text{undefined} & \text{if } n\equiv 2 (\text{mod } 3)\\ [-1] & \text{if } n\equiv 0 (\text{mod } 3)\end{cases}$$

for $n\geq 1$.

3. *If* $k=\pm 2$, *then*

$$\frac{W_{n+1}}{W_n}=\frac{W_{2n+2}}{W_{2n}}=[1;n] \quad \text{and} \quad \frac{W_{2n+1}}{W_{2n-1}}=[1;n-1,2]$$

for $n\geq 2$.

4. *If* $|k|>2$, *then*

$$\frac{W_{n+1}}{W_n}=[k^2-2;\underbrace{-k^2+2,k^2-2}_{(n-2)/2 \text{ times}},-k^2+2]$$

for even $n\geq 2$, *or*

$$\frac{W_{n+1}}{W_n}=[k^2-2;\underbrace{-k^2+2,k^2-2}_{(n-1)/2 \text{ times}}]$$

for odd $n\geq 3$. *Also*

$$\frac{W_{2n+1}}{W_{2n-1}}=[k^4-4k^2+2;\underbrace{-k^4+4k^2-2,k^4-4k^2+2}_{(n-2)/2 \text{ times}},-k^4+4k^2-3]$$

for even $n\geq 2$, *or*

$$\frac{W_{2n+1}}{W_{2n-1}}=[k^4-4k^2+2;\underbrace{-k^4+4k^2-2,k^4-4k^2+2}_{(n-3)/2 \text{ times}},-k^4+4k^2-2,k^4-4k^2+3]$$

for odd $n\geq 3$, *and finally*

$$\frac{W_{2n+2}}{W_{2n}}=[k^4-4k^2+2;\underbrace{-k^4+4k^2-2,k^4-4k^2+2}_{(n-2)/2 \text{ times}},-k^4+4k^2-2]$$

for even $n \geq 2$, *or*

$$\frac{W_{2n+2}}{W_{2n}} = [k^4 - 4k^2 + 2; \underbrace{-k^4 + 4k^2 - 2, k^4 - 4k^2 + 2}_{(n-1)/2 \text{ times}}]$$

for odd $n \geq 3$.

Proof. 1. Let $k = 0$. Then $W_n = (-1)^{n+1}n$. So

$$\frac{W_{n+1}}{W_n} = \frac{(-1)^{n+2}(n+1)}{(-1)^{n+1}n} = -2 + \frac{n-1}{n} = -2 + \frac{1}{1 + \frac{1}{n-1}} = [-2; 1, n-1].$$

Similarly, we deduce that

$$\frac{W_{2n+1}}{W_{2n-1}} = \frac{2n+1}{2n-1} = 1 + \frac{2}{2n-1} = 1 + \frac{1}{n-1+\frac{1}{2}} = [1; n-1, 2]$$

and $\frac{W_{2n+2}}{W_{2n}} = \frac{2n+2}{2n} = 1 + \frac{1}{n} = [1; n]$.

2. Let $k = \pm 1$, then $W_n = 1$ for $n \equiv 1 (\text{mod } 3)$; -1 for $n \equiv 2 (\text{mod } 3)$ or 0 for $n \equiv 0 (\text{mod } 3)$. Hence the result is clear.

3. Let $k = \pm 2$. Then $W_n = n$. So clearly,

$$\frac{W_{n+1}}{W_n} = \frac{n+1}{n} = 1 + \frac{1}{n} = [1; n]$$
$$\frac{W_{2n+1}}{W_{2n-1}} = \frac{2n+1}{2n-1} = 1 + \frac{1}{n-1+\frac{1}{2}} = [1; n-1, 2]$$
$$\frac{W_{2n+2}}{W_{2n}} = \frac{2n+2}{2n} = \frac{n+1}{n} = 1 + \frac{1}{n} = [1; n].$$

4. Let $|k| > 2$. We prove it by induction on n. Let $n = 2$. Then $W_2 = k^2 - 2$ and $W_3 = k^4 - 4k^2 + 3$ and hence

$$\frac{W_3}{W_2} = \frac{k^4 - 4k^2 + 3}{k^2 - 2} = k^2 - 2 + \frac{1}{-k^2 + 2} = [k^2 - 2; -k^2 + 2].$$

So it is true for $n = 2$. Let us assume that it is true for $n - 2$, that is,

$$\frac{W_{n-1}}{W_{n-2}} = [k^2 - 2; \underbrace{-k^2 + 2, k^2 - 2}_{(n-4)/2 \text{ times}}, -k^2 + 2].$$

Then

$$[k^2 - 2; \underbrace{-k^2 + 2, k^2 - 2}_{(n-2)/2 \text{ times}}, -k^2 + 2] = k^2 - 2 + \cfrac{1}{-k^2 + 2 + \cfrac{1}{k^2 - 2 + \cfrac{1}{\ddots \; k^2 - 2 + \frac{1}{-k^2+2}}}}$$

$$= k^2 - 2 + \frac{1}{-k^2 + 2 + \frac{1}{\frac{W_{n-1}}{W_{n-2}}}} =$$
$$= \frac{(k^4 - 4k^2 + 3)W_{n-1} - (k^2 - 2)W_{n-2}}{(k^2 - 2)W_{n-1} - W_{n-2}} =$$
$$= \frac{(k^2 - 2)[(k^2 - 2)W_{n-1} - W_{n-2}] - W_{n-1}}{(k^2 - 2)W_{n-1} - W_{n-2}} =$$
$$= \frac{(k^2 - 2)W_n - W_{n-1}}{(k^2 - 2)W_{n-1} - W_{n-2}} =$$
$$= \frac{W_{n+1}}{W_n}.$$

The other cases can be proved similarly. ■

3.6. Cross-ratio

Theorem 3.10. *Let $[W_n, W_{n+1}; W_{n+2}, W_{n+3}]$ denote the cross–ratio of four consecutive W_n numbers.*

1. *If $k = 0$, then*
$$[W_n, W_{n+1}; W_{n+2}, W_{n+3}] = \frac{4}{(2n+3)^2}.$$

2. *If $k = \pm 1$, then*
$$[W_n, W_{n+1}; W_{n+2}, W_{n+3}] = \begin{cases} \infty & \text{for } n \equiv 0 (mod\ 3) \\ -\infty & \text{for } n \equiv 1, 2 (mod\ 3). \end{cases}$$

3. *If $k = \pm 2$, then*
$$[W_n, W_{n+1}; W_{n+2}, W_{n+3}] = \frac{4}{3}.$$

4. *If $|k| > 2$, then*
$$\lim_{n \to \infty} [W_n, W_{n+1}; W_{n+2}, W_{n+3}] = \frac{k^2}{k^2 - 1}.$$

Proof. Recall that the cross–ratio of four different real numbers a, b, c, d is denoted by $[a, b; c, d]$ and is defined to be

$$[a, b; c, d] = \frac{(a - c)(b - d)}{(b - c)(a - d)}. \tag{3.6}$$

1. Let $k=0$. Then since $W_n=(-1)^{n+1}n$, we have

$$\begin{aligned}
[W_n,W_{n+1};W_{n+2},W_{n+3}] &= \frac{(W_n-W_{n+2})(W_{n+1}-W_{n+3})}{(W_{n+1}-W_{n+2})(W_n-W_{n+3})}= \\
&= \frac{[(-1)^{n+1}n-(-1)^{n+3}(n+2)][(-1)^{n+2}(n+1)-(-1)^{n+4}(n+3)]}{[(-1)^{n+2}(n+1)-(-1)^{n+3}(n+2)][(-1)^{n+1}n-(-1)^{n+4}(n+3)]}= \\
&= \frac{4}{(2n+3)^2}.
\end{aligned}$$

2. Let $k=\pm 1$. Then $W_n=1$ if $n\equiv 1(\text{mod } 3)$, -1 if $n\equiv 2(\text{mod } 3)$ or 0 if $n\equiv 0(\text{mod } 3)$. So $W_n-W_{n+2}=1$ if $n\equiv 0,1(\text{mod } 3)$ or -2 if $n\equiv 2(\text{mod } 3)$; $W_{n+1}-W_{n+3}=1$ if $n\equiv 0,2(\text{mod } 3)$ or -2 if $n\equiv 1(\text{mod } 3)$; $W_{n+1}-W_{n+2}=-1$ if $n\equiv 1,2(\text{mod } 3)$ or 2 if $n\equiv 0(\text{mod } 3)$ and $W_n-W_{n+3}=0$ for every n. Hence $(W_n-W_{n+2})(W_{n+1}-W_{n+3})=1$ if $n\equiv 0(\text{mod } 3)$ or -2 if $n\equiv 1,2(\text{mod } 3)$. So the result is obvious.

3. Let $k=\pm 2$. Then $W_n=n$ and hence

$$[W_n,W_{n+1};W_{n+2},W_{n+3}]=\frac{[n-(n+2)][n+1-(n+3)]}{[n+1-(n+2)][n-(n+3)]}=\frac{4}{3}.$$

4. Let $|k|>2$. Then $W_n=(k^2-2)W_{n-1}-W_{n-2}$. So $W_{n+2}=(k^2-2)W_{n+1}-W_n$ and $W_{n+3}=(k^4-4k^2+3)W_{n+1}-(k^2-2)W_n$. Hence

$$\begin{aligned}
W_n-W_{n+2} &= -(k^2-2)W_{n+1}+2W_n,\\
W_{n+1}-W_{n+3} &= (-k^4+4k^2-2)W_{n+1}+(k^2-2)W_n,\\
W_{n+1}-W_{n+2} &= (-k^2+3)W_{n+1}+W_n,\\
W_n-W_{n+3} &= -(k^4-4k^2+3)W_{n+1}+(k^2-1)W_n.
\end{aligned}$$

So (3.6) becomes

$$\begin{aligned}
&[W_n,W_{n+1};W_{n+2},W_{n+3}]\\
&\quad=\frac{[-(k^2-2)W_{n+1}+2W_n][(-k^4+4k^2-2)W_{n+1}+(k^2-2)W_n]}{[(-k^2+3)W_{n+1}+W_n][-(k^4-4k^2+3)W_{n+1}+(k^2-1)W_n]}.
\end{aligned}\tag{3.7}$$

Taking the limit of both sides of (3.7), we deduce that

$$\lim_{n\to\infty}[W_n,W_{n+1};W_{n+2},W_{n+3}]=\frac{k^2}{k^2-1}$$

since $W_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$.

3.7. Circulant matrix and spectral norm

Theorem 3.11. *Let $|k| > 2$ and let W denote the circulant matrix of W_n numbers. Then*

1. *The eigenvalues of W are*

$$\lambda_j(W) = \frac{(W_{n-1}+1)w^{-j} - W_n}{w^{-2j} - (k^2-2)w^{-j} + 1}$$

for $j = 0, 1, 2, \cdots, n-1$.

2. *The spectral norm of W is*

$$||W||_{spec} = \frac{W_n - W_{n-1} - 1}{k^2 - 4}$$

for $n \geq 1$.

Proof. 1. Recall that a circulant matrix (see [3]) is a matrix M defined as

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-2} & m_{n-1} \\ m_{n-1} & m_0 & m_1 & \cdots & m_{n-3} & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & \cdots & m_{n-4} & m_{n-3} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ m_2 & m_3 & m_4 & \cdots & m_0 & m_1 \\ m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_0 \end{bmatrix},$$

where m_i are constant. In this case, the eigenvalues of M are

$$\lambda_j(M) = \sum_{u=0}^{n-1} m_u w^{-ju}, \tag{3.8}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$. Since $W_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, we get from (3.8) that

$$\begin{aligned} \lambda_j(W) = \sum_{u=0}^{n-1} W_u w^{-ju} &= \\ = \sum_{u=0}^{n-1} \left(\frac{\alpha^u - \beta^u}{\alpha - \beta}\right) w^{-ju} &= \\ = \frac{1}{\alpha - \beta} \left[\sum_{u=0}^{n-1} (\alpha w^{-j})^u - \sum_{u=0}^{n-1} (\beta w^{-j})^u\right] &= \end{aligned}$$

$$= \frac{1}{\alpha-\beta}\left[\frac{\alpha^n-1}{\alpha w^{-j}-1} - \frac{\beta^n-1}{\beta w^{-j}-1}\right] =$$
$$= \frac{1}{\alpha-\beta}\left[\frac{w^{-j}(\alpha^n\beta-\beta-\beta^n\alpha+\alpha)-\alpha^n+\beta^n}{w^{-2j}-(k^2-2)w^{-j}+1}\right] =$$
$$= \frac{w^{-j}\left[(\alpha\beta)\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)+\frac{\alpha-\beta}{\alpha-\beta}\right]-\frac{\alpha^n-\beta^n}{\alpha-\beta}}{w^{-2j}-(k^2-2)w^{-j}+1} =$$
$$= \frac{(W_{n-1}+1)w^{-j}-W_n}{w^{-2j}-(k^2-2)w^{-j}+1}$$

as we claimed.

2. The spectral norm of a matrix $M = [m_{ij}]_{n\times n}$ is defined to be

$$||M||_{spec} = \max_{0\le j\le n-1}\{\sqrt{\lambda_j}\},$$

where λ_j are the eigenvalues of M^*M and M^* denotes the conjugate transpose of M. For the circulant matrix for W_n numbers, we have

$$W^*W = \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1(n-1)} & W_{1n} \\ W_{21} & W_{22} & \cdots & W_{2(n-1)} & W_{2n} \\ . & . & & . & . \\ . & . & \cdots & . & . \\ . & . & & . & . \\ W_{(n-1)1} & W_{(n-1)2} & \cdots & W_{(n-1)(n-1)} & W_{(n-1)n} \\ W_{n1} & W_{n2} & \cdots & W_{n(n-1)} & W_{nn} \end{bmatrix},$$

where

$$\begin{aligned}
W_{11} &= W_0^2 + W_{n-1}^2 + \cdots + W_2^2 + W_1^2, \\
W_{12} &= W_0W_1 + W_{n-1}W_0 + \cdots + W_2W_3 + W_1W_2, \\
&\cdots \\
W_{1(n-1)} &= W_0W_{n-2} + W_{n-1}W_{n-3} + \cdots + W_2W_0 + W_1W_{n-1}, \\
W_{1n} &= W_0W_{n-1} + W_{n-1}W_{n-2} + \cdots + W_2W_1 + W_1W_0, \\
W_{21} &= W_1W_0 + W_0W_{n-1} + \cdots + W_3W_2 + W_2W_1, \\
W_{22} &= W_1^2 + W_0^2 + \cdots + W_3^2 + W_2^2, \\
&\cdots \\
W_{2(n-1)} &= W_1W_{n-2} + W_0W_{n-3} + \cdots + W_3W_0 + W_2W_{n-1}, \\
W_{2n} &= W_1W_{n-1} + W_0W_{n-2} + \cdots + W_3W_1 + W_2W_0, \\
&\cdots \\
W_{(n-1)1} &= W_{n-2}W_0 + W_{n-3}W_{n-1} + \cdots + W_0W_2 + W_{n-1}W_1,
\end{aligned}$$

$$\begin{aligned}
W_{(n-1)2} &= W_{n-2}W_1 + W_{n-3}W_0 + \cdots + W_0W_3 + W_{n-1}W_2, \\
&\cdots \\
W_{(n-1)(n-1)} &= W_{n-2}^2 + W_{n-3}^2 + \cdots + W_0^2 + W_{n-1}^2, \\
W_{(n-1)n} &= W_{n-2}W_{n-1} + W_{n-3}W_{n-2} + \cdots + W_0W_1 + W_{n-1}W_0, \\
W_{n1} &= W_{n-1}W_0 + W_{n-2}W_{n-1} + \cdots + W_1W_2 + W_0W_1, \\
W_{n2} &= W_{n-1}W_1 + W_{n-2}W_0 + \cdots + W_1W_3 + W_0W_2, \\
&\cdots \\
W_{n(n-1)} &= W_{n-1}W_{n-2} + W_{n-2}W_{n-3} + \cdots + W_1W_0 + W_0W_{n-1}, \\
W_{nn} &= W_{n-1}^2 + W_{n-2}^2 + \cdots + W_1^2 + W_0^2.
\end{aligned}$$

The eigenvalues of W^*W are $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$. Here λ_0 is maximum and is

$$\begin{aligned}
\lambda_0 &= W_0^2 + W_1^2 + \cdots + W_{n-2}^2 + W_{n-1}^2 \\
&\quad + 2\left[\begin{array}{r} W_0(W_1 + W_2 + \cdots + W_{n-2} + W_{n-1}) \\ + W_1(W_2 + \cdots + W_{n-2} + W_{n-1}) \\ + \cdots \\ + W_{n-2}W_{n-1} \end{array}\right] \\
&= (W_0 + W_1 + \cdots + W_{n-1})^2.
\end{aligned}$$

Thus the spectral norm of W is hence $||W||_{spec} = \sqrt{\lambda_0} = W_0 + W_1 + \cdots + W_{n-1}$. So $||W||_{spec} = \frac{W_n - W_{n-1} - 1}{k^2 - 4}$ by Theorem 3.1. ■

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A. Akın and A. Tekcan
Uludag University
Faculty of Science
Department of Mathematics
Gorukle, Bursa–Turkiye
arzuakin504@gmail.com
tekcan@uludag.edu.tr