A MULTIPLICATIVE FUNCTION WITH EQUATION
\[ f(p + m^3) = f(p) + f(m^3) \]

Bui Minh Phong (Budapest, Hungary)

Dedicated to the memory of Professor Antal Iványi

Communicated by Imre Kátai
(Received May 10, 2017; accepted June 20, 2017)

Abstract. We prove that if a multiplicative function \( f \) satisfies the conditions
\[
 f(p + m^3) = f(p) + f(m^3) \quad \text{and} \quad f(\pi^2) = f(\pi)^2
\]
for all primes \( p, \pi \) and positive integers \( m \), then \( f(n) = n \) holds for all positive integers \( n \).

1. Introduction

An arithmetic function \( g(n) \neq 0 \) is said to be multiplicative if \((n, m) = 1\) implies that
\[
 g(nm) = g(n)g(m)
\]
and it is completely multiplicative if this relation holds for all positive integers \( n \) and \( m \). Let \( M \) and \( M^* \) denote the class of all complex-valued multiplicative, completely multiplicative functions, respectively.

Let \( P, \mathbb{N} \) be the set of primes, positive integers, respectively. \( n \parallel m \) denotes that \( m \) is a unitary divisor of \( n \), i.e. that \( m|n \) and \( (\frac{n}{m}, m) = 1 \). Let
\[
 M(n) = \max\{q^\gamma : q^\gamma \parallel n, \ q \in \mathcal{P} \}.
\]

Key words and phrases: Multiplicative function, a Dirichlet character, the identity function, functional equation.

2010 Mathematics Subject Classification: 11A07, 11A25, 11N25, 11N64.
In 1992, Spiro [8] showed that if $f \in \mathcal{M}$ and $f(p_0) \neq 0$ for some prime $p_0$, then
\[ f(p + q) = f(p) + f(q) \quad \text{for all } p, q \in \mathcal{P} \]
implies that $f(n) = n$ for all $n \in \mathbb{N}$.

In the paper [3] written with J.-M. De Koninck and I. Kátai, we proved that if $f \in \mathcal{M}$ with $f(1) = 1$ and
\[ f(p + m^2) = f(p) + f(m^2) \quad \text{for all } p \in \mathcal{P}, m \in \mathbb{N}, \]
then $f(n) = n$ for all $n \in \mathbb{N}$. Recently in [7] we improve this result by proving that if $f, g \in \mathcal{M}$ with $f(1) = 1$ satisfy
\[ f(p + m^2) = g(p) + g(m^2) \quad \text{and} \quad g(p^2) = g(p)^2 \]
for all primes $p$ and $m \in \mathbb{N}$, then either
\[ f(p + m^2) = 0, \quad g(p) = -1 \quad \text{and} \quad g(m^2) = 1 \]
for all primes $p$ and $m \in \mathbb{N}$ or
\[ f(n) = n \quad \text{and} \quad g(p) = p, \quad g(m^2) = m^2 \]
for all $p \in \mathcal{P}, n, m \in \mathbb{N}$. The case $f = g \in \mathcal{M}^*$ was investigated in the previous paper [6].

For some generalizations of this topics, we refer to the works mentioned in the references of [7].

In this note, we prove

**Theorem 1.** If $f \in \mathcal{M}$ satisfies the conditions
\begin{align}
(1.1) \quad & f(p + m^3) = f(p) + f(m^3) \quad \text{for all } p \in \mathcal{P}, m \in \mathbb{N} \\
\text{and} \quad & f(\pi^2) = f(\pi)^2 \quad \text{for all } \pi \in \mathcal{P},
\end{align}
then
\[ f(n) = n \quad \text{for all } n \in \mathbb{N}. \]

2. Auxiliary lemmas

**Lemma 1.** We have
\[ \mathcal{S}(3) := \sum_{p \in \mathcal{P}} \frac{1}{p^3} < 0.1747626338. \]
Proof. Let
\[ G(x, 3) := \sum_{p \in \mathcal{P}, \, p \leq x} \frac{1}{p^3}. \]

It is clear that
\[ G(3) - G(x, 3) = \sum_{p \in \mathcal{P}, \, p > x} \frac{1}{p^3} < \sum_{n=\lfloor x \rfloor + 1}^{\infty} \frac{1}{n^3} < \frac{1}{2(x)^2} < \frac{1}{2(x-1)^2}, \]

consequently
\[ G(3) < G(x, 3) + \frac{1}{2(x-1)^2}. \]

One can check with Maple that
\[ G(10^6 + 1, 3) < 0.1747626336, \]

which shows that
\[ G(3) < 0.1747626336 + \frac{1}{2 \cdot 10^{12}} < 0.1747626338. \]

Lemma 1 is proved.

Lemma 2. If \( f \in \mathcal{M} \) satisfies (1.1) and (1.2), then
\[(2.1) \quad f(n) = n \quad \text{for all} \quad n \leq 565 \cdot 10^{10}.\]

Proof. First we prove that (2.1) holds for \( n = 2 \).

Let \( f(2) := x \). Then we infer from (1.1) that
\[
\begin{align*}
    f(3) &= f(2 + 1^3) = x + 1, \\
    f(4) &= f(3 + 1^3) = f(3) + 1 = x + 2, \\
    f(5) &= f(5 + 1^3) - 1 = f(2) f(3) - 1 = x^2 + x - 1, \\
    f(8) &= f(2 + 2^3) - f(2) = f(2) f(5) - f(2) = x^3 + x^2 - 2x, \\
    f(11) &= f(11 + 1^3) - 1 = f(3) f(4) - 1 = x^2 + 3x + 1, \\
    f(19) &= f(11 + 2^3) - 1 = f(11) + f(8) = x^3 + 2x^2 + x + 1, \\
    f(27) &= f(19 + 2^3) = f(19) + f(8) = 2x^3 + 3x^2 - x + 1, \\
    f(29) &= f(2 + 3^3) = f(2) + f(27) = 2x^3 + 3x^2 + 1.
\end{align*}
\]

On the other hand, we also get from (1.1)
\[
\begin{align*}
    f(19) &= f(19 + 1) - 1 = f(4) f(5) - 1 = x^3 + 3x^2 + x - 3
\end{align*}
\]
and
\[ f(29) = f(29 + 1) - 1 = f(2)f(3)f(5) - 1 = x^4 + 2x^3 - x - 1. \]
Thus, we have
\[ x^3 + 2x^2 + x + 1 = x^3 + 3x^2 + x - 3, \quad (x - 2)(x + 2) = 0 \]
and
\[ 2x^3 + 3x^2 + 1 = x^4 + 2x^3 - x - 1, \quad (x - 2)(x^3 + 2x^2 + x + 1) = 0, \]
which imply \( x = 2 \). The assertion (2.1) is proved for \( n = 2 \).

As we seen above, we have
\[ f(n) = n \quad \text{if} \quad n \in \{1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 15, 19, 20, 27\}. \]
These with (1.1) imply
\[ f(7) = f(7 + 1^3) - 1 = f(8) - 1 = 7, \]
\[ f(17) = \frac{f(34)}{f(2)} = \frac{f(34)}{2} = \frac{f(7) + f(27)}{2} = \frac{7 + 27}{2} = 17, \]
\[ f(9) = \frac{f(18)}{f(2)} = \frac{f(17) + 1}{2} = \frac{17 + 1}{2} = 9, \]
\[ f(13) = f(5 + 2^3) = f(5) + f(8) = 13, \]
\[ f(16) = \frac{f(48)}{f(3)} = \frac{f(47) + 1}{3} = \frac{f(55) - 7}{3} = \frac{f(5)f(11) - 7}{3} = 16, \]
and so
\[ f(n) = n \quad \text{for all} \quad n \in \{1, 2, \ldots, 22\}. \]

Now we prove Lemma 2.

Assume by contradiction that there is a number \( Q \in \mathbb{N} \), \( 22 < Q < 565 \cdot 10^{10} \) such that \( f(n) = n \) for all \( n < Q \) and \( f(Q) \neq Q \). Then \( Q = \pi^n \geq 23 \) is a prime power.

If \( \alpha = 1 \), then \( Q + 1 \) is even and \( f(Q + 1) = f(Q) + 1 \neq Q + 1 \). Since \( Q + 1 \) is an even composite, then either \( Q + 1 = 2^e \), \( e \geq 5 \) or \( Q + 1 = uv \) with \( 1 < u < v < Q + 1 \), \( (u, v) = 1 \). If \( Q + 1 = 2^e \), then
\[ Q + 27 = 2(2^{e-1} + 13) = f(2)f(2^{e-1} + 13) = f(Q + 27) = f(Q) + f(27) = f(Q) + 27, \]
because \( 2^{e-1} + 13 < 2^e - 1 = Q \). The last relation is impossible. In the case \( Q + 1 = uv \) with \( 1 < u < v < Q + 1 \), \( (u, v) = 1 \), we also get a contradiction, because
\[ Q + 1 \neq f(Q + 1) = f(uv) = f(u)f(v) = uv = Q + 1. \]
A multiplicative function with equation \( f(p + m^3) = f(p) + f(m^3) \)

If \( \alpha = 2 \), then we obtain from (1.2) that \( Q \neq f(Q) = f(\pi^2) = f(\pi)^2 = \pi^2 = Q \), which is impossible. If \( \alpha = 3\beta \), then
\[
\pi + \pi^\alpha = \pi(1 + \pi^{3\beta-1}) = f(\pi)f(1 + \pi^{3\beta-1}) = f(\pi + \pi^{3\beta}) = f(\pi) + f(\pi^\alpha) = \pi + f(\pi^\alpha),
\]
consequently \( f(Q) = f(\pi^\alpha) = \pi^\alpha = Q \). This is a contradiction.

Assume now \( \alpha \geq 4 \), \( 3 \nmid \alpha \). Then there are 394 such prime powers \( \pi^\alpha \leq 565 \cdot 10^{10} \), for which \( \alpha \geq 4 \) and \( 3 \nmid \alpha \) hold. Since \( \pi^4 \leq \pi^\alpha \leq 565 \cdot 10^{10} \), we have \( \pi \leq 1541 \). With the help of Maple program, for each prime power \( \pi^\alpha \leq 565 \cdot 10^{10} \) there is a positive \( x_{\pi^\alpha} \in \{1, \ldots, 237\} \) (see Table 1 for the smallest value of \( x_{\pi^\alpha} \) which is \( \geq 30 \) ) such that

\[
p_{\pi^\alpha} := \pi^\alpha x_{\pi^\alpha} - 1 \in \mathcal{P}, \quad x_{\pi^\alpha} < \pi^\alpha, \quad (x_{\pi^\alpha}, \pi) = 1
\]

and
\[
M(p_{\pi^\alpha} + 8) = M(\pi^\alpha x_{\pi^\alpha} + 7) < \pi^\alpha.
\]

These with the fact that \( f(n) = n \) for all \( n < Q = \pi^\alpha \) imply
\[
x_{\pi^\alpha} f(\pi^\alpha) = f(x_{\pi^\alpha}) f(\pi^\alpha) = f(\pi^\alpha x_{\pi^\alpha}) = f(p_{\pi^\alpha} + 1) = f(p_{\pi^\alpha}) + 1
\]
and
\[
p_{\pi^\alpha} + 8 = f(p_{\pi^\alpha} + 8) = f(p_{\pi^\alpha}) + 8, \quad f(p_{\pi^\alpha}) = p_{\pi^\alpha}.
\]
Thus \( f(Q) = f(\pi^\alpha) = \pi^\alpha = Q \), which is a contradiction.

Lemma 2 is proved.

Let \( \mathfrak{M} \) be the set of those subset of \( \mathcal{L} \) of \( \mathbb{N} \) for which
\[
n, m \in \mathcal{L}, \quad (n, m) = 1 \Rightarrow nm \in \mathcal{L}.
\]

**Lemma 3.** Assume that \( \mathcal{L} \in \mathfrak{M} \) and an integer \( T \geq 11000 \). If

\[
\pi \in \mathcal{L} \quad \text{for all} \quad \pi \in \mathcal{P}, \quad \pi < T
\]
and
\[
2^\alpha \in \mathcal{L}, \quad 3^\beta \in \mathcal{L}, \quad 5^\gamma \in \mathcal{L} \quad (0 \leq \alpha \leq 10, \quad 0 \leq \beta \leq 8, \quad 0 \leq \gamma \leq 5),
\]
then for each \( Q < T, \) \( 6 \nmid Q \) we have
\[
A_Q := \{p \in \mathcal{P} \mid p < T, \ p + Q \in \mathcal{L}\} \neq \emptyset.
\]

**Proof.** This is Lemma 7 in [7].
Let $B.M. Phong$

This statement is proved in [2].

**Proof.** We shall use the following explicit inequality on the distribution of primes.

\[
\pi(x) < \frac{x}{\log x} + 1.2762 \frac{x}{(\log x)^2} \quad \text{if } x \geq 1.
\]

**Lemma 4.** Let $\pi(x)$ be the number of primes $p \leq x$. We have

\[
\pi(x) < \frac{x}{\log x} + 1.2762 \frac{x}{(\log x)^2} \quad \text{if } x \geq 1.
\]

**Proof.** This statement is proved in [2].

**Lemma 5.** Assume that $\mathcal{H} \in \mathfrak{M}$ and $U$ is a cube-free, $U > 565 \cdot 10^{10}$, $(U, 2) = 1$. If

(a) $\pi \in \mathcal{H}$ for all $\pi \in \mathcal{P}$, $\pi < U$,
A multiplicative function with equation $f(p + m^3) = f(p) + f(m^3)$

(b) $p^2 \in \mathcal{H}$ for all $p \in \mathcal{P}, p < U$

and

(c) $2^\delta \in \mathcal{H}$ $(1 \leq \delta \leq 23)$,

then we have

(2.4) $B_U := \{ n \in \mathbb{N} \mid n < 3\sqrt[3]{U}, U + n^3 \in \mathcal{H} \} \neq \emptyset$.

**Proof.** Assume that the conditions of Lemma 5 are satisfied.

We define the function $\kappa \in \mathcal{M}$ as follows:

$$\kappa(p^\alpha) = \begin{cases} 
1 & \text{if } p = 2 \\
1 & \text{if } p \neq 2, \alpha \leq 2 \\
0 & \text{otherwise.}
\end{cases}$$

Let $h(n) = U + n^3$ and let $H = \lfloor U^{1/3} \rfloor$.

First we consider the congruence

(2.5) $h(x) = x^3 + U \equiv 0 \pmod{\pi^3}, (2 < \pi \leq H, \pi \in \mathcal{P})$.

If $(\pi, U) = 1$, then the congruence (2.5) has at most $\left(3, \pi^2(\pi - 1)\right) \leq 3$ solutions (modulo $\pi^3$). If $(\pi, U) > 1$, then for each solution $n \pmod{\pi^3}$ of the congruence (2.5), we have $\pi \mid n$ and $\pi^3 \mid U$. Since $U$ is cube-free, (2.5) has no solutions in the case $\pi \mid U$. Let

$$U := \sum_{n \equiv 1 \pmod{2}} \kappa(h(n)).$$

We infer from Lemma 1, Lemma 4 and $U > 565 \cdot 10^{10}$ that

$$U \geq \frac{H}{2} - \sum_{3 < \pi \leq \sqrt[3]{H/2}} 3 \left( \frac{H}{2\pi^3} + 1 \right) - 3 \sum_{\sqrt[3]{H/2} < \pi < \sqrt[3]{2U}} 1 \geq$$

$$\geq \left[ \frac{1}{2} - \frac{3}{2} (S(3) - \frac{1}{8}) \right] H - 3\pi(\sqrt[3]{2U}) + 3 \geq$$

$$\geq 0.4253560493H - \frac{3\sqrt[3]{2U}}{\log \sqrt[3]{2U}} - \frac{3c\sqrt[3]{2U}}{(\log \sqrt[3]{2U})^2} + 3 \geq$$

$$\geq \left[ 0.4253560493 - \frac{3\sqrt[3]{2}}{\log \sqrt[3]{2U}} - \frac{3c\sqrt[3]{2}}{(\log \sqrt[3]{2U})^2} \right] \sqrt[3]{U} + 2,$$

where $c := 1.2762$. 
Now we give the upper estimate for
\[ E_\alpha = \sum_{n \leq H, \ n \equiv 1 \ (\text{mod } 2)} h(n) \equiv 0 \ (\text{mod } 2^\alpha). \]

One easily check that if \((U, 2) = 1\), then the congruence
\[ h(n) = U + n^3 \equiv 0 \ (\text{mod } 2^\alpha) \ (\alpha \geq 2) \]
has at most two solutions (modulo \(2^\alpha\)). Thus, we have
\[ E_\alpha \leq 2 \left( \frac{H}{2^\alpha} + 1 \right) < \frac{\sqrt[3]{U}}{2^{\alpha-1}}. \]

Consequently, \( U > 565 \cdot 10^{10} \) gives
\[ U - E_{24} = \sum_{n \equiv 1 \ (\text{mod } 2)} \kappa(h(n)) - \sum_{n \leq H, \ n \equiv 1 \ (\text{mod } 2^4)} 1 \geq \left[ 0.4253560493 - \frac{3\sqrt{2}}{\log \sqrt[3]{2U}} - \frac{3c\sqrt{2}}{(\log \sqrt[3]{2U})^2} - \frac{1}{2^{23}} \right] \sqrt[3]{U} > 0. \]

We may now complete the proof of (2.4).

Since \( U - E_{24} > 0 \), there is a \( n \in \mathbb{N}, \ n^3 < U, \ 2 \nmid n \) such that
\[ U + n^3 = 2^\delta \eta, \ 1 \leq \delta \leq 23, \kappa(\eta) = 1, \eta < U. \]
Since \( U > 565 \cdot 10^{10} \) and \( \delta \leq 23 \), we have \( 1 < \eta < U \) and so by our assumptions, using the conditions (a)-(c) we get
\[ U + n^3 = 2^\delta \eta \in \mathcal{H}, \]
which proves Lemma [5]. \[ \Box \]

**Remark.** By using the method of C. Hooley [4, 5] one can prove that
\[ U = U \prod_{p > 2} \left( 1 - \frac{\rho_U(p^2)}{p^2} \right) + O(U(\log U)^{-1/2}), \]
where \( \rho_U(m) \) is the number of those residues for which \((2n + 1)^3 + U \equiv 0 \ (\text{mod } m)\) and the constant implied by error term is absolute.

Consequently, one can prove the following assertion: There exists a constant \( c_0 \) such that if \( f \in \mathcal{M} \) satisfies the condition (1.1), furthermore \( f(n) = n \) for \( n \leq c_0 \), then \( f(n) = n \) for all \( n \in \mathbb{N} \).

The constant \( c_0 \) is effective, and perhaps \( f(n) = n \ (n \leq c_0) \) holds but too much numerical computation would be necessary.
A multiplicative function with equation \( f(p + m^3) = f(p) + f(m^3) \)

**Lemma 6.** Assume that \( f \in \mathcal{M} \) satisfy (1.1) and (1.2). Then
\[
f(p) = p \quad \text{for all} \quad p \in \mathcal{P}
\]
and
\[
f(m^3) = m^3 \quad \text{for all} \quad m \in \mathbb{N}.
\]

**Proof.** We apply Lemma 3 and Lemma 5 with
\[
\mathcal{L} = \mathcal{H} := \{ n \in \mathbb{N} \mid f(n) = n \}.
\]

Assume that \( f(p) = p \) for all prime \( p < R \), where \( R \in \mathcal{P} \). We may assume that \( R > 565 \cdot 10^{10} \) (see Lemma 2). From (1.2) we have
\[
p, \ p^2 \in \mathcal{L} \quad \text{for all} \quad p \in \mathcal{P}, \ p < R.
\]

Let \( Q := \pi^{3\alpha} < R \), where \( \pi \in \mathcal{P} \) and \( \alpha \in \mathbb{N} \). Since \( 6 \nmid Q \), by applying Lemma 3 with \( T = R \), there is \( p \in \mathcal{P} \), \( p < R \) such that \( p + \pi^{3\alpha} \in \mathcal{L} \), which gives
\[
p + \pi^{3\alpha} = f(p + \pi^{3\alpha}) = f(p) + f(\pi^{3\alpha}) = p + f(\pi^{3\alpha}).
\]
Therefore
\[
f(\pi^{3\alpha}) = \pi^{3\alpha} \quad \text{for all} \quad \pi^{3\alpha} < R,
\]
consequently
\[
f(m^3) = m^3 \quad \text{for all} \quad m \in \mathbb{N}, \ m^3 < R.
\]

Now we apply Lemma 5 with \( U = R \). Then there is an \( m \in \mathbb{N}, \ m^3 < R \) such that \( R + m^3 \in \mathcal{L} \), which with (1.1) and (2.6) gives
\[
R + m^3 = f(R + m^3) = f(R) + f(m^3) = f(R) + m^3.
\]
Consequently \( f(R) = R \), and so \( f(p) = p \) is satisfied for all primes \( p \). The assertion (2.6) is proved.

Finally, the assertion (2.7) follows directly from (2.6) and (3.1).

Lemma 6 is proved. \[\blacksquare\]

### 3. Proof of Theorem 1

From Lemma 6 we obtain that \( f(p) = p \) and \( g(m^3) = m^3 \) for all \( p \in \mathcal{P}, \ m \in \mathbb{N} \). Thus
\[
f(p + m^3) = p + m^3 \quad \text{for all} \quad p \in \mathcal{P}, \ m \in \mathbb{N}.
\]
Let \( E_k(x) \) be the number of those \( n \leq x \) which cannot be written as a sum of a prime and a \( k \)-th power of an integer, and \( n \neq m^k \). Here \( k \geq 2, k \in \mathbb{N} \).

Devenport and Heilbronn proved in [1] that for each \( k \geq 2 \) there is a constant 
\( c = c(k) > 0 \) such that
\[
E_k(x) = O\left( \frac{x}{(\log x)^c} \right).
\]

For us it is enough to know that \( E_3(x)/x \to 0 \ (x \to \infty) \).

Let \( \mathcal{B} \) be the set of those \( n \) which can be written as \( n = p + m^3 \) \((p \in \mathcal{P}, m \in \mathbb{N})\). It follows from (3.1) that \( f(n) = n \) if \( n \in \mathcal{B} \).

Let \( \pi^\alpha \) be an arbitrary prime power. Consider the set of integers \( \pi^\alpha \nu \leq x \), \((\nu, \pi) = 1\). The size of this set is \( \geq \frac{x}{\pi^\alpha} \left( 1 - \frac{1}{\pi} \right) - 1 \). The number of those \( \nu \) for which \( \nu \notin \mathcal{B} \), or \( \pi^\alpha \nu \notin \mathcal{B} \) is \( \leq E(x) + E\left( \frac{x}{\pi^\alpha} \right) \). Thus, if \( x \) is large enough, then we can find such a \( \nu \in \mathcal{B} \) for which \( \pi^\alpha \nu \in \mathcal{B} \) and \((\pi, \nu) = 1\). Consequently
\[
\pi^\alpha \nu = f(\pi^\alpha \nu) = f(\pi^\alpha)f(\nu) = f(\pi^\alpha)\nu,
\]
which proves
\[
f(\pi^\alpha) = \pi^\alpha.
\]
Thus
\[
f(n) = n \quad \text{for all} \quad n \in \mathbb{N}
\]
holds, and so our theorem is proved. \( \blacksquare \)

References

A multiplicative function with equation $f(p + m^3) = f(p) + f(m^3)$


Bui Minh Phong
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C
Hungary
bui@compalg.inf.elte.hu