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ON SIMULTANEOUS NUMBER SYSTEMS WITH 3 BASES

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Dedicated to the memory of Professor Antal Iványi

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Abstract. Simultaneous number systems with 2 bases were studied before in different structures. One way of generalization is using 3 bases instead of 2. In this paper we consider the ring of integers of the quadratic field $\mathbb{Q}[\sqrt{5}]$. The difficulty is the amount of triplets we have to check to validate the number system property, so the development of a software which can check these sets in acceptable times (using parallelization, GPU computing) was necessary. The existence of simultaneous number systems with three bases was confirmed by our software for some concrete choices of bases and digit sets.

1. Introduction

Simultaneous expansions were studied before in different structures: in the ring of rational integers by Indlekofer, Kátai and Racskó in [1], in the ring of Gaussian integers by Nagy in [8] and by Kovács in [3],[4], in the ring of Eisenstein integers by Kovács in [5], of the real numbers by Komornik and Pethő in [2]. Indlekofer, Kátai and Racskó examined for what N_1, N_2 will $(-N_1, -N_2, \mathcal{A}_c)$ be a simultaneous number system, where $2 \leq N_1 < N_2$ rational integers and $\mathcal{A}_c = \{0, 1, \ldots, N_1N_2 - 1\}$.

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Definition 1. The triplet $(-N_1, -N_2, \mathcal{A}_c)$ is called a simultaneous number system, if there exists $d_j \in \mathcal{A}_c$ $(j = 0, 1, \ldots, k)$ for all n_1, n_2 rational integers such that

$$n_1 = \sum_{j=0}^{k} d_j (-N_1)^j, \quad n_2 = \sum_{j=0}^{k} d_j (-N_2)^j.$$

The result of Indlekofer, Kátai and Racskó is the following:

Theorem 1. *$(-N_1, -N_2, \mathcal{A}_c)$ is a simultaneous number system if and only if $N_2 = N_1 + 1$.*

Analogous definition can be formulated for simultaneous number systems of Gaussian integers:

Definition 2. Let α_1 and α_2 be Gaussian integers and let $\mathcal{A}$ be a proper digit set. The triplet $(\alpha_1, \alpha_2, \mathcal{A})$ is called a simultaneous number system, if there exists $d_j \in \mathcal{A}$ $(j = 0, 1, \ldots, k)$ for all $z_1, z_2 \in \mathbb{Z}[i]$ such that:

$$z_1 = \sum_{j=0}^{k} d_j \alpha_1^j, \quad z_2 = \sum_{j=0}^{k} d_j \alpha_2^j.$$

It is easy to prove (see [8]) that if $(\alpha_1, \alpha_2, \mathcal{A})$ is a simultaneous number system of Gaussian integers, then the difference of the bases is unit, namely $\alpha_1 - \alpha_2 \in \{\pm 1, \pm i\}$. Also in [8] Nagy proved that the Gaussian integers with absolute value greater than 14.61 can always serve as base for a simultaneous number system using a digit set construction based on K-type digit sets. Kovács proved in [3] that except 43 cases the Gaussian integers can always serve as base for a simultaneous number system using the dense digit set (a dense digit set consists of elements with the smallest norm from each congruent class). Moreover he showed in [4] which Gaussian integers can not serve as a base for a simultaneous number system for any digit set, and gave proper digit sets for the capable ones.

A possible way of generalization is considering number systems with 3 bases.

Definition 3. Let $Z_1, Z_2, Z_3 \in \mathbb{Z}[i]$ and let $\mathcal{A}$ be a proper digit set. The quadrille $(Z_1, Z_2, Z_3, \mathcal{A})$ is called a simultaneous number system, if there exists $d_j \in \mathcal{A}$ $(j = 0, 1, \ldots, k)$ for all $b_1, b_2, b_3 \in \mathbb{Z}[i]$ such that:

$$b_1 = \sum_{j=0}^{k} d_j Z_1^j, \quad b_2 = \sum_{j=0}^{k} d_j Z_2^j, \quad b_3 = \sum_{j=0}^{k} d_j Z_3^j.$$

Unfortunately the necessity condition that the pairwise difference of the bases must be unit means that there is no simultaneous number system of the Gaussian integers with 3 bases.

If we try to solve the equation $u_1 + 1 = u_2$ in the ring of integers of a real quadratic field $\mathbb{Q}[\sqrt{D}]$, where u_1, u_2 are units, we get that if it is solvable, then $D \equiv 1\ (4)$, moreover it has solution for $D = 5$. So our goal was to find simultaneous number system with 3 bases in the ring of integers of $\mathbb{Q}[\sqrt{5}]$.

Let I be the ring of integers of $\mathbb{Q}[\sqrt{5}]$. Every element of I can be written in the form $a_1 + a_2\omega$, where $a_1, a_2 \in \mathbb{Z}$ and $\omega = \frac{1+\sqrt{5}}{2}$, so the elements of I can be represented by pairs of rational integers. From now on $(a_1, a_2) \in \mathbb{Z}^2$ means the number $a_1 + a_2\omega \in \mathbb{Q}[\sqrt{5}]$, and $||a_1 + a_2\omega|| = a_1^2 + a_1a_2 - a_2^2$.

We examined the cases where the bases are in the following form:

1. (a, b), $(a + 1, b)$, $(a + 1, b + 1)$;
2. (a, b), $(a, b + 1)$, $(a + 1, b + 1)$;
3. (a, b), $(a - 1, b)$, $(a - 1, b + 1)$;
4. (a, b), $(a, b + 1)$, $(a - 1, b + 1)$.

The definition of simultaneous number system with 3 bases in this case is analogous to the previous one.

We can use the dense digit set, or we can create a proper digit set based on the K-type digit set similarly to the case of 2 bases:

Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be K-type digit sets belonging to given $Z_1, Z_2, Z_3 \in I$. Define $\mathcal{A}$ in the following way:

$$\mathcal{A}_{1,2} := \bigcup_{d \in \mathcal{A}_2} (\mathcal{A}_1 + dZ_1),$$
$$\mathcal{A} := \bigcup_{d \in \mathcal{A}_{1,2}} (\mathcal{A}_3 + dZ_3).$$

A dense digit set consists of elements with the minimal norm from each congruent class. We construct one by checking the elements of I on a spiral starting from 0: $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$,... We start with $\mathcal{A} = \emptyset$ and if there is no $d \in \mathcal{A}$ from the same congruent class as the examined element e, then we expand $\mathcal{A}$ with e.

Example 1. A dense digit set belonging to $(-1, 3)$ is

$$\mathcal{A} = \{(-1,-1),\ (-1,0),\ (-1,1),\ (0,-1),\ (0,0),\ (0,1),\ (1,-1),\ (1,0),\ (1,1),\ (2,-1),\ (2,0)\}.$$

Let $Z_1, Z_2, Z_3 \in I$ and let $\mathcal{A}$ be a proper digit set. For all $z_1, z_2, z_3 \in I$ there uniquely exists $d \in \mathcal{A}$ such that $Z_1 | z_1 - d$ and $Z_2 | z_2 - d$ and $Z_3 | z_3 - d$.

Let

$$J(z_1,z_2,z_3)=J(z_1,z_2,z_3,d)=\left(\frac{z_1-d}{Z_1},\frac{z_2-d}{Z_2},\frac{z_3-d}{Z_3}\right)=$$
$$=(J_1(z_1,d),J_2(z_2,d),J_3(z_3,d))=(J_1(z_1),J_2(z_2),J_3(z_3)).$$

With the notations $Z_1=(z_{11},z_{12})$, $z_1=(a_1,b_1)$, $d=(d_1,d_2)$ and $J_1(z_1)=$ $=(a_2,b_2)$ we get that:

$$a_1+b_1\frac{1+\sqrt5}{2}=d_1+d_2\frac{1+\sqrt5}{2}+\left(a_2+b_2\frac{1+\sqrt5}{2}\right)\cdot Z_1,$$

$$a_2+b_2\frac{1+\sqrt5}{2}=\frac{a_1-d_1+(b_1-d_2)\frac{1+\sqrt5}{2}}{z_{11}+z_{12}\frac{1+\sqrt5}{2}}=$$
$$=\frac{(a_1-d_1)+\frac{b_1-d_2}{2}+\frac{b_1-d_2}{2}\sqrt5}{z_{11}+\frac{z_{12}}{2}+\frac{z_{12}}{2}\sqrt5}\cdot\frac{z_{11}+\frac{z_{12}}{2}-\frac{z_{12}}{2}\sqrt5}{z_{11}+\frac{z_{12}}{2}-\frac{z_{12}}{2}\sqrt5}=$$
$$=\frac{(a_1-d_1)z_{11}+\frac{(a_1-d_1)z_{12}}{2}+\frac{(b_1-d_2)z_{11}}{2}+\frac{(b_1-d_2)z_{12}}{4}-5\frac{(b_1-d_2)z_{12}}{4}}{z_{11}^2+\frac{z_{12}^2}{4}+z_{11}z_{12}-\frac54 z_{12}^2}+$$
$$+\sqrt5\cdot\frac{\frac{(b_1-d_2)z_{11}}{2}+\frac{(b_1-d_2)z_{12}}{4}-\frac{(a_1-d_1)z_{12}}{2}-\frac{(b_1-d_2)z_{12}}{4}}{||Z_1||}=$$
$$=\frac{(a_1-d_1)z_{11}+(a_1-d_1)z_{12}-(b_1-d_2)z_{12}}{||Z_1||}+$$
$$+\frac{\sqrt5+1}{2}\cdot\frac{(b_1-d_2)z_{11}-(a_1-d_1)z_{12}}{||Z_1||}.$$

Let K be the maximum of the absolute values of the coordinates of the elements of $\mathcal{A}$ (so $d_1,d_2\in[-K,K]$). The solutions of the equations

$$L_1=\frac{(K+L_1)|z_{11}|+(K+L_1)|z_{12}|+(K+L_1)|z_{12}|}{|||Z_1|||}$$

and

$$L_1=\frac{(K+L_1)|z_{11}|+(K+L_1)|z_{12}|}{|||Z_1|||}$$

are

$$L_{11}=\frac{K(|z_{11}|+2|z_{12}|)}{|||Z_1|||-|z_{11}|-2|z_{12}|}$$

and

$$L_{12}=\frac{K(|z_{11}|+|z_{12}|)}{|||Z_1|||-|z_{11}|-|z_{12}|}$$

respectively, so with the notation $M_1 = \max\{L_{11}, L_{12}\}$ and assuming $a_1, b_1 \in \in [-M_1, M_1]$ we get that $a_2, b_2 \in [-M_1, M_1]$ as well and if (z_1, z_2, z_3) is a periodic element, then the coordinates of z_1 are in $[-M_1, M_1]$. Similarly we can calculate M_2 and M_3 which are upper bounds for the absolute values of the coordinates of z_2 and z_3 respectively.

This means that all the periodic elements are in a finite set defined by M_1, M_2 and M_3. Unfortunately the size of the set for bases with small norm is already so large that checking its elements feels hopeless with existing programs. So we developed a new one.

2. Description of the program and the methods used in it

The main problem was quickly finding the digit sequence corresponding to a number triplet while checking for non-trivial periodic elements. This is accomplished by the iteration of the previously defined J_i functions. In every iteration step, the following system of congruences must be solved.

$$d \equiv z_1 \pmod{Z_1},$$

$$d \equiv z_2 \pmod{Z_2},$$

$$d \equiv z_3 \pmod{Z_3}.$$

On average, $\frac{|\mathcal{A}|}{2}$ elements from $\mathcal{A}$ need to be tried (checking every element of the digit set until a solution is found), which is too slow for our purposes. Instead, it is faster to solve the following congruences:

$$d_1 \equiv z_1 \pmod{Z_1},$$

$$d_2 \equiv z_2 \pmod{Z_2},$$

$$d_3 \equiv z_3 \pmod{Z_3},$$

where d_1 is an element of a complete residue system modulo Z_1, and d_2 and d_3 belong to complete residue systems modulo Z_2 and Z_3, respectively. Z_1, Z_2 and Z_3 are pairwise relative primes, so d_1, d_2 and d_3 determines the single element of $\mathcal{A}$ (d) which solves the previous system of congruences. The cardinality of a complete residue system modulo Z_i is $||Z_i||$, while the cardinality of $\mathcal{A}$ is $|||Z_1|| \cdot ||Z_2|| \cdot ||Z_3|||$. This is a significant speed-up in calculations, since it is sufficient to try on average $|\frac{||Z_1||}{2}| + |\frac{||Z_2||}{2}| + |\frac{||Z_3||}{2}|$ solutions instead of $|\frac{||Z_1||\cdot||Z_2||\cdot||Z_3||}{2}|$. This process can be further optimized by code generation: since $\mathcal{A}$ and the residue systems modulo the bases do not change during the calculations, we can pre-compute lookup tables for all values of d_1, d_2 and d_3 using perfect hashing.

This solution can be applied to all digit set constructions. It has the advantage of high speed and relatively simple implementation, however for every change in the bases or the digit set construction the program needs to be recompiled. We note that there are other fast methods for this challenge [6].

After we found out that number systems with 3 bases exist we also wanted to determine whether we can say something about the suitable bases. In this case we used a probabilistic method: if we check a relatively large amount of randomly chosen number triplets for every base, we can state whether the base triplets form a simultaneous number system with high confidence, since the probability of false results goes down with the amount of number triplets checked. If a non-trivial periodic element is found, we can say that the base triplets do not form a number system with the given digit set construction. If no non-trivial periodic element is found then we assume that we found a number system.

The method is the same as before, however since the base triplets and digit sets change for every case and the amount of number triplets that need to be checked using a base triplet is smaller (only a small, randomly chosen part of the full set is checked), it is not feasible to recompile the program every time the base triplets are changed. This means that code generation and lookup-tables are not suitable, the data structures need to be built dinamically at runtime. This way, more time is needed for the computations, but this did not cause problems since the amount of number triplets were smaller compared to checking a full set for a single base triplet and digit set construction. As distance increases from the origin, generally the norms of the bases increase too, which means larger digit sets that need more time to generate and more memory to store. Soon a state was reached where we could not continue the computations with the available resources. Fortunately, we could still inspect enough base triplets for our purposes. This means approximately one thousand base triplets with K-type digit set and a few hundred with dense digit set. The mapped area is approximately a square centered on the origin, the base triplets were generated starting from the origin and progressing outwards in a spiral path.

If the base triplet and digit set construction is known, we can determine the maximal and minimal coordinate values that can occur during the iteration of the J_i functions based on the previously stated estimation for the set of number triplets to be checked. This is important for assuring the correctness of our results, since if the machine's integer type can represent these values then we can be sure that no under- or overflow occurs during the computation. Of course, this is only possible if the estimation is applicable for the given base triplet. If the estimation fails (zero division appears in the formula), we can't determine the set of number triplets to be checked, so we can't verify the existence of a number system in those cases. Because of this we've chosen only such base triplets where the estimation is applicable.

Let $a = (a_1, a_2)$ be an element of the set of number triplets to be checked, and $J_i(a) = (b_1, b_2)$. Based on the previously stated estimation, (b_1, b_2) is also in the set. From the estimation, it follows that if the absolute values of the coordinates of the elements of the set are smaller than $M \in \mathbb{N}^+$ and M is representable as a machine integer, then the only possible cause for overflow is the evaluation of J_i. Since

$$J_i(a) = \frac{a-d}{Z_i} \ (d = (d_1, d_2) \in \mathcal{A}),$$

it is sufficient to give an upper bound for values occuring during the subtraction and the division. Let D be the maximal absolute value of coordinates in $\mathcal{A}$ and B the maximal absolute value of coordinates in the base triplets. Then for the $a - d$ difference we can say that

$$-(M + D) \leq a_1 - d_1, a_2 - d_2 \leq M + D$$

and for the norms of the bases

$$-2B^2 \leq ||Z_i|| \leq 2B^2$$

is true. The formula for division is the following:

$$\frac{k}{l} = \frac{(k_1, k_2)}{(l_1, l_2)} = \left(\frac{k_1 l_1 + k_1 l_2 - k_2 l_2}{||l||}, \frac{k_2 l_1 - k_1 l_2}{||l||} \right).$$

Using the previous two bounds:

$$-3(M + D)B \leq (a_1 - d_1)z_{i_1} + (a_1 - d_1)z_{i_2} - (a_2 - d_2)z_{i_2} \leq 3(M + D)B$$

and

$$-2(M + D)B \leq (a_2 - d_2)z_{i_1} - (a_1 - d_1)z_{i_2} \leq 2(M + D)B.$$

This is sufficient, since we know that the result of the division will be an element of the set of number triplets, so its coordinates are bounded by M. This means that if $\pm \max\{2B^2, 3(M + D)B\}$ is representable by machine integers, then there will be no under- or overflow during the computation.

3. Results

We successfully verified the existence of simultaneous number systems with 3 bases in I, both with dense and K-type digit sets. The base triplets were selected based on the size of the number set that needed to be checked and in accordance with the bounds on the smallest and largest numbers occurring during the computation (in order to avoid erronous results due to integer under- or overflow). The results are given in Table 1.

Bases	**Digit set construction**	**Checked triplets**	**Runtime**	**Max. value**
$(-1,3)$, $(0,3)$, $(0,4)$	dense (1 584 digits)	42 487 515 625	~16 hours	864
$(-1,4)$, $(0,4)$, $(0,5)$	dense (7 600 digits)	379 205 945 209	~70 hours	2265
$(-5,0)$, $(-4,0)$, $(-4,1)$	K-type (4 400 digits)	24 008 572 809	~4 hours	1875
$(-6,0)$, $(-5,0)$, $(-5,1)$	K-type (17 100 digits)	31 739 204 025	~8 hours	2592
$(-6,0)$, $(-6,1)$, $(-5,1)$	K-type (19 836 digits)	412 853 796 369	~28 hours	3438
$(-5,-1)$, $(-5,0)$, $(-6,0)$	K-type (26 100 digits)	23 315 152 249	~7 hours	2484
$(-6,-1)$, $(-6,0)$, $(-5,0)$	K-type (36 900 digits)	64 650 690 225	~27 hours	3078
$(6,0)$, $(5,0)$, $(5,1)$	K-type (26 100 digits)	26 629 344 225	~8 hours	3556
$(5,0)$, $(6,0)$, $(6,1)$	dense (36 900 digits)	13 509 180 441	~6 hours	2358
$(-6,0)$, $(-5,0)$, $(-5,1)$	dense (17 100 digits)	9 957 844 521	~4 hours	2124
$(-5,0)$, $(-6,0)$, $(-6,1)$	K-type (26 100 digits)	60 908 759 209	~22 hours	2898
$(-6,-1)$, $(-5,-1)$, $(-5,0)$	dense (29 725 digits)	34 784 861 049	~12 hours	2538
$(5,0)$, $(5,1)$, $(6,1)$	dense (29 725 digits)	34 784 861 049	~13 hours	2538
$(6,0)$, $(5,0)$, $(5,1)$	dense (26 100 digits)	9 006 959 025	~3 hours	2124
$(-5,-1)$, $(-5,0)$, $(-6,0)$	dense (26 100 digits)	9 006 959 025	~5 hours	2124
$(6,-1)$, $(6,0)$, $(5,0)$	dense (26 100 digits)	13 198 563 225	~7 hours	2232

Table 1: Simultaneous number systems with 3 bases in $\mathbb{Q}[\sqrt{5}]$. The last column contains the largest possible (in absolute value) number which can occur during the calculations. No under- or overflows will occur if this can be represented by the machine's integer type (this is not a problem in small cases like these). Runtimes are not directly proportional with the number of triplets to be checked. This is caused by different sized digit sets and changes in the computing environment (hardware).

Regarding the position on the plane of the suitable bases 1000 base-triplets were checked with K-type digit set and 180 with dense digit set for all four base-triplet constructions. Generally the size of the digit sets and the amount of time needed grows with the absolute value of the coordinates of the base triplets, with more computational resources it will be possible to check more triplets with larger digit sets. The first bases of the triplets were generated starting from the origin and progressing outwards in a spiral, and the second and third bases were determined by the first base using the following schemes:

1. (a,b), $(a+1,b)$, $(a+1,b+1)$;
2. (a,b), $(a,b+1)$, $(a+1,b+1)$;
3. (a,b), $(a-1,b)$, $(a-1,b+1)$;
4. (a,b), $(a,b+1)$, $(a-1,b+1)$.

For every base triplet 10 000 number triplets were checked, the results are shown on Figure 1 and Figure 2 (the plots only show the position of the first base for every base triplet, since it uniquely determines the other two bases in the triplet).

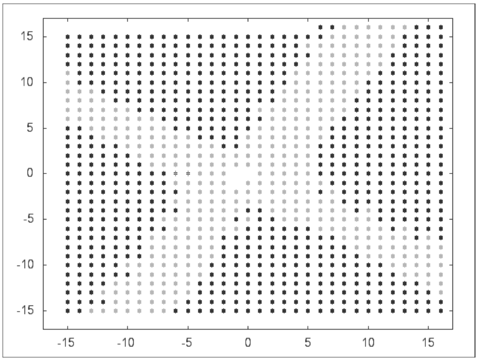

Figure 1: Simultaneous number system bases with K-type digit set. A grey lattice point (a,b) in the picture denotes that the bases $(a,b),(a+1,b),(a+1,b+1)$ do not form a number system with the K-type digit set. A black lattice point (a,b) in the picture denotes that the bases $(a,b),(a+1,b),(a+1,b+1)$ probably form a number system with the K-type digit set. A striped lattice point (a,b) in the picture denotes that the bases $(a,b),(a+1,b),(a+1,b+1)$ form a number system with the K-type digit set. Similar picture belongs to the other 3 base schemes.

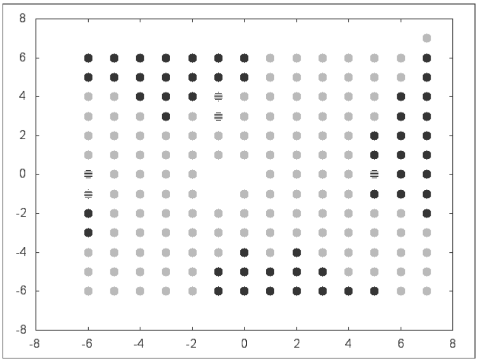

Figure 2: Simultaneous number system bases with dense digit set. A grey lattice point (a,b) in the picture denotes that the bases $(a,b),(a+1,b),(a+1,b+1)$ do not form a number system with the dense digit set. A black lattice point (a,b) in the picture denotes that the bases $(a,b),(a+1,b),(a+1,b+1)$ probably form a number system with the dense digit set. A striped lattice point (a,b) in the picture denotes that the bases $(a,b),(a+1,b),(a+1,b+1)$ form a number system with the dense digit set. Similar picture belongs to the other 3 base schemes.

Compared to Figure 3 we can formulate the following:

Conjecture 1. *If the norms of the bases are big enough, then they form a number system both with the K-type and dense digit sets.*

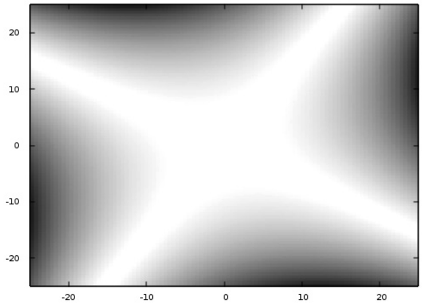

Figure 3: The absolute value of $||a + b\omega||$ for (a, b). Darker colour denotes greater value.

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