THE INTEGER SEQUENCE a_n AND ITS RELATIONSHIP WITH PELL, PELL–LUCAS AND BALANCING NUMBERS

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Abstract. In this work we deduce some algebraic relations on integer sequence a_n and its relationship with Pell, Pell–Lucas and balancing numbers. Further we formulate the eigenvalues and spectral norm of the circulant matrix of a_n numbers.

1. Preliminaries

Let p and q be two non–zero integers and let $d = p^2 - 4q \neq 0$ (to exclude a degenerate case). We set the sequences U_n and V_n to be

$$U_n = U_n(p, q) = pU_{n-1} - qU_{n-2}$$
$$V_n = V_n(p, q) = pV_{n-1} - qV_{n-2}$$

for $n \geq 2$ with $U_0 = 0, U_1 = 1, V_0 = 2, V_1 = p$. The characteristic equation of them is $x^2 - px + q = 0$ and hence the roots are $\alpha = \frac{p+\sqrt{d}}{2}$ and $\beta = \frac{p-\sqrt{d}}{2}$. Their Binet formulas are $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n = \alpha^n + \beta^n$. Note

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that $U_n(1,-1) = F_n$ Fibonacci numbers (A000045 in OEIS), $V_n(1,-1) = L_n$ Lucas numbers (A000032 in OEIS), $U_n(2,-1) = P_n$ Pell numbers (A000129 in OEIS) and $V_n(2,-1) = Q_n$ Pell–Lucas numbers (A002203 in OEIS) (for further details see [2, 5, 6, 7, 12]).

Recently, balancing numbers have been defined by Behera and Panda in [1]. A positive integer n is called a balancing number if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

holds for some positive integer r which is called cobalancing number (or balancer). From (1.1) one has $\frac{(n-1)n}{2} = rn + \frac{r(r+1)}{2}$ and so

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \quad \text{and} \quad n = \frac{2r+1+\sqrt{8r^2+8r+1}}{2}. \tag{1.2}$$

Let B_n denote the n^{th} balancing number, and let b_n denote the n^{th} cobalancing number. Then from (1.2) we see that B_n is a balancing number iff $8B_n^2+1$ is a perfect square, and b_n is a cobalancing number iff $8b_n^2+8b_n+1$ is a perfect square. So $C_n = \sqrt{8B_n^2+1}$ and $c_n = \sqrt{8b_n^2+8b_n+1}$ are integers called the n^{th} Lucas–balancing and n^{th} Lucas–cobalancing number, respectively. Binet formulas for all balancing numbers are $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$, $b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$, $C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$, where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ which are the roots of the characteristic equation of Pell numbers (for further details see [8, 9, 10, 11]).

2. Main results

In [13], Santana and Diaz–Barrero set a sequence

$$a_n = P_{2n} + P_{2n+1} \tag{2.1}$$

in order to determine the sum of first $4n+1$ nonzero terms of Pell numbers. They proved that $a_{n+1} = 6a_n - a_{n-1}$ for $n \geq 1$, where $a_0 = 1, a_1 = 7$. Since $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$, the Binet formula for a_n numbers is

$$a_n = \frac{\alpha^{2n+1} + \beta^{2n+1}}{2} \tag{2.2}$$

for $n \geq 0$. For the sums of a_n numbers, we can give the following result.

Theorem 2.1. *Let a_n denote the n^{th} number. Then*

$$\sum_{i=0}^{n} a_i = \frac{a_{n+1} - a_n - 2}{4} \quad \textit{for } n \geq 1,$$

$$\sum_{i=1}^{n} a_{2i-1} = \frac{33a_{2n-1} - a_{2n-3} - 8}{32} \quad \textit{for } n \geq 2,$$

$$\sum_{i=0}^{n} a_{2i} = \frac{33a_{2n} - a_{2n-2} - 8}{32} \quad \textit{for } n \geq 1.$$

Proof. Since $a_n = \frac{\alpha^{2n+1}+\beta^{2n+1}}{2}$, we easily get

$$\begin{aligned}
\sum_{i=0}^{n} a_i &= a_0 + a_1 + \cdots + a_n = \\
&= \frac{\alpha+\beta}{2} + \frac{\alpha^3+\beta^3}{2} + \cdots + \frac{\alpha^{2n+1}+\beta^{2n+1}}{2} = \\
&= \frac{\alpha^{2n+2}+\beta^{2n+2}-2}{4} = \\
&= \frac{\frac{\alpha^{2n+1}[2(1+\sqrt{2})]+\beta^{2n+1}[2(1-\sqrt{2})]}{2} - 2}{4} = \\
&= \frac{\frac{\alpha^{2n+1}(\alpha^2-1)+\beta^{2n+1}(\beta^2-1)}{2} - 2}{4} = \\
&= \frac{\frac{\alpha^{2n+3}+\beta^{2n+3}}{2} - \frac{\alpha^{2n+1}+\beta^{2n+1}}{2} - 2}{4} = \\
&= \frac{a_{n+1} - a_n - 2}{4}.
\end{aligned}$$

The others can be proved similarly. ■

For the relationship with Pell, Pell–Lucas and balancing numbers, we can give the following result.

Theorem 2.2. *Let a_n denote the n^{th} number. Then for $n \geq 1$, we have*

1. *The sum of $(n+1)^{st}$ and n^{th} balancing numbers is equal to the n^{th} a_n number, that is,*

$$B_{n+1} + B_n = a_n.$$

2. *The half of the difference of $(n+2)^{nd}$ and n^{th} cobalancing numbers is equal to the n^{th} a_n number, that is,*

$$\frac{b_{n+2} - b_n}{2} = a_n.$$

3. *The half of the difference of $(n+1)^{st}$ and n^{th} Lucas–balancing numbers is equal to the n^{th} a_n number, that is,*

$$\frac{C_{n+1}-C_n}{2}=a_n.$$

4. *The $(n+1)^{st}$ Lucas–cobalancing number is equal to the n^{th} a_n number, that is,*

$$c_{n+1}=a_n.$$

5. *The half of $(2n+1)^{st}$ Pell–Lucas number is equal to the n^{th} a_n number, that is,*

$$\frac{Q_{2n+1}}{2}=a_n.$$

Proof. 1. Since $B_n=\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$, we easily get

$$\begin{aligned}
B_{n+1}+B_n &= \frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}}+\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}= \\
&= \frac{\alpha^{2n}(\alpha^2+1)+\beta^{2n}(-\beta^2-1)}{4\sqrt{2}}= \\
&= \frac{\alpha^{2n}(1+\sqrt{2})+\beta^{2n}(1-\sqrt{2})}{2}= \\
&= \frac{\alpha^{2n+1}+\beta^{2n+1}}{2}= \\
&= a_n
\end{aligned}$$

since $\alpha^2+1=2\sqrt{2}(1+\sqrt{2})$ and $-\beta^2-1=2\sqrt{2}(1-\sqrt{2})$.

The other cases can be proved similarly. ∎

We have already $P_{2n}+P_{2n+1}=a_n$ by (2.1). Also we can give the following theorem.

Theorem 2.3. *Let a_n denote the n^{th} number. Then for $n\geq 0$,*

1. *The sum of $(2n+1)^{st}$ balancing and cobalancing numbers is a perfect square and is*

$$B_{2n+1}+b_{2n+1}=a_n^2.$$

2. *The sum of $(2n+1)^{st}$ Lucas–balancing and Lucas–cobalancing numbers is equal to product of four times of n^{th} a_n number and $(2n+1)^{st}$ Pell number, that is,*

$$C_{2n+1}+c_{2n+1}=4a_nP_{2n+1}.$$

Proof. 1. Recall that $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$ and $b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$. So

$$\begin{aligned}
B_{2n+1} + b_{2n+1} &= \frac{\alpha^{4n+2} - \beta^{4n+2}}{4\sqrt{2}} + \frac{\alpha^{4n+1} - \beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2} = \\
&= \frac{\alpha^{4n}(\alpha^2 + \alpha) - \beta^{4n}(\beta^2 + \beta) - 2\sqrt{2}}{4\sqrt{2}} = \\
&= \frac{\alpha^{4n+2} + 2(\alpha\beta)^{2n+1} + \beta^{4n+2}}{4} = \\
&= \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right)^2 = \\
&= a_n^2
\end{aligned}$$

since $\alpha\beta = -1$.

The other case can be proved similarly. ■

Panda and Ray ([10]) considered the sums of Pell numbers and proved that the sum of first $2n-1$ Pell numbers is equal to the sum of n^{th} balancing number and its balancer, that is,

$$\sum_{i=1}^{2n-1} P_i = B_n + b_n.$$

Later Gözeri, Özkoç and Tekcan ([4]) proved that the sum of Pell–Lucas numbers from 0 to $2n-1$ is equal to the sum of n^{th} Lucas–balancing and Lucas–cobalancing number, that is,

$$\sum_{i=0}^{2n-1} Q_i = C_n + c_n.$$

Similarly, we can give the following result.

Theorem 2.4. *Let a_n denote the n^{th} number.*

1. *The sum of a_n numbers from 0 to n is equal to sum of the $(n+1)^{st}$ balancing number and its balancer, that is,*

$$\sum_{i=0}^{n} a_i = B_{n+1} + b_{n+1}.$$

2. *The sum of even a_n numbers from 1 to n is equal to the product of $(n+1)^{st}$ a_n number and n^{th} balancing number, that is,*

$$\sum_{i=1}^{n} a_{2i} = a_{n+1}B_n.$$

3. *The sum of odd a_n numbers from 1 to n is equal to the product of n^{th} a_n number and n^{th} balancing number, that is,*

$$\sum_{i=1}^{n} a_{2i-1} = a_n B_n.$$

4. *The sum of even a_n numbers from 1 to $2n+1$ is equal to the product of n^{th} and $(2n+2)^{nd}$ a_n numbers and $(2n+1)^{st}$ Pell number, that is,*

$$\sum_{i=1}^{2n+1} a_{2i} = a_n a_{2n+2} P_{2n+1}.$$

5. *The sum of odd a_n numbers from 1 to $2n+1$ is equal to the product of n^{th} and $(2n+1)^{st}$ a_n numbers and $(2n+1)^{st}$ Pell number, that is,*

$$\sum_{i=1}^{2n+1} a_{2i-1} = a_n a_{2n+1} P_{2n+1}.$$

6. *The sum of odd balancing numbers from 0 to $2n$ is equal to the product of $(2n+1)^{st}$ Pell number, n^{th} a_n number and $(2n+1)^{st}$ balancing number, that is,*

$$\sum_{i=0}^{2n} B_{2i+1} = P_{2n+1} a_n B_{2n+1}.$$

7. *The sum of even balancing numbers from 1 to $2n$ is equal to the product of two times of n^{th} a_n number, n^{th} balancing number, n^{th} Lucas–balancing number and $(2n+1)^{st}$ Pell number, that is,*

$$\sum_{i=1}^{2n} B_{2i} = 2a_n B_n C_n P_{2n+1}.$$

8. *The sum of even Pell numbers from 1 to $2n$ is equal to the product of two times of n^{th} balancing and n^{th} a_n number, that is,*

$$\sum_{i=1}^{2n} P_{2i} = 2B_n a_n.$$

9. *The sum of odd Pell numbers from 0 to $2n$ is equal to the product of $(2n+1)^{st}$ Pell number and n^{th} a_n number, that is,*

$$\sum_{i=0}^{2n} P_{2i+1} = P_{2n+1} a_n.$$

10. *The sum of Pell–Lucas numbers from* 0 *to* $2n$ *is equal to the quotient of two times of* $(2n+1)^{st}$ *balancing number by the* n^{th} a_n *number, that is,*

$$\sum_{i=0}^{2n} Q_i = \frac{2B_{2n+1}}{a_n}.$$

11. *The sum of even Pell–Lucas numbers from* 0 *to* $2n$ *is equal to the product of* n^{th} a_n *number and the difference of* n^{th} *and* $(n-1)^{st}$ a_n *numbers, that is,*

$$\sum_{i=0}^{2n} Q_{2i} = a_n(a_n - a_{n-1}).$$

Proof. 1. Since $a_0 + a_1 + \cdots + a_n = \frac{a_{n+1}-a_n-2}{4}$ by Theorem 2.1, we easily get

$$\begin{aligned}
\sum_{i=0}^{n} a_i = \frac{a_{n+1}-a_n-2}{4} &= \\
&= \frac{\frac{\alpha^{2n+3}+\beta^{2n+3}}{2} - \frac{\alpha^{2n+1}+\beta^{2n+1}}{2} - 2}{4} = \\
&= \frac{\alpha^{2n+1}(\alpha^2-1)+\beta^{2n+1}(\beta^2-1)}{8} - \frac{1}{2} = \\
&= \frac{\alpha^{2n+1}(1+\sqrt{2})+\beta^{2n+1}(1-\sqrt{2})}{4} - \frac{1}{2} = \\
&= \frac{\alpha^{2n+2}(1+\alpha^{-1})+\beta^{2n+2}(-1-\beta^{-1})}{4\sqrt{2}} - \frac{1}{2} = \\
&= \frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}} + \frac{\alpha^{2n+1}-\beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2} = \\
&= B_{n+1} + b_{n+1}
\end{aligned}$$

as we claimed.

The other cases can be proved similarly. ■

For the perfect squares, we can give the following result.

Theorem 2.5. *Let* a_n *denote the* n^{th} a_n *number. Then*

1. $1+8B_{n+1}^2$ *is a perfect square and is* $\sqrt{1+8B_{n+1}^2} = 2a_n + C_n$.

2. $P_{2n+1}^2 + P_{2n}P_{2n+2}$ *is a perfect square and is* $\sqrt{P_{2n+1}^2 + P_{2n}P_{2n+2}} = a_n$.

3. $B^2_{2n+1}+4a^2_n B_n B_{n+1}$ *is a perfect square and is* $\sqrt{B^2_{2n+1}+4a^2_n B_n B_{n+1}} = a^2_n$.

Proof. 1. Applying Binet formulas, we get

$$\begin{aligned}
\sqrt{1+8B^2_{n+1}} &= \sqrt{1+8\left(\frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}}\right)^2} = \\
&= \sqrt{\frac{\alpha^{4n+4}+\beta^{4n+4}+2}{2}} = \\
&= \frac{\alpha^{2n+2}+\beta^{2n+2}}{2} = \\
&= \frac{\alpha^{2n}(2\alpha+1)+\beta^{2n}(2\beta+1)}{2} = \\
&= 2\left(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\right)+\frac{\alpha^{2n}+\beta^{2n}}{2} = \\
&= 2a_n + C_n
\end{aligned}$$

since $2\alpha+1=\alpha^2$ and $2\beta+1=\beta^2$. The other cases can be proved similarly. ■

In [13], Santana and Diaz–Barrero proved that the sum of first nonzero $4n+1$ terms of Pell numbers is a perfect square and is

$$\sum_{i=1}^{4n+1} P_i = \left(\sum_{i=0}^{n}\begin{pmatrix} 2n+1 \\ 2i \end{pmatrix} 2^i\right)^2.$$

In fact, this sum is equals to a^2_n, that is,

$$\sum_{i=1}^{4n+1} P_i = a^2_n.$$

Tekcan and Tayat ([14]) proved that the sum of first nonzero $2n+1$ terms of Pell numbers is

$$\sum_{i=1}^{2n+1} P_i = \begin{cases} \left(\frac{\alpha^{n+1}+\beta^{n+1}}{2}\right)^2, & \text{for even } n \geq 0 \\ \frac{\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{2}}\right)^2}{2}, & \text{for odd } n \geq 1 \end{cases}$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. By considering this result, they set two integer sequences $X_n = \frac{\alpha^{n+1}+\beta^{n+1}}{2}$ and $Y_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{2}}$ and proved that the sum of first nonzero $4n+1$ terms of Pell numbers is

$$\sum_{i=1}^{4n+1} P_i = (2X^2_n - 2X_n Y_{n-1} + (-1)^{n+1})^2.$$

Similarly we can give the following result.

Theorem 2.6. *Let a_n denote the n^{th} a_n number. Then*

1. *The sum of a_n numbers from 0 to $2n$ is a perfect square and is*
$$\sum_{i=0}^{2n} a_i = a_n^2$$
for $n \geq 1$.

2. *The half of the sum of odd Pell–Lucas numbers from 0 to $2n$ is a perfect square and is*
$$\frac{\sum\limits_{i=0}^{2n} Q_{2i+1}}{2} = a_n^2$$
for $n \geq 1$.

3. *The sum of a_n numbers from 1 to n and adding 2 is a perfect square and is*
$$2 + \sum_{i=1}^{n} a_i = C^2_{\frac{n+1}{2}}$$
for odd $n \geq 1$, or the sum of a_n numbers from 1 to n and adding 1 is a perfect square and is
$$1 + \sum_{i=1}^{n} a_i = c^2_{\frac{n+2}{2}}$$
for even $n \geq 2$.

Proof. 1. Since $a_0 + a_1 + \cdots + a_n = \frac{a_{n+1} - a_n - 2}{4}$, we deduce that

$$\begin{aligned}
\sum_{i=0}^{2n} a_i &= \frac{a_{2n+1} - a_{2n} - 2}{4} = \frac{\frac{\alpha^{4n+3}+\beta^{4n+3}}{2} - \frac{\alpha^{4n+1}+\beta^{4n+1}}{2} - 2}{4} = \\
&= \frac{\frac{\alpha^{4n+1}(\alpha^2-1)+\beta^{4n+1}(\beta^2-1)}{2} - 2}{4} = \frac{\alpha^{4n+1}(1+\sqrt{2}) + \beta^{4n+1}(1-\sqrt{2}) - 2}{4} = \\
&= \frac{\alpha^{4n+2} + \beta^{4n+2} - 2}{4} = \frac{\alpha^{4n+2} + 2(\alpha\beta)^{2n+1} + \beta^{4n+2}}{4} = \\
&= \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right)^2 = \\
&= a_n^2.
\end{aligned}$$

The other cases can be proved similarly. ■

A circulant matrix (see [3]) is a matrix M defined as

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-2} & m_{n-1} \\ m_{n-1} & m_0 & m_1 & \cdots & m_{n-3} & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & \cdots & m_{n-4} & m_{n-3} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ m_2 & m_3 & m_4 & \cdots & m_0 & m_1 \\ m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_0 \end{bmatrix},$$

where m_i are constant. In this case, the eigenvalues of M are

$$\lambda_j(M) = \sum_{u=0}^{n-1} m_u w^{-ju}, \tag{2.3}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$. The spectral norm of a matrix $Q = [q_{ij}]_{n\times n}$ is defined to be

$$||Q||_{spec} = \max_{0\leq j\leq n-1}\{\sqrt{\lambda_j}\},$$

where λ_j are the eigenvalues of Q^*Q and Q^* denotes the conjugate transpose of Q.

Theorem 2.7. *Let a denote the circulant matrix of a_n numbers. Then*

1. *The eigenvalues of a are*

$$\lambda_j(a) = \frac{(a_{n-1}+1)w^{-j}+1-a_n}{w^{-2j}-6w^{-j}+1}$$

for $j = 0, 1, 2, \cdots, n-1$.

2. *The spectral norm of a is*

$$||a||_{spec} = \frac{a_n - a_{n-1} - 2}{4}.$$

Proof. 1. Applying (2.3), we deduce that

$$\begin{aligned}
\lambda_j(a) &= \sum_{u=0}^{n-1} a_u w^{-ju} = \\
&= \sum_{u=0}^{n-1} \left(\frac{\alpha^{2u+1}+\beta^{2u+1}}{2} \right) w^{-ju} = \\
&= \frac{1}{2}\left[\alpha \sum_{u=0}^{n-1} (\alpha^2 w^{-j})^u + \beta \sum_{u=0}^{n-1} (\beta^2 w^{-j})^u \right] = \\
&= \frac{1}{2}\left[\alpha \frac{\alpha^{2n}-1}{\alpha^2 w^{-j}-1} + \beta \frac{\beta^{2n}-1}{\beta^2 w^{-j}-1} \right] = \\
&= \frac{1}{2}\left[\frac{(\alpha^{2n+1}-\alpha)(\beta^2 w^{-j}-1)+(\beta^{2n+1}-\beta)(\alpha^2 w^{-j}-1)}{(\alpha^2 w^{-j}-1)(\beta^2 w^{-j}-1)} \right] = \\
&= \frac{w^{-j}\left[(\alpha\beta)^2 \left(\frac{\alpha^{2n-1}+\beta^{2n-1}}{2} \right) + \frac{-\alpha\beta^2-\beta\alpha^2}{2} \right] + \frac{\alpha+\beta}{2} - \frac{\alpha^{2n+1}+\beta^{2n+1}}{2}}{w^{-2j}-6w^{-j}+1} = \\
&= \frac{(a_{n-1}+1)w^{-j}+1-a_n}{w^{-2j}-6w^{-j}+1}
\end{aligned}$$

since $\alpha\beta = -1$, $\frac{-\alpha\beta^2-\beta\alpha^2}{2} = 1$ and $\frac{\alpha+\beta}{2} = 1$.

2. For the circulant matrix

$$a = \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\
a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0
\end{bmatrix},$$

for a_n numbers, we have

$$(a)^* a = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\
. & . & & . & . \\
. & . & \cdots & . & . \\
. & . & & . & . \\
a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\
a_{n1} & a_{n2} & \cdots & a_{n(n-1)} & a_{nn}
\end{bmatrix},$$

where

$$\begin{aligned}
a_{11} &= a_0^2 + a_{n-1}^2 + \cdots + a_2^2 + a_1^2, \\
a_{12} &= a_0a_1 + a_{n-1}a_0 + \cdots + a_2a_3 + a_1a_2, \\
&\vdots \\
a_{1(n-1)} &= a_0a_{n-2} + a_{n-1}a_{n-3} + \cdots + a_2a_0 + a_1a_{n-1}, \\
a_{1n} &= a_0a_{n-1} + a_{n-1}a_{n-2} + \cdots + a_2a_1 + a_1a_0, \\
a_{21} &= a_1a_0 + a_0a_{n-1} + \cdots + a_3a_2 + a_2a_1, \\
a_{22} &= a_1^2 + a_0^2 + \cdots + a_3^2 + a_2^2, \\
&\vdots \\
a_{2(n-1)} &= a_1a_{n-2} + a_0a_{n-3} + \cdots + a_3a_0 + a_2a_{n-1} \\
a_{2n} &= a_1a_{n-1} + a_0a_{n-2} + \cdots + a_3a_1 + a_2a_0, \\
&\vdots \\
a_{n1} &= a_{n-1}a_0 + a_{n-2}a_{n-1} + \cdots + a_1a_2 + a_0a_1, \\
a_{n2} &= a_{n-1}a_1 + a_{n-2}a_0 + \cdots + a_1a_3 + a_0a_2, \\
&\vdots \\
a_{n(n-1)} &= a_{n-1}a_{n-2} + a_{n-2}a_{n-3} + \cdots + a_1a_0 + a_0a_{n-1}, \\
a_{nn} &= a_{n-1}^2 + a_{n-2}^2 + \cdots + a_1^2 + a_0^2.
\end{aligned}$$

The eigenvalues of a^*a are $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$. Here λ_0 is maximum and is

$$\begin{aligned}
\lambda_0 &= a_0^2 + a_1^2 + \cdots + a_{n-2}^2 + a_{n-1}^2 + \\
&+ 2\begin{bmatrix} a_0(a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}) + \\ + a_1(a_2 + \cdots + a_{n-2} + a_{n-1}) + \\ + \cdots + \\ + a_{n-2}a_{n-1} \end{bmatrix} = \\
&= (a_0 + a_1 + \cdots + a_{n-1})^2.
\end{aligned}$$

Thus the spectral norm of a is hence $||a||_{spec} = \sqrt{\lambda_0} = a_0 + a_1 + \cdots + a_{n-1}$. So the result is clear from Theorem 2.1. ■

Example 2.8. Let $n = 5$. Then the circulant matrix is

$$a = \begin{bmatrix} 1 & 7 & 41 & 239 & 1393 \\ 1393 & 1 & 7 & 41 & 239 \\ 239 & 1393 & 1 & 7 & 41 \\ 41 & 239 & 1393 & 1 & 7 \\ 7 & 41 & 239 & 1393 & 1 \end{bmatrix}.$$

The eigenvalues of

$$a^*a = \begin{bmatrix} 1999301 & 344413 & 68817 & 68817 & 344413 \\ 344413 & 1999301 & 344413 & 68817 & 68817 \\ 68817 & 344413 & 1999301 & 344413 & 68817 \\ 68817 & 68817 & 344413 & 1999301 & 344413 \\ 344413 & 68817 & 68817 & 344413 & 1999301 \end{bmatrix}$$

are $\lambda_0 = 2825761, \lambda_1 = 1792686+137798\sqrt{5}, \lambda_2 = 1792686-137798\sqrt{5}, \lambda_3 = \lambda_1$ and $\lambda_4 = \lambda_2$. So the spectral norm of a is $||a||_{spec} = \sqrt{\lambda_0} = 1681$. On the other hand $\frac{a_5-a_4-2}{4} = 1681$.

From Theorem 2.7, we can give the following result.

Corollary 2.1. *The spectral norm of a is*

$$||a||_{spec} = \begin{cases} (\sqrt{2}P_n)^2, & \textit{for even } n \geq 2 \\ (a_{\frac{n-1}{2}})^2, & \textit{for odd } n \geq 1. \end{cases}$$

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