ON THE UNIQUENESS PROBLEM OF
NON-ARCHIMEDEAN MEROMORPHIC FUNCTIONS
AND THEIR DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we discuss the uniqueness problem for differential polynomials \((P^n(f))^{(k)}, (Q^n(g))^{(k)}\), sharing the same value, where \(P, Q\) are polynomials of Fermat-Waring type, \(f\) and \(g\) are meromorphic functions on a non-Archimedean field.

1. Introduction

Let \(K\) be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value. We denote by \(\mathcal{A}(K)\) the ring of entire functions in \(K\), by \(\mathcal{M}(K)\) the field of meromorphic functions, i.e., the field of fractions of \(\mathcal{A}(K)\), and \(\hat{K} = K \cup \{\infty\}\). We assume that the reader is familiar with the notations in the non-Archimedean Nevanlinna theory (see [18]). Let \(f\) be a non-constant meromorphic function on \(K\). For every \(a \in K\), define the function \(\nu_a^f : \hat{K} \to \mathbb{N}\) by

\[
\nu_a^f(z) = \begin{cases} 
0 & \text{if } f(z) \neq a \\
m & \text{if } f(z) = a \text{ with multiplicity } m,
\end{cases}
\]

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and set \( \nu_f^\infty = \nu_{f_0}^0 \). For \( f \in \mathcal{M}(\mathbb{K}) \) and \( S \subset \mathbb{K} \cup \{ \infty \} \), we define

\[
E_f(S) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in \mathbb{K}\}.
\]

Let \( \mathcal{F} \) be a nonempty subset of \( \mathcal{M}(\mathbb{K}) \). Two functions \( f, g \) of \( \mathcal{F} \) are said to share \( S \), counting multiplicity, if \( E_f(S) = E_g(S) \). Let \( a \subset \mathbb{K} \cup \{ \infty \} \), and \( S \subset \mathbb{K} \cup \{ \infty \} \),

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we define

\[
E_f(S) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in \mathbb{K}\}.
\]
Theorem 1. Let \( f, g \) be two non-constant meromorphic functions on \( \mathbb{K} \) and let \( P(z), Q(z) \) be defined in (1.1). Assume that \( n \geq 3k + 5, d \geq 2m + 8 \) and either \( m \geq 3 \) or \( (d, m) = 1 \) and \( m \geq 2 \). If \( (P^n(f))^{(k)} \) and \( (Q^n(g))^{(k)} \) share 1 CM, then \( g = hf \) for a constant \( h \) such that \( h^d = \frac{b_2}{b_1}, h^m = \frac{a_2}{a_1} \).

Theorem 2. Let \( f, g \) be two non-constant meromorphic functions on \( \mathbb{K} \) and let \( P(z), Q(z) \) be defined in (1.1). Assume that \( d \geq 2m + 8 \) and either \( m \geq 3 \) or \( (d, m) = 1 \) and \( m \geq 2 \). If \( P(f) \) and \( Q(g) \) share 0 CM, then \( g = hf \) for a constant \( h \) such that \( h^d = \frac{b_2}{b_1}, h^m = \frac{a_2}{a_1} \).

As immediate consequences of Theorem 2, we have

Corollary 3. Let \( S, T \) be the zero sets of the above polynomials \( P(z), Q(z) \), respectively, and let \( f, g \) be two non-constant meromorphic functions on \( \mathbb{K} \). Assume that \( d \geq 2m + 8 \) and either \( m \geq 3 \) or \( (d, m) = 1 \) and \( m \geq 2 \). If \( E_f(S) = E_g(T) \), then \( g = hf \) for a constant \( h \) such that \( h^d = \frac{b_2}{b_1}, h^m = \frac{a_2}{a_1} \).

Corollary 4. Let \( S \) be the zero sets of the polynomial \( P(z) \), and let \( f, g \) be two non-constant meromorphic functions on \( \mathbb{K} \). Assume that \( d \geq 2m + 8 \) and \( (d, m) = 1 \) and \( m \geq 2 \). If \( E_f(S) = E_g(S) \), then \( f = g \).

2. Lemmas

We assume that the reader is familiar with the notations in the non-Archimedean Nevanlinna theory (see [4], [8], [9] and [18]).

We first need the following Lemmas.

Lemma 2.1. ([18]) Let \( f \) be a non-constant meromorphic function on \( \mathbb{K} \) and let \( a_1, a_2, \ldots, a_q \) be distinct points of \( \mathbb{K} \cup \{\infty\} \). Then

\[
(q - 2)T(r, f) \leq \sum_{i=1}^{q} N_1(r, \frac{1}{f - a_i}) - \log r + O(1).
\]

Lemma 2.2. ([18]) Let \( f \) be a non-constant meromorphic function on \( \mathbb{K} \) and let \( a_1, a_2, \ldots, a_q \) be distinct points of \( \mathbb{K} \cup \{\infty\} \). Suppose either \( f - a_i \) has no zeros, or \( f - a_i \) has zeros, in which case all the zeros of the functions \( f - a_i \) have multiplicity at least \( m_i, i = 1, \ldots, q \). Then

\[
\sum_{i=1}^{q} (1 - \frac{1}{m_i}) < 2.
\]
Lemma 2.3. ([12]) Let $f$ and $g$ be non-constant meromorphic functions on $\mathbb{K}$. If $E_f(1) = E_g(1)$, then one of the following three cases holds:

1. $T(r,f) \leq N_2(r,f) + N_2(r,\frac{1}{f}) + N_2(r,g) + N_2(r,\frac{1}{g}) - \log r + O(1)$, and the same inequality holds for $T(r,g)$;
2. $fg = 1$;
3. $f = g$.

Lemma 2.4. ([12]) Let $f$ be a non-constant meromorphic function on $\mathbb{K}$ and $n,k$ be positive integers, $n > k$ and $a$ be a pole of $f$. Then

1. $(f^n)^{(k)} = \frac{\varphi_k}{(z-a)^np+k}$, where $p = \nu_f^\infty(a), \varphi_k(a) \neq 0$.
2. $\frac{f^n}{f^{n-k}} = \frac{h_k}{(z-a)^{pk+k}}$, where $p = \nu_f^{\infty}(a), h_k(a) \neq 0$.

Lemma 2.5. Let $f, (f)^{(k)}$ be non-constant meromorphic functions on $\mathbb{K}$ and $k$ be a positive integer. Then

$$T(r,(f)^{(k)}) \leq (k+1)T(r,f) + O(1).$$

Proof. By Lemma 2.4, and noting that $m(r,\frac{(f)^{(k)}}{f}) = O(1)$ we get

$$T(r,(f)^{(k)}) = m(r,(f)^{(k)}) + N(r,(f)^{(k)}) \leq n(m(r,f) + N(r,f) + kN_1(r,f) + O(1) \leq T(r,f) + kT(r,f) + O(1) = (k+1)T(r,f) + O(1).$$

Lemma 2.5 is proved. □

Lemma 2.6. ([12]) Let $f$ be a non-constant meromorphic function on $\mathbb{K}$ and $n,k$ be positive integers, $n \geq k + 1$. Then

$$T(r,f) \leq T(r,(f^n)^{(k)}) + O(1),$$

in particular, $(f^n)^{(k)}$ is not a constant.

Lemma 2.7. ([12]) Let $f$ be a non-constant meromorphic function on $\mathbb{K}$ and $n,k$ be positive integers, $n > 2k$. Then

1. $(n-2k)T(r,f) + kN(r,f) + N(r,\frac{1}{(f^n)^{(k)}}) \leq T(r,(f^n)^{(k)}) + O(1)$;
2. $N(r,\frac{1}{(f^n)^{(k)}}) \leq kT(r,f) + kN_1(r,f) + O(1)$. 

Lemma 2.8. Let $f$ be a non-constant meromorphic function on $\mathbb{K}$ and $n, k$ be positive integers, $n > 2k$, and let $P(z)$ be a polynomial of degree $d > 0$. Then

1. $(n-2k)dT(r,f) + kN(r,P(f)) + N(r, \frac{1}{((P(f))^n)^{(k)}}) \leq T(r,((P(f))^n)^{(k)}) +$ 
$+ O(1) \leq (k+1)ndT(r,f) + O(1)$.

2. $N(r, \frac{1}{((P(f))^n)^{(k)}}) \leq kdT(r,f) + kN_1(r,P(f)) + O(1) =$ 
$= kdT(r,f) + kN_1(r,f) + O(1) \leq k(d+1)T(r,f) + O(1)$.

Proof. 1. Set $A = ((P(f))^n)^{(k)}, C = P(f)$. Then $T(r,C) = T(r,P(f)) = dT(r,f) + O(1), T(r,P^n(f)) = ndT(r,f) + O(1)$. Therefore, $C, C^n$ are not constants. By Lemma 2.6 we see that $A = (C^n)^{(k)}$ is not a constant. On the other hand, by Lemma 2.7 and Lemma 2.5 we get

$(n-2k)T(r,C) + kN(r,C) + N(r, \frac{1}{C^{n-k}}) \leq T(r,A) + O(1) \leq (k+1)T(r,C^n) + O(1)$,

i.e.

$(n - 2k)dT(r,f) + kN(r,P(f)) + N(r, \frac{1}{((P(f))^n)^{(k)}}) \leq$ 
$\leq T(r,((P(f))^n)^{(k)}) + O(1) \leq (k+1)ndT(r,f) + O(1)$.

2. By Lemma 2.7 we have

$N(r, \frac{1}{A}) \leq kT(r,C) + kN_1(r,C) + O(1)$.

On the other hand,

$T(r,C) = dT(r,f) + O(1), N_1(r,C) = N_1(r,f) \leq N(r,f) \leq T(r,f) + O(1)$.

Therefore,

$N(r, \frac{1}{((P(f))^n)^{(k)}}) \leq kdT(r,f) + kN_1(r,P(f)) + O(1) =$ 
$= kdT(r,f) + kN_1(r,f) + O(1) \leq k(d+1)T(r,f) + O(1)$.

Lemma 2.8 is proved. \hfill \blacksquare
Lemma 2.9. Let $d \geq 2m+3$ and either $m \geq 3$ or $(d, m) = 1$ and $m \geq 2$, $c \neq 0$, and let $P(z), Q(z)$ be defined by (1). Assume that the equation $P(f) = cQ(g)$ has a non-constant meromorphic solution $(f, g)$. Then $g = hf$ for a constant $h$ such that $h^d = \frac{1}{c} = \frac{b_2}{b_1}$, $h^m = \frac{a_2}{a_1}$.

Proof. Since $P(f) = Q(g)$ we get

$$f^d + a_1 f^{d-m} + b_1 = c(g^d + a_2 g^{d-m} + b_2), \quad dT(r, f) + O(1) = dT(r, g),$$

(2.1) \quad $T(r, f) + O(1) = T(r, g)$.

Equation (2.1) can be rewritten as

$$f_1 + f_2 = cb_2 - b_1, \quad \text{where } f_1 = f^{d-m}(f^{m} + a_1), \quad f_2 = -cg^{d-m}(g^{m} + a_2).$$

If $cb_2 - b_1 \neq 0$, then by Lemma 2.1, we have

$$T(r, f_1) \leq N_1(r, f_1) + N_1(r, \frac{1}{f_1}) + N_1(r, \frac{1}{f_1} - (cb_2 - b_1)) - \log r + O(1),$$

$$dT(r, f) \leq N_1(r, f) + N_1(r, \frac{1}{f}) + N_1(r, \frac{1}{f^{m} + a_1}) + N_1(r, \frac{1}{g}) +$$

$$+ N_1(r, \frac{1}{g^{m} + a_2}) - \log r + O(1),$$

$$dT(r, f) \leq (2m + 3)T(r, f) - \log r + O(1), \quad (d - 2m - 3)T(r, f) \leq$$

$$\leq - \log r + O(1),$$

which contradicts to $d \geq 2m + 3$. Hence $cb_2 - b_1 = 0$. Thus, (2.1) becomes

(2.2) \quad $f^d + a_1 f^{d-m} = cg^d + ca_2 g^{d-m}$.

For simplicity, set $h = \frac{g}{f}$, and $\alpha = \frac{1}{c} \neq 0; \beta = \frac{a_1}{ca_2} \neq 0$. Then we obtain

$$f^{m}(ch^d - 1) = -(ca_2 h^{d-m} - a_1), \quad f^{m}(h^d - \alpha) = -a_2(h^{d-m} - \beta),$$

(2.3) \quad $f^{m} = -a_2 \frac{h^{d-m} - \beta}{h^d - \alpha}$.

Assume that $h$ is not a constant. Consider the following possible cases:

Case 1. $m \geq 2$, $(m, d) = 1$. If $h^d - \alpha$ and $h^{d-m} - \beta$ have no common zeros, then all zeros of $h^d - \alpha$ have multiplicities $\geq m$. Then

$$N_1(r, \frac{1}{h^d - \alpha}) \leq \frac{1}{m} N(r, \frac{1}{h^d - \alpha}).$$
By Lemma 2.1 we obtain
\[ T(r, h^d) \leq N_1(r, h^d) + N_1(r, \frac{1}{h^d - \alpha}) - \log r + O(1), \]
\[ dT(r, h) \leq 2T(r, h) + \frac{1}{m}N(r, \frac{1}{h^d - \alpha}) - \log r + O(1) \leq (2 + \frac{d}{m})T(r, h) - \log r + O(1), \]
\[ (d - 2 - \frac{d}{m})T(r, h) \leq -\log r + O(1), \]
which leads to \( d(m - 1) < 2m \), a contradiction to the condition \( d \geq 2m + 3 \).

If \( h^d - \alpha \) and \( h^{d-m} - \beta \) have common zeros, then there exists \( z_0 \) such that \( h^d(z_0) = \alpha, h^{d-m}(z_0) = \beta \). From (2.3) we get
\[ \alpha f^m \left( \frac{h}{h(z_0)} \right)^d - 1 = -\beta a_2 \left( \frac{h}{h(z_0)} \right)^{d-m} - 1. \]
Since \((m, d) = 1\), the equations \( z^d - 1 = 0 \) and \( z^{d-m} - 1 = 0 \) have different roots, except for \( z = 1 \). Let \( r_i, i = 1, \ldots, 2d - m - 2 \), be all the roots of them. Then all zeros of \( \frac{h}{h(z_0)} - r \) have multiplicities \( \geq m \). Therefore, by Lemma 2.2 we obtain
\[ (1 - \frac{1}{m})(2d - m - 2) < 2, 2d(m - 1) < m^2 + 3m - 2, \]
which contradicts \( d \geq 2m + 3, m \geq 2 \). Thus, \( h \) is a constant.

**Case 2.** \( m \geq 3 \). Note that equation \( z^d - \alpha = 0 \) has \( d \) simple zeros, equation \( z^{d-m} - \beta = 0 \) has \( d - m \) simple zeros. Then \( z^d - \alpha = 0, z^{d-m} - \beta = 0 \) have at most \( d - m \) common simple zeros. Therefore, the equation \( z^d - \alpha = 0 \) has at least \( m \) distinct roots, which are not roots of \( z^{d-m} - \beta = 0 \). Let \( r_1, r_2, \ldots, r_m \) be all these roots. Then all zeros of \( h - r_j, j = 1, \ldots, m \), have multiplicities \( \geq m \). By Lemma 2.2 we have \( m(1 - \frac{1}{m}) < 2 \). Therefore, \( m < 3 \). From \( m \geq 3 \), we obtain a contradiction. Thus \( h \) is a constant.

### 3. Proof of main results

#### 3.1. Proof of Theorem 1
We have
\[ P(f) = (f - e_1) \cdots (f - e_d), e_i \in \mathbb{K}, e_i \neq 0, \]
\[ (P(f))^n = (f - e_1)^n \cdots (f - e_d)^n, \]
\[ Q(g) = (g - k_1) \cdots (g - k_d), k_i \in \mathbb{K}, k_i \neq 0, \]
\[ (Q(g))^n = (g - k_1)^n \cdots (g - k_d)^n. \]
Set
\[ A = ((P(f))^n)^{(k)}, \quad B = ((Q(g))^n)^{(k)}, \quad C = P(f), \]
\[ D = Q(g), \quad F = \frac{A}{C^{n-k}}, \quad Q = \frac{B}{D^{n-k}}. \]

Then
\[ C = (f - e_1) \cdots (f - e_d), \quad D = (g - k_1) \cdots (g - k_d), \]
\[ A = (C^n)^{(k)} = FC^{n-k}, \quad B = (D^n)^{(k)} = QD^{n-k}. \]

Applying Lemma 2.3 to \((C^n)^{(k)}, (D^n)^{(k)}\) we have one of the following possibilities:

**Case 1.**
\[
T(r, A) \leq N_2(r, A) + N_2(r, \frac{1}{A}) + N_2(r, B) + N_2(r, \frac{1}{B}) - \log r + O(1),
\]
\[
T(r, B) \leq N_2(r, A) + N_2(r, \frac{1}{A}) + N_2(r, B) + N_2(r, \frac{1}{B}) - \log r + O(1).
\]

We see that, if \(a\) is a pole of \(A\), then \(C(a) = \infty\) with \(\nu^\infty_A(a) \geq n + k \geq 2\). Therefore,
\[
N_1(r, C) = N_1(r, (f - e_1) \cdots (f - e_d)) = N_1(r, f) \leq T(r, f) + O(1),
\]
\[
N_1(r, \frac{1}{C}) = \sum_{i=1}^d N_1(r, \frac{1}{f - e_i}) \leq dT(r, f) + O(1),
\]
\[
N_2(r, A) = 2N_1(r, C) \leq 2T(r, f) + O(1),
\]
\[
N_2(r, \frac{1}{A}) \leq N_2(r, \frac{1}{C^{n-k}}) + N(r, \frac{1}{F}) = 2N_1(r, \frac{1}{C}) + N(r, \frac{1}{F}) \leq
\]
\[
\leq 2dT(r, f) + N(r, \frac{1}{F}) \leq 2dT(r, f) + kN_1(r, C) +
\]
\[+ kdT(r, f) + O(1) = d(k + 2)T(r, f) + kN_1(r, C) + O(1).\]

Similarly,
\[
N_2(r, B) \leq 2T(r, g) + O(1),
\]
\[
N_2(r, \frac{1}{B}) \leq 2dT(r, g) + N(r, \frac{1}{Q}) \leq d(k + 2)T(r, g) + kN_1(r, D) + O(1).
\]

Combining the above inequalities, we get
\[
T(r, A) \leq (2+2d+kd)T(r, f)+(2+2d)T(r, g)+kN_1(r, C)+N(r, \frac{1}{Q})-\log r+O(1),
\]
\[ T(r, B) \leq (2 + 2d + kd)T(r, g) + (2 + 2d)T(r, f) + kN_1(r, D) + N(r, \frac{1}{F}) - \log r + O(1). \]

\[ T(r, A) + T(r, B) \leq (4 + 4d + kd)(T(r, f) + T(r, g)) + kN_1(r, C) + N(r, \frac{1}{f}) - 2\log r + O(1). \]

By Lemma 2.8 we obtain

\[ (n - 2k)dT(r, f) + kN(r, C) + N(r, \frac{1}{F}) \leq T(r, A) + O(1), \]

\[ (n - 2k)dT(r, g) + kN(r, D) + N(r, \frac{1}{Q}) \leq T(r, B) + O(1). \]

Thus,

\[
(n - 2k)d(T(r, f) + T(r, g)) + kN(r, C) + N(r, \frac{1}{F}) + kN(r, D) + N(r, \frac{1}{Q}) \leq
\]

\[ \leq T(r, A) + T(r, B) + O(1), \]

\[ (n - 2k)d(T(r, f) + T(r, g)) + kN(r, C) + N(r, \frac{1}{F}) + kN(r, D) + N(r, \frac{1}{Q}) \leq \]

\[ \leq (4 + 4d + kd)(T(r, f) + T(r, g)) + kN_1(r, C) + N(r, \frac{1}{Q}) + \]

\[ + kN_1(r, D) + N(r, \frac{1}{F}) - 2\log r + O(1). \]

Therefore,

\[ (n - 2k)d(T(r, f) + T(r, g)) \leq (4 + 4d + kd)(T(r, f) + T(r, g)) - 2\log r + O(1), \]

\[ ((n - 2k)d - 4 - 4d - kd)(T(r, f) + T(r, g)) \leq -2\log r + O(1). \]

Since \( n \geq 3k + 5 > 2k + \frac{4 + 4d + kd}{d} \), we obtain a contradiction.

\textbf{Case 2.} \( (P(f))^n)^{(k)} \ (Q(g))^n)^{(k)} = 1 \). Then we have

\[ C = P(f) = (f - e_1) \cdots (f - e_d), (C^n)^{(k)} = C^{n-k}F, D = Q(g). \]

Therefore

\[ (f - e_1)^{n-k} \cdots (f - e_d)^{n-k}F.(D^n)^{(k)} = (C^n)^{(k)}(D^n)^{(k)} = 1. \]

Because \( n \geq 3k + 5 \) we see that, if \( z_0 \) is a zero of \( f - e_i \) with \( 1 \leq i \leq d \), then \( z_0 \) is a zero of \( C \), and therefore, \( z_0 \) is a zero of \( (C^n)^{(k)} \), and then \( z_0 \) is a pole
of \((D^n)^{(k)}\) and \(\nu_{(D^n)^{(k)}}(z_0) = (n-k)\nu_f^{\infty}(z_0)\). Thus, \(z_0\) is a pole of \(g\), and by Lemma 2.4 we get

\[\nu_{(D^n)^{(k)}}(z_0) = n\nu_g^{\infty}(z_0) + k \geq nd + k.\]

So \(\nu_f^{\infty}(z_0) = \frac{n\nu_g^{\infty}(z_0) + k}{n-k} \geq \frac{nd+k}{n-k}, i = 1, 2, \ldots, d.\) Applying Lemma 2.2, we obtain:

\[\sum_{i=1}^{d} (1 - \frac{n-k}{nd+k}) < 2.\]

From this we have \(n(d^2 - 3d) < 2k(1-d)\), and so we obtain a contradiction to \(d \geq 12.\)

**Case 3.** \((P(f))^n)^{(k)} = ((Q(g))^n)^{(k)}\). Then \((P(f))^n - s = (Q(g))^n\), where \(s\) is a polynomial of degree \(< k\). We prove \(s \equiv 0\). If it is not the case, then

\[\frac{(P(f))^n}{s} - 1 = \frac{(g-k_1)^n \cdots (g-k_d)^n}{s},\]

\[\frac{(g-k_1)^n \cdots (g-k_d)^n}{s} + 1 = \frac{(f-e_1)^n \cdots (f-e_d)^n}{s}.\]

Set \(H = \frac{C^n}{s}, G = \frac{D^n}{s}\). Since \(f, g\) are not constants, and so are \(C, D, C^n, D^n, H, G\). Applying Lemma 2.1 to \(H\) with values \(\infty, 0, 1\), we get

\[T(r, H) \leq N_1(r, H) + N_1(r, \frac{1}{H}) + N_1(r, \frac{1}{H-1}) - \log r + O(1).\]

On the other hand,

\[T(r, C^n) = nT(r, C) + O(1) \leq T(r, H) + T(r, s) = T(r, H) + (k-1) \log r + O(1),\]

\[nT(r, C) - (k-1) \log r \leq nT(r, H) + O(1), \quad ndT(r, f) - (k-1) \log r \leq T(r, H) + O(1).\]

\[N_1(r, H) \leq N_1(r, C^n) + N_1(r, \frac{1}{s}) \leq N_1(r, f) + (k-1) \log r \leq T(r, f) + (k-1) \log r,\]

\[N_1(r, \frac{1}{H}) \leq N_1(r, \frac{1}{C^n}) = N_1(r, \frac{1}{C}) \leq T(r, C) + O(1) = dT(r, f) + O(1),\]

\[N_1(r, \frac{1}{H-1}) = N_1(r, \frac{1}{D^n}) \leq N_1(r, \frac{1}{D}) \leq T(r, D) + O(1) = dT(r, g) + O(1), ndT(r, f) - (k-1) \log r \leq T(r, f) + (k-1) \log r + d(T(r, f) + T(r, g)) + O(1).\]

From this, and noting that \(dT(r, f) \leq T(r, f) + d(T(r, f) + T(r, g)) + O(1).\)
Applying Lemma 2.1 to \( G \) with values \( \infty, 0, -1 \), and noting that \( \log r \leq T(r, g) \) we obtain

\[
T(r, G) \leq N_1(r, G) + N_1(r, \frac{1}{G}) + N_1(r, \frac{1}{G+1}) - \log r + O(1),
\]

\[
ndT(r, g) - (k-1) \log r \leq T(r, g) + dT(r, f) + T(r, g) - \log r + O(1),
\]

\[
(nd - 2(k - 1))T(r, g) \leq T(r, g) + d(T(r, f) + T(r, g)) - \log r + O(1).
\]

So

\[
(nd - 2(k - 1))(T(r, f) + T(r, g)) \leq T(r, f) + T(r, g) + 2d(T(r, f) + T(r, g)) - 2\log r + O(1),
\]

\[
(nd - 2d - 2k + 1)(T(r, f) + T(r, g)) + 2\log r \leq O(1).
\]

We obtain a contradiction to \( n \geq 3k + 5 > \frac{2d + 2k - 1}{d} \). So \( s = 0 \). Then \( (P(f))^n = (Q(g))^n \). Therefore, \( P(f) = cQ(g) \), \( c^n = 1 \). From this and by Lemma 2.9, we obtain the conclusion of Theorem 1.

### 3.2 Proof of Theorem 2

Set

\[
C = P(f) = f^d + a_1 f^{d-m} + b_1, \quad D = Q(g) = g^d + a_2 g^{d-m} + b_2,
\]

\[
M = -\frac{f^{d-m}(f^m + a_1)}{b_1}, \quad N = -\frac{g^{d-m}(g^m + a_2)}{b_2}.
\]

Since \( P(f) \) and \( Q(g) \) share 0 \( CM \), we get \( E_M(1) = E_N(1) \). Applying Lemma 2.3 to \( M, N \), we have one of the following possibilities:

**Case 1.**

\[
T(r, M) \leq N_2(r, M) + N_2(r, \frac{1}{M}) + N_2(r, N) + N_2(r, \frac{1}{N}) - \log r + O(1),
\]

\[
T(r, N) \leq N_2(r, M) + N_2(r, \frac{1}{M}) + N_2(r, N) + N_2(r, \frac{1}{N}) - \log r + O(1).
\]

Moreover,

\[
T(r, M) = dT(r, f) + O(1) \quad N_1(r, M) = N_1(r, f) \leq T(r, f) + O(1),
\]

\[
N_2(r, M) = 2N_1(r, f) \leq 2T(r, f) + O(1),
\]

\[
N_2(r, \frac{1}{M}) \leq 2N_1(r, \frac{1}{f}) + N_2(r, \frac{1}{f^{m+a_1}}) \leq 2T(r, f) + nT(r, f) + O(1).
\]

Similarly

\[
N_2(r, N) \leq 2T(r, g) + O(1), \quad N_2(r, \frac{1}{N}) \leq 2T(r, g) + mT(r, g) + O(1).
\]
Therefore,

\[ T(r, M) = dT(r, f) + O(1) \leq 4(T(r, f) + T(r, g)) + m(T(r, f) + T(r, g)) - \log r + O(1). \]

Similarly

\[ T(r, N) = dT(r, g) + O(1) \leq 4(T(r, f) + T(r, g)) + m(T(r, f) + T(r, g)) - \log r + O(1). \]

Combining the above inequalities we get

\[ d(T(r, f) + T(r, g)) \leq 8(T(r, f) + T(r, g)) + 2m(T(r, f) + T(r, g)) - 2\log r + O(1), \]

\[ (d - 2m - 8)(T(r, f) + T(r, g)) + 2\log r \leq O(1). \]

We obtain a contradiction to \( d \geq 2m + 8. \)

**Case 2.** \( M, N = 1, \) i.e. \( f \equiv m_a g^{a_2} = b_1/b_2. \)

Note that equation \( z^m + a_1 = 0 \) has \( m \) simple zeros. Let \( r_1, r_2, ..., r_m \) be all these roots. Therefore

\[ (3.1) \quad f^{d-m}(f - r_1)...(f - r_1)g^{d-m}(g^m + a_2) = b_1/b_2. \]

From (3.1) it follows that all zeros of \( f - r_j, j = 1, ..., m \), have multiplicities \( \geq d \), and all zeros of \( f \) have multiplicities \( \geq d/m \). By Lemma 2.2 we have

\[ 1 - \frac{d - m}{d} + m(1 - \frac{1}{d}) < 2 \text{. Then } m < 2. \text{ Since } m \geq 2, \text{ we obtain a contradiction.} \]

**Case 3.** \( M = N, \) i.e. \( f^{d-m}(f^m + a_1)g^d = b_1/b_2. \) Then

\[ (3.2) \quad f^d + a_1f^{d-m} + b_1 = b_1/b_2(g^d + a_2g^{d-m} + b_2). \]

Applying Lemma 2.9 to (3.2), we obtain we obtain the conclusion of Theorem 2.

3.3 Proof of Corollary 3. Since \( P(z), Q(z) \) have no multiple zeros, we see that \( E_f(S) = E_q(T) \) if and only if \( P(f) \) and \( Q(g) \) share \( 0 \) CM. From this and Theorem 2, we obtain the conclusion of Corollary 3.

3.4 Proof of Corollary 4. By \( E_f(S) = E_q(S) \) and Corollary 3, we obtain \( g = hf \) for a constant \( h \), such that \( h^d = b_2/b_1, h^m = a_2/a_1 \) with \( b_1 = b_2, a_1 = a_2. \) Therefore, \( h^d = 1 \) and \( h^m = 1 \). Because \( (d, m) = 1 \) we have \( h = 1. \) So \( f = g. \)
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