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ON EXPONENTIAL SUMS INVOLVING THE DIVISOR FUNCTION OVER $\mathbb{Z}[i]$

Antonina Radova and Sergey Varbanets

(Odessa, Ukraine)

Dedicated to the memory of Professor Antal Iványi

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Abstract. We apply the van der Corput transform to investigate the sums of view $\sum r(n)g(n)e(f(n)))$, where $r(n)$ is the number of representations of n as the sum of two squares of integer numbers. Such sums have been studied by M. Jutila, O. Gunyavy, M. Huxley and etc. Depending on differential properties of the functions $g(n)$ and $f(n)$ there have been obtained different kinds of error terms in bounds of the considered sums. In the special case, O. Gunyavy improved the result of M. Jutila in the problem on estimate the exponential sum involving the divisor function $\tau(n)$. We obtain the asymptotic formula of the sum $\sum \tau_k(\alpha)e\left(\frac{a}{q}N(\alpha)\right)$, $k = 2, 3$ over the ring of Gaussian integers which is an analogue of the asymptotic formulas obtained by M. Jutila and O. Gunyavy.

1. Introduction

In 1985 M. Jutila constructed an asymptotic formula for the sum

$$\sum_{n\leq N} \tau(n)e\left(\frac{an}{q}\right),$$

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which is a special case of more general sum

$$\sum_{n\leq N} a(n)g(n)e\left(\frac{f(n)}{q}\right),$$

where $a(n)$ is a multiplicative function, and $f(x)$, $g(x)$ are real-value differentiable functions on $[1,\infty)$. Such sums were studied in the works of M. Jutila[4] [5], A. Krätzel[6], M. Haxley[3] etc.

The asymptotic formula of M. Jutila for $a(n) = \tau(n)$ is nontrivial for $q \ll \ll x^{\frac{2}{5}-\varepsilon}$. O. Gunyavy[2] has extended the field of nontriviality for this formula up to $q \ll x^{\frac{1}{2}+\varepsilon}$.

The primary purpose of our paper is to obtain the analogical asymptotic formulas for divisor function $\tau_k(\alpha)$ over the ring of Gaussian integers.

Notations. We will frequently use the Landau and Vinogradov asymptotic notations.

The big "O" notation $f(x) = O(g(x))$ (equivalently, $f(x) \ll g(x)$) means that there exists some constant C such that $|f(x)| \leq C|g(x)|$ on the domain in question.

By $f(x) \asymp g(x)$, we shall mean that $g(x) \ll f(x) \ll g(x)$.

We denote $e^{2\pi ix}$ as $e(x)$.

2. Main lemma

Consider real-valued functions $f(x)$ and $g(x)$, $x \in [N, N_1]$, $N \leq N_1 \leq 2N$. We will suppose that the following conditions

(i) $g'(x)$, $f''(x)$ are monotonic;

(ii) $g(x) \ll g(N)$, $g'(x) \ll \frac{g(N)}{N}$, $\frac{1}{N} \ll |f'(N)| \ll |f'(x)| \ll |f'(N)|$;

(iii) $|f'(N)| \ll |f'(x) + 2xf''(x)| \ll |f'(N)|$, $|f^{(3)}(x)| \ll \frac{|f'(N)|}{N^2}$

are true.

We prove the following statement that allows to study the distribution of values of the divisor function over the ring of Gaussian integers.

Lemma. *Let the functions $f(x)$ and $g(x)$ satisfy the conditions above, and let $r(n)$ be the number of the representation of n by form $n = u^2 + v^2$, $u, v \in \mathbb{Z}$.*

Then we have

$$\sum_{N\le n\le N_1\le 2N} r(n)g(n)e^{2\pi i f(n)} =$$

$$= \omega_f \sum_{\frac{1}{2}X\le n\le 2X} r(n)g(4\varphi(n))\sqrt{|\varphi'(n)|}e^{2\pi i(f(4\varphi(n))-2\sqrt{n\varphi(n)})}+$$

$$+O\left(N^{\varepsilon}\left|\frac{g(N)}{f'(N)}\right|\right)+O(N^{\varepsilon}\,|g(N)|)+$$

$$+O\left(|g(N)|\,N^{\frac{1}{2}+\varepsilon}\min\left(\sqrt{|f'(N)|},\ \frac{1}{\sqrt{|f'(N)|}}\right)\right),$$

where

$$\omega_f=\begin{cases} -1 & if \quad f'(N)(f'(n)+2Nf''(N))>0,\\ i & if \quad f'(N)>0,\ f'(N)+2Nf''(N)<0,\\ -i & if \quad f'(N)<0,\ f'(N)+2Nf''(N)>0,\end{cases}$$

$X=2Nf'^2(N)$, *$\varepsilon>0$ is an arbitrary small, and constatnts in symbols "O" depend only on ε.*

Proof. Applying the asymptotic representation of summatory function $r(n)$ (see, [6], Ch. VIII, formula (685))

$$\sum_{n\le x} r(n)=\pi x+\sqrt{x}\sum_{n=1}^{\infty}\frac{r(n)}{\sqrt{n}}\Im_1(2\pi\sqrt{nx}),$$

where $\Im_\nu(u)$, $\nu=0,1,2,\dots$ is the Bessel function of kind one and ν-th order, we can write for arbitrary $X>1$

$$\sum_{N\le N_2\le N_1<2N} r(n)g(n)e(f(n)) =$$

$$(1)\qquad \begin{aligned} &=\int_N^{N_1} g(x)e(f(x))\frac{d}{dx}\left(\pi x+\sqrt{x}\sum_{n\le X}\frac{r(n)}{\sqrt{n}}\Im_1(2\pi\sqrt{nx})\right)+\\ &+\left(\sqrt{x}\sum_{n>X}\frac{r(n)}{\sqrt{n}}\Im_1(\sqrt{nx})\right)g(x)e(f(x))\Bigg|_N^{N_1}+\\ &+\int_N^{N_1}\frac{d}{dx}\left(g(x)e(f(x))\right)\sqrt{x}\sum_{n>X}\frac{r(n)}{\sqrt{n}}\Im_1(2\pi\sqrt{nX})dx.\end{aligned}$$

First we consider the case $\frac{1}{N}\ll f'(N)\ll 1$. Put $X=2Nf'^2(N)$. Estimating the trigonometrical integral with respect to first derivation, we have

$$(2)\qquad \int g(x)e(f(x))\pi dx\ll\frac{g(N)}{f'(N)}.$$

From the estimate

$$R(x) := \sqrt{x}\sum_{n>X}\frac{r(n)}{\sqrt{n}}\mathfrak{J}_1(2\pi\sqrt{nx}) \ll X^{\varepsilon}\left(1+\sqrt{\frac{x}{X}}\right)$$

we obtain

$$R(x)g(x)e(f(x)) \ll \frac{X^{\varepsilon}g(N)}{f'(N)}.$$

Next, since $g'(x)$ is monotone function, then

$$\begin{aligned}
&\int_N^{N_1} g'(x)e(f(x))R(x)dx \ll N^{\varepsilon}\int_N^{N_1}|g'(x)|\sqrt{\frac{x}{X}}dx \ll \\
&\ll \frac{N^{\varepsilon}}{f'(N)}\int_N^{N_1}|g'(x)|dx \ll \frac{N^{\varepsilon}g(N)}{f'(N)}.
\end{aligned}\tag{3}$$

Taking into account that $\frac{d}{dx}\left(x^{\frac{1}{2}}\mathfrak{J}_1(2\pi\sqrt{nx})\right) = \pi\sqrt{n}\mathfrak{J}_0(2\pi\sqrt{nx})$ we have

$$\begin{aligned}
&\sum_{n=N}^{N_1}\tau(n)g(n)e(f(n)) = \pi\sum_{n\le X}r(n)\int_N^{N_1}\mathfrak{J}_0(2\pi\sqrt{nx})g(x)e(f(x))dx+ \\
&+2\pi i\sum_{n>X}\frac{r(n)}{\sqrt{n}}\int_N^{N_1}g(x)f'(x)e(f(x))\sqrt{x}\mathfrak{J}_1(2\pi\sqrt{nx})dx+ \\
&+O\left(\frac{N^{\varepsilon}g(N)}{f'(N)}\right) = S_1+S_2+O\left(\frac{N^{\varepsilon}g(N)}{f'(N)}\right),
\end{aligned}\tag{4}$$

say.

Select $X = \max\{Nf'^2(N), N_1f'^2(N_1)\}$.

By the formulas for the Bessel functions

$$\mathfrak{J}_{\nu}(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{x}\right)\right),\ x\gg 1,\ \nu=0,1,\dots,$$

we at once derive that the critical points of subintegral functions in S_1 and S_2 lie beyond interval of integration if $n\notin\left[\frac{1}{2}X, X\right]$. Thus for such values of n an integration by parts gives

$$
\begin{aligned}
(5)\qquad &\sum_{n\le\frac{1}{2}X} r(n)\int\limits_N^{N_1}\mathfrak{I}_0(2\pi\sqrt{nx})g(x)e(f(x))dx \ll\\
&\ll \frac{g(N)}{N^{\frac{1}{2}}f'(N)}+g(N)\sqrt{Nf'(N)}\log N,\\
&\sum_{n>X}\frac{r(n)}{\sqrt{n}}\int\limits_N^{N_1}x^{\frac{1}{2}}\mathfrak{I}_1(2\pi\sqrt{nx})g(x)f'(x)e(f(x))dx \ll\\
&\ll \frac{N^{\varepsilon}g(N)}{f'(N)}+N^{\varepsilon}g(N)\sqrt{Nf'(N)}.
\end{aligned}
$$

For the other values of n we have

$$
\begin{aligned}
(6)\qquad &\sum_{\frac{1}{2}X<n\le X} r(n)\int\limits_N^{N_1} g(x)e(f(x))\mathfrak{I}_0(2\pi\sqrt{nx})dx=\\
&=-\frac{e^{\frac{\pi i}{4}}}{\sqrt{2}}\sum_{\frac{1}{2}X<n\le X}\frac{r(n)}{n^{\frac{1}{4}}}\int\limits_N^{N_1}\frac{g(x)f'(x)}{x^{\frac{1}{4}}}e(f(x)-\sqrt{nx})dx+\\
&+O\left(N^{\varepsilon}g(N)\sqrt{Nf'(N)}\right),
\end{aligned}
$$

The integral on the right-hand side in (6) will be calculated by the van der Corput method (see, [8], Ch. 4. Lemma 3 or [9]).

We have

$$
\begin{aligned}
(7)\qquad &I(n):=\int\limits_N^{N_1}x^{-\frac{1}{4}}g(x)e(f(x))dx=4\int\limits_{\frac{1}{4}N}^{\frac{1}{4}N_1}\frac{g(4x)e(f_n(x))}{\sqrt{2}x^{\frac{1}{4}}}dx=\\
&=\omega_f\frac{g(4\varphi(n))e(f_n(\varphi(n)))}{\varphi(n)^{\frac{1}{4}}\sqrt{|f_n''(\varphi(n))|}}+O\left(\frac{g(N)}{N^{\frac{1}{4}}(\varphi(N)-N)f_n''(\varphi(n))}\right)+\\
&+O\left(\frac{g(N)}{N^{\frac{1}{4}}Nf_n''(N)}\right),
\end{aligned}
$$

where $f_n(x)=f(4x)-2\sqrt{nx}$, $\varphi(n)$ is the solution of the equality $f_n'(x)=0$, i.e. $16xf'^2(x)=n$.

Simple calculations show that for $n \in \left[\frac{1}{2}X, X\right]$

$$
(8)\qquad \begin{aligned} f_n''(x) = f''(x) + \frac{\sqrt{n}}{4x^{\frac{3}{2}}} = \frac{1}{4x}\left(4xf''(x) + \sqrt{\frac{n}{x}}\right) \asymp \\ \asymp \frac{1}{N}(f'(N) + Nf''(N)) \asymp \frac{f'(N)}{N} > 0. \end{aligned}
$$

So, from (1)-(8) we find

$$
\begin{aligned} \sum_{n=N}^{N_1} r(n)g(n)e(f(n)) = \\ = \omega_f \sum_{\frac{1}{2}X<n\le X} r(n)g(\varphi(n))\sqrt{|\varphi'(n)|}e(f(\varphi(n)) - \sqrt{n\varphi(n)})+ \\ +O\left(\frac{N^\varepsilon g(N)}{f'(N)}\right) + O\left(g(N)N^2\sqrt{Nf'(N)}\right), \end{aligned}
$$

where

$$
\omega_f = \begin{cases} 1 & if \quad f'(N)(f'(n) + 2Nf''(N)) > 0, \\ i & if \quad f'(N) > 0,\ f'(N) + 2Nf''(N) < 0, \\ -i & if \quad f'(N) < 0,\ f'(N) + 2Nf''(N) > 0, \end{cases}
$$

In the case $f'(N) \gg 1$ we consider the expression

$$
\overline{\omega}_f = \sum r(n)g(\varphi(n))\sqrt{|\varphi'(n)|}e(-\widetilde{f}(\varphi(n))),
$$

where bar denotes the complex conjugate value, and

$$
\widetilde{f}(n) = -f(\varphi(n)) + \varphi(n)f'(\varphi(n)).
$$

Its clear that the following relations

$$
\widetilde{f}'(x) = \frac{1}{f'(\varphi(x))} \ll 1,
$$

$$
\widetilde{f}'(x) + 2x\widetilde{f}''(x) = 4[f'(\varphi(x)) + 2\varphi(x)f''(\varphi(x))]^{-1},\ 4x\widetilde{f}'^2(x) = \varphi(x).
$$

are true.

Hence, for $\frac{1}{2}X \le n \le X$

$$
g(\varphi(n))\sqrt{\varphi'(n)} \ll g(N)(f'(N))^{-1}.
$$

So, we have

$$
\begin{aligned} \sum_{N\le n\le N_1} r(n)g(n)e(f(n)) = \omega_f \sum_{\frac{1}{2}X\le n\le X} r(n)g(\varphi(n))\sqrt{|\varphi'(n)|}e(\widetilde{f}(\varphi(n)))+ \\ + O\left(N^\varepsilon g(N)\right) + O\left(N^\varepsilon g(N)\frac{\sqrt{N}}{\sqrt{f'(N)}}\right). \end{aligned}
$$

For $f'(N) < 0$ we instead of sum $\sum r(n)g(n)e(f(n))$ consider the complex conjugate sum $\sum r(n)g(n)e(-f(n))$.

So, lemma is proved. $\blacksquare$

This lemma allows to prove the following statements.

3. Main results

By an analogy with the work of M. Jutila[5] it can be shown that for $(a, q) = = 1$. We have

$$A(x, \frac{a}{q}) = \sum_{n \le x} r(n) e^{2\pi i \frac{an}{q}} = \pi \frac{x}{q} + R\left(x, \frac{a}{q}\right), \tag{9}$$

where for $1 \le N << x$

$$R\left(x, \frac{a}{q}\right) = x^{\frac{1}{4}} \sqrt{q} \sum_{n \le N} r(n) e^{2\pi i \frac{an}{q}} \frac{\cos\left(2\pi \frac{\sqrt{nx}}{q} - \frac{\pi}{4}\right)}{n^{\frac{3}{4}}} + O\left(q \frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{2}}}\right). \tag{10}$$

The following theorem is an analogue of the Jutila theorem for divisor function $\tau(n)$. For that we will use Main lemma proved above.

Theorem 1. *Let a, q be the positive integers, $1 \le a \le q$, $(a, 2q) = 1$. Then for an arbitrary $0 < \varepsilon < \frac{1}{2}$ we have*

$$A(x; \frac{a}{q}) := \sum_{n \le x} r(n) e\left(\frac{a}{q} n\right) = \frac{\pi x}{q} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

(*This asymptotic formula is nontrivial for $q \le x^{\frac{1}{2}-\varepsilon}$*).

Proof. We have

$$\begin{aligned} A(x; \frac{a}{q}) = \sum_{n \le x} r(n) e^{2\pi i \frac{an}{q}} &= \sum_{k=0}^{K} \sum_{2^k \le n < 2^{k+1}} r(n) e^{2\pi i \frac{an}{q}} + \\ &+ \sum_{2^K < n \le x} r(n) e^{2\pi i \frac{an}{q}}, \end{aligned} \tag{11}$$

where K is such that $2^K \le x < 2^{K+1}$.

The function $f(x) = \frac{a}{q}x$ suffices the conditions of Main lemma. In sums over n on the right side of the expression above we apply the Main lemma, and then sum over all k.

We deduce

$$A(x; \frac{a}{q}) = \omega_f \sum_{n \le \frac{a^2}{q^2}x} r(n) \frac{q}{a} e^{2\pi i - (-\frac{4q}{a}n)} + O\left(x^{\varepsilon}\frac{q}{a}\right) + O\left(x^{\varepsilon}\sqrt{\frac{xa}{q}}\right). \tag{12}$$

(Here we take into account that in the Main lemma for our case $\varphi(n) = \frac{q^2}{a^2}n$).

Note that in this case $\omega_f = 1$ (because $f'(N) \cdot (f'(N) + 2Nf''(N)) > 0$).

So, we obtain

$$A(x; \frac{a}{q}) = \frac{q}{a} A\left(\frac{a^2}{q^2}x, -\frac{4q}{a}\right) + O\left(x^{\varepsilon}\frac{q}{a}\right) + O\left(x^{\varepsilon}\sqrt{\frac{xa}{q}}\right).$$

Define a_0, $1 \le a_0 < a$, from the congruence $a_0 \equiv -4q \pmod{a}$.

Therefore, we can write

$$A(x; \frac{a}{q}) = \frac{q}{a} A\left(\frac{a^2}{q^2}x, \frac{a_0}{a}\right) + O\left(x^{\varepsilon}\frac{q}{a}\right) + O\left(x^{\varepsilon}\sqrt{\frac{xa}{q}}\right).$$

By proceeding this reasoning we infer the sequence of equalities

$$\frac{q}{a} A\left(\frac{xa^2}{q^2}, \frac{a_0}{a}\right) = \frac{q}{a_0} A\left(\frac{xa_0^2}{q^2}, \frac{-a}{a_0}\right) + O\left(x^{\varepsilon}\sqrt{\frac{q}{a_0}}\right) + O\left(x^{\varepsilon}\sqrt{\frac{xa_0}{a}}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{q}{a_{i-1}} A\left(\frac{xa_{i-1}^2}{q^2}, \frac{a_i}{a_{i-1}}\right) = \frac{q}{a_i} A\left(\frac{xa_i^2}{q^2}, \frac{-a_{i-1}}{a_i}\right) + O\left(x^{\varepsilon}\sqrt{\frac{q}{a_i}}\right) + O\left(x^{\varepsilon}\sqrt{\frac{xa_i}{a_{i-1}}}\right).$$

By virtue of the fact that $a_i \in \mathbb{N}$, $a_0 > a_1 > \ldots$, and $a_{i+1} \equiv -4a_i \pmod{a_{i-1}}$, we draw a conclusion that for the some index i_0 will have $a_{i_0} = 1$.

It is clear that $i_0 \le n_0$, where n_0 is an integer of Fibonacci number nearest to q. Hence, $i + 0 \ll \log q \ll x^{\varepsilon}$. Now we obtain after $\log q$ steps the following relation

$$A\left(x, \frac{q}{q}\right) = \frac{q}{1} A\left(\frac{x}{q}, 1\right) + O\left(x^{\varepsilon}\sum_{j=0}^{\log q}\frac{q}{a_j}\right) + O\left(x^{\varepsilon}\sum_{j=0}^{\log q}\left(\frac{xa_{j+1}}{a_j}\right)^{\frac{1}{2}}\right) =$$
$$= \frac{\pi x}{q} + O\left(qx^{\varepsilon}\right) + A\left(x^{\frac{1}{2}+\varepsilon}\right). \qquad \blacksquare$$

The result of this theorem has larger region of nontriviality of the asymptotic formula than region of nontriviality which can be obtained from (9)–(10).

With help of Theorem 1 we can study the distribution of the values of the divisor function $\tau(\alpha)$ over the ring of Gaussian integers, where

$$\tau(\alpha) = \sum_{\substack{\alpha_1,\alpha_2\in\mathbb{Z}[i]\\ \alpha_1\alpha_2=\alpha}} 1.$$

Theorem 2. *Let α_0, β be the Gaussian integers, $(\alpha_0,\beta)=1$, and $\tau(\alpha)$ be the divisor function over the ring of Gaussian numbers. Then for $N(\beta) \ll x^{\frac{1}{4}-\varepsilon}$ the following asymptotic formula*

$$\sum_{N(\alpha)\le x} \tau(\alpha) e^{2\pi i N(\frac{\alpha_0\alpha}{\beta})} = C_1(\beta)\frac{x\log x}{N(\beta)} + C_2(\beta)\frac{x}{N(\beta)} + O\left(x^{\frac{3}{4}+\varepsilon}\right) + O\left(x^{\frac{1}{2}+\varepsilon}N(\beta)\right)$$

where $C_i(\beta)$ are computable constants, $N(\beta)^{-\varepsilon} \ll C_i(\beta) \ll N(\beta)^{\varepsilon}$, $i=1,2$, holds.

Proof. Denote $N(\alpha_0)=a$, $N(\beta)=q$. Then, we have

$$\sum_{N(\alpha)\le x} \tau(\alpha) e^{2\pi i N\left(\frac{\alpha\alpha_0}{\beta}\right)} = \sum_{\substack{\alpha_1,\alpha_2\in\mathbb{Z}[i]\\ N(\alpha_1\alpha_2)\le x}} e^{2\pi i \frac{aN(\alpha_1)N(\alpha_2)}{q}} = \sum_{mn\le x} r(m)r(n)e^{2\pi i\frac{amn}{q}} =$$

$$= 2\sum_{d|q}\sum_{\substack{(m,q)=d\\ m\le x^{\frac{1}{2}}}} r(m)\sum_{n\le\frac{x}{m}} r(n)e^{2\pi i\frac{amn}{q}} - \sum_{d|q}\sum_{\substack{m\le x^{\frac{1}{2}}\\ (m,q)=1}}\sum_{n\le x^{\frac{1}{2}}} r(m)r(n)e^{2\pi i\frac{amn}{q}} =$$

$$= 2\sum\nolimits_1 - \sum\nolimits_2,$$

say.

For $(m,q)=d$ let us suppose $m=m_1d$, $q=q_1d$, $(m_1,q_1)=1$. Then, from the Theorem 1, we get

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{d|q}\sum_{m_1\le\frac{x^{\frac{1}{2}}}{d}} r(m_1d)\sum_{n\le\frac{x}{m_1d}} r(n)e^{2\pi i\frac{am_1n}{q_1}} = \\
&= \sum_{d|q}\left[\frac{\pi x}{dq_1}\sum_{m_1\le\frac{x^{\frac{1}{2}}}{d}}\frac{r(m_1d)}{m_1} + O\left(\sum_{\substack{m_1\le\frac{x^{\frac{1}{2}}}{d}\\ (m_1,q_1)=1}}\left(\frac{x}{m_1d}\right)^{\frac{1}{2}+\varepsilon} + q_1\left(\frac{x}{m_1d}\right)^{\varepsilon}\right)\right] = \\
&= \sum_{d|q}\left\{\left[\frac{\pi x}{q}\operatorname*{res}_{s=1}\left(\zeta(s)L(s,\chi_4)\right)\sum_{d_1|d}\chi_4(d_1)\prod_{p|\frac{d}{d_1}}\left(1-\frac{\chi_4(d_1)}{p^s}\right)\frac{x^{s-1}}{s-1}\right] + \right. \\
&\qquad\left. + O\left(\left(\frac{x^{\frac{1}{2}}}{d}\right)^{\frac{1}{3}+\varepsilon}\right) + O\left(x^{\frac{3}{4}+\varepsilon}d^{-1-\varepsilon}\right) + O\left(qx^{\frac{1}{2}+\varepsilon}\right)\right\} = \\
&= A_1(q)\frac{x\log x}{q} + A_2(q)\frac{x}{q} + O\left(x^{\frac{1}{2}+\varepsilon}q\right) + O\left(x^{\frac{3}{4}+\varepsilon}\right),
\end{aligned} \tag{13}$$

where $A_1(q)$, $A_2(q)$ be the computable constants, $q^{-\varepsilon} \ll A_i(q) \ll q^{\varepsilon}$, $i = 1, 2$.

As before, we obtain

$$\begin{aligned}\sum\nolimits_2 = \sum_{d|q} \sum_{\substack{m_1 \le \frac{x^{\frac12}}{d} \\ (m_1,q_1)=1}} r(m) \left(\frac{\pi x^{\frac12}}{q_1} + O\left(x^{\frac14+\varepsilon}\right) + O\left(q_1 x^{\varepsilon}\right)\right) = \\ = B(q)\frac{x}{q} + O\left(x^{\frac34+\varepsilon}\right) + O\left(q_1 x^{\varepsilon}\right),\end{aligned} \tag{14}$$

where $q^{-\varepsilon} \ll B(q) \ll q^{\varepsilon}$.

Collecting out estimates(13)–(14) together, we obtain the desired result of theorem. ∎

Let $\tau_3(\alpha)$ be the number of representations of Gaussian integer α in form $\alpha = \alpha_1\alpha_2\alpha_3$, $\alpha_j \in \mathbb{Z}[j]$. We proved the following statement.

Theorem 3. *Let a and q be the positive integers, $(a, q) = 1$. Then for $x \to \infty$*

$$\sum_{N(\alpha)\le x} \tau_3(\alpha) e\left(\frac{aN(\alpha)}{q}\right) = \frac{x}{q} P_2(\log x) + O\left(x^{\theta_0}\right),$$

where $P_2(u)$ is a polynomial of two degree with the fixed coefficients $\theta_0 = \frac{2-\theta}{3-2\theta}$, $\theta = \frac{1792}{3615}$.

Proof. For $1 \le X < x$ we have

$$\begin{aligned} T_3(x; a, q) &= \sum_{N(\alpha)=x} \tau_3(\alpha) e\left(\frac{aN(\alpha)}{q}\right) = \sum_{\substack{\alpha_1,\alpha_2,\alpha_3 \\ \alpha_1\alpha_2\alpha_3=\alpha \\ N(\alpha_1\alpha_2\alpha_3)\le x}} e\left(\frac{aN(\alpha_1\alpha_2\alpha_3)}{q}\right) = \\ &= \sum_{\substack{\alpha_1,\alpha_2 \\ \alpha_1,\alpha_2=\alpha \\ N(\alpha_1\alpha_2)\le x}} 1 \sum_{\substack{\alpha_3\in\mathbb{Z}[i] \\ N(\alpha_3)\le \frac{x}{N(\alpha_1\alpha_2)}}} e\left(\frac{aN(\alpha_1\alpha_2)N(\alpha_3)}{q}\right) = \\ &= \sum_{N(\alpha)\le X} \tau(\alpha) \sum_{m\le \frac{x}{N(\alpha)}} r(m) e\left(\frac{aN(\alpha)}{q} m\right) + \\ &+ \sum_{X<N(\alpha)\le x} \tau(\alpha) \sum_{X<m\le \frac{x}{N(\alpha)}} r(m) e\left(\frac{aN(\alpha)}{q} m\right) = \\ &= \sum_{N(\alpha)\le X} \tau(\alpha) \sum_{m\le \frac{x}{N(\alpha)}} r(m) e\left(\frac{aN(\alpha)}{q} m\right) + \\ &+ \sum_{m\le \frac{x}{X}} r(m) \sum_{X<N(\alpha)\le \frac{x}{m}} \tau(\alpha) e\left(\frac{aN(\alpha)}{q}\right). \end{aligned}$$

Now the application of theorems 1 and 2 gives

$$\sum_{N(\alpha)\le x}\tau_3(\alpha)e\left(\tfrac{aN(\alpha)}{q}\right)=\sum_{N(\alpha)\le X}\tau(\alpha)\left\{\tfrac{\pi}{q}\cdot\tfrac{x}{N(\alpha)}+O\left(\left(\tfrac{x}{N(\alpha)}\right)^{\frac12+\varepsilon}\right)\right\}+$$

$$+\sum_{m\le\frac{x}{X}}r(m)\left\{\tfrac{C_1(q)}{q}\left(\tfrac{x}{m}\log\tfrac{x}{m}-X\log X\right)+\tfrac{C_2(q)}{q}\left(\tfrac{x}{m}-X\right)+\right.$$

$$\left.+O\left(\left(\tfrac{x}{m}\right)^{\frac34+\varepsilon}\right)+O\left(\left(\tfrac{x}{m}\right)^{\frac12+\varepsilon}q\right)\right\}.$$

In the right side of the first sum above, we will use the following asymptotic formula (see, [1])

$$\sum_{N(\alpha)\le x}\tau(\alpha)=\frac{\pi^2}{16}x\log x+\left(\frac{\pi}{2}L'(1,\chi_4)+\frac{\pi^2\gamma}{8}+\frac{\pi^2}{16}\right)x+O\left(x^{\theta}\right),$$
$$\left(\theta=\frac{1792}{3615}<0,496\right),$$

where $L(s,\chi_4)$ is the Dirichlet L-function with non-principal character mod4, γ is the Euler constant.

Then after simple calculation we deduce

$$\sum_{N(\alpha)\le x}\tau_3(\alpha)e\left(\frac{aN(\alpha)}{q}\right)=\frac{\pi x}{q}P_2(\log x)+O\left(x^{\frac34+\varepsilon}\right)+$$
$$+O\left(x^{\frac12+\varepsilon}X^{\frac12}\right)+O\left(xX^{\theta-1}\right).$$

Putting $X=x^{\frac{1}{3-2\theta}}$, we obtain

$$\sum_{N(\alpha)\le x}\tau_3(\alpha)e\left(\frac{aN(\alpha)}{q}\right)=\frac{\pi x}{q}P_2(\log x)+O\left(x^{\theta_0}\right)+O\left(x^{\frac34+\varepsilon}\right),$$

where $\theta_0=\frac{2-\theta}{3-2\theta}$, $\theta=\frac{1792}{3615}$.

Because, $0.88<\theta_0<0.89$, $\theta_0>\frac34+\varepsilon$, $0<\varepsilon<0.01$, the theorem is proved. ∎

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A. Radova and S. Varbanets
Department of Computer Algebra and Discrete Mathematics
I.I. Mechnikov Odessa National University
Odessa, 65026 Ukraine
radova_as@mail.ru
varb@sana.od.ua