AN INEQUALITY AND SOME EQUALITIES FOR THE MIDRADIUS OF A TETRAHEDRON

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Dedicated to the memory of Professor Antal Iványi

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Abstract. An inequality containing the circumradius, midradius, and inradius of a tetrahedron will be "experimentally analysed" and proved in two special cases. These refer to the regular triangular pyramid and some tetrahedra having four equal edges. In particular, equalities (like Euler triangle formula in the plane) will be proved.

1. Introduction

For tetrahedra with circumradius R and inradius r, Euler's inequality [9]

$$R \geq 3r$$

holds with equality if and only if the tetrahedron is regular. For the so-called Crelle's tetrahedra having a midsphere (tangent to all the six edges), one also has a midradius ρ. With this the above inequality can be sharpened to

$$R^2 \geq 3\rho^2 \quad \text{and} \quad \rho^2 \geq 3r^2,$$

cf. [10]. Therefore it is natural to ask for the infimum of the function

$$f(R,\rho,r) = \frac{R^2 - 3\rho^2}{\rho^2 - 3r^2} \tag{1.1}$$

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over Crelle's tetrahedra. (From upper side it is not bounded, as seen on a degenerate tetrahedron.) Observe the equivalence

$$f(R,\rho,r) \geq \gamma \iff \frac{R^2+3\gamma r^2}{(3+\gamma)\rho^2} \geq 1.$$

Numerical experiments show that $\inf f \in [6,7)$, at least we get

$$\min \frac{R^2+18r^2}{9\rho^2} = 1$$

for Crelle's tetrahedra, while the corresponding relation for $\gamma = 7$ does not hold. The next conjecture will be proved in section 3 for two special cases.

Conjecture 1.1. *For Crelle's tetrahedra with circumradius R, midradius ρ, and inradius r it holds that*

$$\min \frac{R^2-3\rho^2}{\rho^2-3r^2} \equiv \gamma^* = 6.2145156604... \tag{1.2}$$

To prove it in two special cases, we characterize Crelle's tetrahedra by the help of four parameters, enabling us an algebraic manipulation with Maple.

2. Generating tetrahedra with a midsphere

The midsphere of a tetrahedron is a sphere tangent to every edge. Since not all tetrahedra have a midsphere, it is important to give characterizations. The following is found in [10], (P_i) denote the vertices of the tetrahedron.

Theorem ([10]). $\mathcal{P}$ has the tangent sphere of the edge if and only if x_i exist and satisfy $a_{ij} = x_i + x_j$ for $0 \leq i < j \leq 3$, where $P_iP_j = a_{ij}$.

Note that this theorem is valid for an *existing* tetrahedron. However, if only edge lengths (a_{ij}) are given, further information is necessary. Let us introduce an index-free formalism: denote by a, b, c the side lengths of three edges forming a (basic) triangle, and by a', b', c' the lengths of opposite edges, meeting in a common vertex.

To be concrete, we will use – as a main condition – equations

$$\begin{aligned} a &= y+z, \quad & b &= x+z, \quad & c &= x+y, \\ a' &= x+w, \quad & b' &= y+w, \quad & c' &= z+w, \end{aligned} \tag{2.1}$$

in accordance with the theorem cited above, with positive numbers x, y, z, w.

As a supplementary condition we formulate two requirements, the first of which is motivated by the idea in [8]. Their example

$$a = b = c = a' = b' = 4, \quad c' = 7$$

shows that the triangle inequality holding for the four faces is not sufficient for the six lengths to make up a tetrahedron at all. Their main theorem [8], Theorem 3.1. asserts that a sextuple $\{a, b, c, a', b', c'\}$ defines a tatrahedron (not necessarily a Crelle's one) if and only if the four triangle inequalities hold *and* the Cayley-Menger determinant [3] is positive. However, this can be evoked by *any* of the following geometrical type requirements:

1. no edge of $(a, b, \dots c')$ can be too large,
2. no value of (x, y, z, w) can be too small.

Remark 2.1. To make condition one exact, assume we have to determine the maximum of a'. This is reached when by enlarging a' with the other five edges fixed, the tetrahedron becomes degenerate, giving inequality $a' < a'_{\max}$.

(As a point of interest notice that if for given positive x, y, z, w the Cayley-Menger determinant with edges (2.1) is negative then always exactly *three* of the inequalities $a < a_{\max},\ b < b_{\max}, \dots, c' < c'_{\max}$ will be violated, namely those with sides meeting at a common vertex.)

The next lemma gives the formula for the maximum of a' by means of the edge lengths – and subsequently by the generating parameters.

Lemma 2.1. *With the above notations we have*

$$a'^2 \leq a'^2_{\max} \equiv \frac{p + \sqrt{q}}{2a^2},$$

where

$$p = a^2(b^2 + c^2 + b'^2 + c'^2 - a^2) + (b^2 - c^2)(b'^2 - c'^2),$$

$$q = \big((b+c)^2 - a^2\big)\big(a^2 - (b-c)^2\big)\big((b'+c')^2 - a^2\big)\big(a^2 - (b'-c')^2\big).$$

If the tetrahedron has a midsphere – and hence is generated by parameters x, y, z, w –, then the following equivalent formulae also are valid:

$$p = (y^2 - z^2) + 2(y+z)^2\big(x^2 + w^2 + (x+w)(y+z)\big) - (y-z)^2(2x+y+z)(2w+y+z),$$

$$q = 256(x+y+z)xy^2z^2w(y+z+w).$$

Proof. Assume the edge lengths a, b, c, b', c' are fixed, and enlarge a' to get a plane figure – a quadrilateral – with sides b, c, b', c' and diagonals a and $a'_{\max}$. Calculating the area of this quadrilateral by Bretschneider's formula [2] on the one hand, and by Heron's formula applied to $\Delta\{a, b, c\}$ and $\Delta\{a, b', c'\}$ on the other hand, the equality of these areas yields $a'_{\max}$. ■

Remark 2.2. As regards condition two, take e.g. the triangle with sides a, b, c, draw the circles centered at the vertices with radii x, y, z (where x is the radius of the circle opposite to side a, etc.), and determine the radius $\underline{w}$ of the Soddy circle [7]. Then the inequality obtained for w reads $w > \underline{w}$. This value of $\underline{w}$ is given below – the remaining three formulae are similar.

Lemma 2.2. *For a Crelle's tetrahedron with fixed generating parameters (x, y, z) the minimal value of w is*

$$\underline{w} = \frac{xyz}{xy + yz + zx + 2\sqrt{xyz(x + y + z)}}.$$

Proof. The proof runs as indicated in the above remark. Note that the quantities $\{a, b, c\}$ and $\{x, y, z\}$ are in one-to-one correspondence. ■

Figure 2.1 shows an isosceles triangle of sides $a = 6$, $b = c = 5$ generated by $x = 2$, $y = z = 3$. The center of the Soddy circle is $(0, \frac{8}{5})$, its radius is $\frac{2}{5}$.

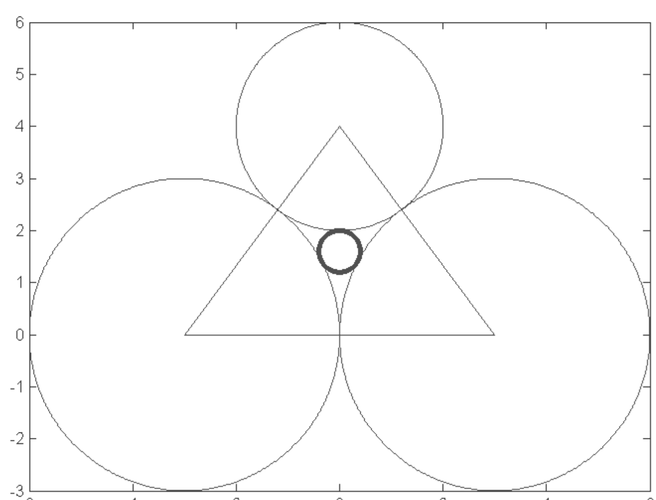

Figure 2.1. The Soddy circle for $x = 2$, $y = z = 3$.

One can employ either of Lemma 2.1 or Lemma 2.2 to guarantee an edge-touching tetrahedron.

3. The proof in two special cases

For Crelle's tetrahedra there is a one-to-one correspondence between the four generating parameters and the six edge lengths, symbolized as

$$(x, y, z, w) \Longleftrightarrow \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}.$$

(Direction $\Rightarrow$ is trivial, while $\Leftarrow$ follows by observing that the underlying system of linear equation has a unique solution.)

To make our task more manageable, we restrict ourselves to problems with only *one* free parameter. First consider the possible two parameter-families

$$(x, x, x, w) \Longleftrightarrow \begin{pmatrix} a & a & a \\ a' & a' & a' \end{pmatrix}, \text{ and } (x, x, w, w) \Longleftrightarrow \begin{pmatrix} a & a & c \\ a & a & c' \end{pmatrix}.$$

By homogeneity of (1.1), one of the two parameters can be chosen to be unity, thus we have in fact cases $(1, 1, 1, w)$ and $(x, x, 1, 1)$ to be examined.

Type 1: (1, 1, 1, w). It is easy to see that these parameters generate a regular triangular pyramid (in short: RTP) with $a = b = c = 2$, while the family includes the regular tetrahedron for $w = 1$.

First we report on some interesting numerical experiments. Attempts to minimize function f for variable w result in the *almost* optimal arguments

$$\left(1, 1, 1, \frac{1}{1+\sqrt{3}}\right).$$

Calculating ratio (1.1) for these values yields

$$\frac{1}{1336}\left(558 + 2013\sqrt{3} + (42\sqrt{3} + 1602)\sqrt{3 + 2\sqrt{3}}\right) \approx 6.2145156617...,$$

a thorough investigation however reveals that this is not the optimum!

Theorem 3.1. *For a Crelle's tetrahedron of type* 1 (*i.e. for an RTP*) *with circumradius* R, *midradius* ρ, *and inradius* r *it holds that*

$$\min_w \frac{R^2 - 3\rho^2}{\rho^2 - 3r^2} = \gamma^* = 6.2145156604...$$

Proof. By means of Maple (1.1) can be factored to be

$$\frac{1}{4}\,\frac{\left(\sqrt{3} + 3\sqrt{(w+2)w}\right)^2 (w^2 + 6w + 1)(w-1)^2}{8\sqrt{3w(w+2)}w^2 + 3w^4 - 12w^3 - 26w^2 + 12w - 1}.$$

Removing the singularity at $w = 1$ yields

$$f = \frac{\left(\sqrt{3} + 3\sqrt{(w+2)w}\right)^2 (w^2 + 6w + 1) * \mathrm{conj}}{-4(w^2 - 10w + 1)(3w^2 + 6w - 1)^2},$$

where

$$\mathrm{conj} = (-8\sqrt{3w(w+2)}w^2 + 3w^4 - 12w^3 - 26w^2 + 12w - 1$$

is the conjugate of the original denominator. Factoring the derivative of f gives (among others) a sixth degree polynomial

$$w^6 - 10w^5 - 73w^4 - 84w^3 + 39w^2 - 2w + 1$$

with a positive zero $w^* \approx 0.366033$, which at the same time minimizes the function f. This value is surprisingly close to $\frac{1}{1+\sqrt{3}} \approx 0.366025$. However the minimum value $\gamma^* \approx 6.2145156604$ is a very little less than $6.2145156617\ldots$ obtained for $w = \frac{1}{1+\sqrt{3}}$ – a remarkable fact! ■

Type 2: (x, x, 1, 1). The edge lengths become then by (2.1)

$$a = a' = b = b' = x + 1, \quad c = 2x, \ c' = 2. \tag{3.1}$$

Hence this case is characterized by the fact that two pairs of opposite edges are of the same length, while the remaining two (opposite) sides may be different. Curiously, the attributive 'opposite' can be omitted.

Lemma 3.2. *Crelle's tetrahedra with four equal edges are type 2 tetrahedra. Moreover, the remaining two edges are perpendicular skew lines.*

Proof. To become four equal edges, we have following possibilities:

$$\alpha : \begin{pmatrix} a & a & c \\ a & a & c' \end{pmatrix}, \quad \beta : \begin{pmatrix} a & a & c \\ a & b' & a \end{pmatrix}, \quad \gamma : \begin{pmatrix} a & a & a \\ a & b' & c' \end{pmatrix}.$$

Solving $z + y = x + z = x + w = z + w = a$ as well as solving $z + y = x + z = = x + y = x + w = a$ gives gives $x = y = z = w$, therefore in cases β and γ the tetrahedron is necessarily regular. Thus tetrahedra, different from regular, are found only in family α, which was to be shown. The second assertion follows by elementary observations. ■

Theorem 3.3. *For a Crelle's tetrahedron of type* 2 *with circumradius R, midradius ρ, and inradius r it holds that*

$$\min_w \frac{R^2 - 3\rho^2}{\rho^2 - 3r^2} = 9. \tag{3.2}$$

Proof. *The function f to be minimized now assumes*

$$\frac{(x^2+6x+1)(x-1)^2\left(\sqrt{x+2}+\sqrt{x(2x+1)}\right)^2}{8x^2\left(\sqrt{x(x+2)(2x+1)}+x^2-5x+1\right)}.$$

The singularity at $x=1$ can be removed as well to yield

$$\frac{(x^2+6x+1)\left(\sqrt{x+2}+\sqrt{x(2x+1)}\right)^2\left(x^2-5x+1-\sqrt{x(x+2)(2x+1)}\right)}{8(x^2-10x+1)},$$

a convex function for $x>0$, with global minimum 9 at $x=1$. ∎

Remark 3.1. The formula for the midradius ρ is in this case very simple: $2\rho^2=x$, or, using generating parameters (x,x,y,y) instead of $(x,x,1,1)$,

$$\rho^2=\frac{xy}{2}. \tag{3.3}$$

This motivated us to formulate the statement separately, too.

Lemma 3.4. *Let A and B be opposite points on sphere S of radius ρ. Assume A and B are midpoints of the segments A_1A_2 and B_1B_2 resp., and these determine perpendicular skew lines touching S at points A and B. Now, if $\overline{AA_1}*\overline{BB_1}=$ $=2\rho^2$, then S is the midsphere of tetrahedron $A_1A_2B_1B_2$.*

As an illustration for type 2, see Figure 3.1.

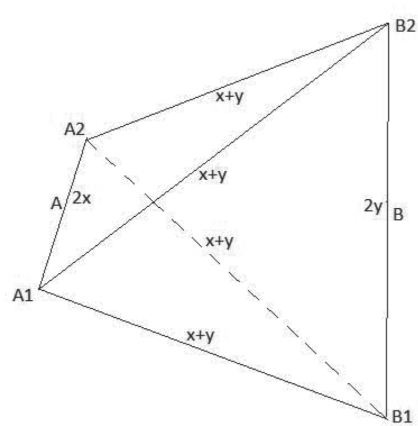

Figure 3.1. Type 2 tetrahedron with parameters (x,x,y,y).
Segments A_1A_2 and B_1B_2 are perpendicular.

The following geometric interpretation gives another proof for (3.3), avoiding in this way long computations.

Lemma 3.5. *A type 2 tetrahedron is obtained from a (right) truncated square pyramid (in short: TSP) in the following way. Let $ABCDA'B'C'D'$ be a TSP with base square $\{A, B, C, D\}$, top square $\{A', B', C', D'\}$, and slant edges AA', BB', CC', DD'.*

Then $AB'CD'$ (and also $A'BC'D$) is a type 2 tetrahedron, and (3.3) holds.

Proof. It suffices to show the validity of (3.3). Let the lengths of edges AC and $B'D'$ be $2x$ and $2y$ resp. Then the four edges AB', AD', CB', CD' have lengths $x + y$. Analyze first the facial trapezoid $ABB'A'$ with bases of side lengths $\sqrt{2}x$, $\sqrt{2}y$ resp., see Figure 3.2.

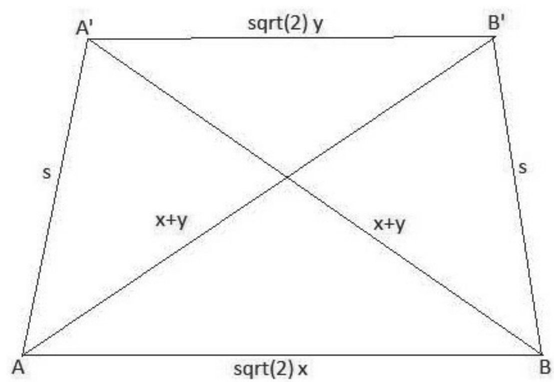

Figure 3.2. Isosceles trapezoid: one of the 4 congruent faces of the TSP.

The diagonals and legs are of length $x+y$ and $s = \overline{AA'} = \overline{BB'}$, whence the law of cosines yields

$$s^2 = (x + y)^2 + 2x^2 - 2\sqrt{2}(x + y)x \cos\alpha,$$
$$s^2 = (x + y)^2 + 2y^2 - 2\sqrt{2}(x + y)y \cos\alpha,$$

giving

$$s^2 = x^2 + y^2.$$

On the other hand, consider another trapezoid $ACC'A'$ with bases $2x, 2y$, legs s and height 2ρ. (This trapezoid is inside of the TSP and halves it.) Applying Pythagoras' theorem for the right triangle APA', where $A'P$ is the altitude of the trapezoid gives

$$s^2 = (x - y)^2 + 4\rho^2.$$

The last two equations imply $2\rho^2 = xy$, which was to be proved. ∎

4. Identities – as in the plane!

While for triangles Euler's identity

$$d^2 = R(R - 2r)$$

is valid with circumradius R, inradius r and distance d between the centers of the circles, a similar equality formula for tetrahedra does not hold [1]. This makes the following theorem interesting: the Crelle's tetrahedra just examined still satisfy some *equality* conditions!

Theorem 4.1. *For type* 1 *tetrahedra it holds that*

$$d^2 = R\big(R - \sqrt{3}\rho\big), \tag{4.1}$$

while for type 2 *tetrahedra we have*

$$d^2 = \big(\sqrt{R^2 + \rho^2} - \rho\big)^2 - \rho^2. \tag{4.2}$$

Here $d = |\mathrm{cir} - \mathrm{mid}|$ *is the distance between the circumcentre and midcentre.*

Equality $d = 0$ *holds if and only if* $R = \sqrt{3}\rho$, *in correspondence with* [10].

Proof. (A) Type 1 tetrahedra. Let a (not necessarily equal to 2) be the length of the side of the regular basic triangle, and h be the height of the tetrahedron. Denote by s the lengths of the three sides meeting at the apex, i.e. let

$$s = \sqrt{h^2 + \frac{a^2}{3}}.$$

Take the basic triangle to be horizontal and let its center be the origin. The third coordinate of the circumcenter and the midcenter are calculated to be

$$\mathrm{cir} = \frac{h^2 - \frac{a^2}{3}}{2h}, \quad \mathrm{mid} = \frac{a(3s - 2a)}{6h}$$

with a distance

$$d = |\mathrm{cir} - \mathrm{mid}| = \frac{s(s - a)}{2h}$$

between them. The circumradius and midradius become

$$R = \frac{s^2}{2h} \quad \text{and} \quad \rho = \frac{a(2s - a)}{2h\sqrt{3}}\, 2h.$$

Now we are able to find a relation between R, ρ and d. First we have

$$\frac{R}{d} = \frac{s}{s-a}, \quad \text{whence} \quad s = \frac{aR}{R-d} \quad \text{and} \quad 2s - a = a\,\frac{R+d}{R-d}.$$

Substituting this s (squared) and $2s-a$ into

$$\frac{R}{\rho} = \frac{\sqrt{3}s^2}{a(2s-a)}$$

proves the statement for type 1.

(B) Type 2 tetrahedra. With the parametrization in Remark 3.1 we have

$$R^2 = \frac{(x^2+y^2)(x^2+4xy+y^2)}{8xy}, \quad \rho^2 = \frac{xy}{2}.$$

Choosing the origin at point $(A+B)/2$, lying on the line defined by A and B (cf. Figure 3.1), we have mid=0 as well as

$$\text{cir} = \frac{y^2-x^2}{4\rho}, \quad d = |\text{cir} - \text{med}| = \frac{|y^2-x^2|}{4\rho},$$

and hence

$$d^2 = \frac{(x^2+y^2)^2 - 4x^2y^2}{16\rho^2}.$$

Since $8xy = 16\rho^2$, all three quantities R, ρ, d only depend on $t \equiv x^2+y^2$. We get

$$16\rho^2 R^2 = t(t+8\rho^2), \quad 16\rho^2 d^2 = t^2 - 16\rho^4,$$

whence, by eliminating t, the statement follows. ■

Example 4.2. We give now a type 2 tetrahedron with fairly simple data. Let the generating parameters be $(2,2,1,1)$. The opposite perpendicular edges are of lengths 4 and 2, while there are four edges of the same length 3. The circumradius is $R = \frac{\sqrt{65}}{4}$, the midradius equals just $\rho = 1$, while the inradius is $r = \frac{2}{3}(\sqrt{2}+\sqrt{5})$. The distance between the circumcentre and midcentre amounts to $d = \frac{3}{4}$. Since

$$R^2 + \rho^2 = \left(\frac{9}{4}\right)^2,$$

the validity of (4.2) is easily checked.

Remark 4.1. Interestingly, two further identities are valid for Crelle's tetrahedra of type 1, which however do not hold for tetrahedra of type 2.

Lemma 4.3. *For regular triangular pyramids it holds that*

$$(\mathrm{cir} - \mathrm{inc})^2 = (R - r)^2 - 4r^2, \tag{4.3}$$

which is the equality form of the Grace-Danielsson inequality [6], [4] *and*

$$(\mathrm{mid} - \mathrm{inc})^2 = \rho * (\rho - \sqrt{3}r), \tag{4.4}$$

a natural companion of (4.1). *Here cir, mid and inc stand for the circum-, mid-, and the incentre.*

Proof. We give the formulae needed. For an RTP with base side $a = 2$, the circumradius, midradius and inradius are (as functions of height h):

$$R = \frac{3h^2 + 4}{6h}, \quad \rho = \frac{2\left(\sqrt{3h^2+4} - \sqrt{3}\right)}{3h}, \quad r = \frac{h}{1 + \sqrt{3h^2+1}},$$

while the circumcenter, midcenter and incenter assume

$$\mathrm{cir} = \frac{3h^2 - 4}{6h}, \quad \mathrm{mid} = \frac{\sqrt{9h^2+12} - 4}{3h}, \quad \mathrm{inc} = \frac{h}{1 + \sqrt{3h^2+1}}.$$

(Equality r=inc is a consequence of choosing the origin in the centre of the basic triangle.) Identities (4.3) and (4.4) then immediately follow. ■

Problem. Since the minimum of function f in Theorem 3.3 is a natural number (namely 9), we may ask for an immediate (geometrical) proof of (3.2)!

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