ON CONJECTURE CONCERNING THE FUNCTIONAL EQUATION

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Dedicated to the memory of Professor Antal Iványi

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Abstract. We determine all solutions of those \( f : \mathbb{N} \to \mathbb{C} \) for which
\[
f(n^2 + Dm^2) = f^2(n) + Df^2(m)
\]
is satisfied for all positive integers \( n, m \), where \( D \) is a given positive integer. This solves a problem of Kátai and Phong.

1. Introduction

Let, as usual, \( \mathcal{P}, \mathbb{N}, \mathbb{Z}, \mathbb{C} \) be the set of primes, positive integers, integers and complex numbers, respectively.

In 1992, C. Spiro \[15\] proved that if a multiplicative function \( f : \mathbb{N} \to \mathbb{C} \) satisfies the relations
\[
f(p_0) \neq 0 \quad \text{for some} \quad p_0 \in \mathcal{P}
\]
and
\[
f(p + q) = f(p) + f(q) \quad \text{for every} \quad p, q \in \mathcal{P},
\]
then \( f(n) = n \) for all \( n \in \mathbb{N} \).

In 1997 J.-M. De Koninck, I. Kátai and B. M. Phong \[5\] proved that if a multiplicative function \( f : \mathbb{N} \to \mathbb{C} \) satisfies the relation
\[
f(p + n^2) = f(p) + f(n^2) \quad \text{for every} \quad p \in \mathcal{P}, \ n \in \mathbb{N},
\]
then $f$ is the identity function. K.-H. Indlekofer and B. M. Phong [9] proved that if $k \in \mathbb{N}$, $f \in \mathcal{M}$ satisfy $f(2)f(5) \neq 0$ and $f(n^2 + m^2 + k + 1) = f(n^2 + 1) + f(m^2 + k)$ for all $n, m \in \mathbb{N}$, then $f(n) = n$ for all $n \in \mathbb{N}$, $(n, 2) = 1$.

For some generalizations of the above results, we refer the other works of P. V. Chung [2], B. M. Phong [10], [11], [12].

In 2014 B. Bojan determined all solutions of those $f: \mathbb{N} \to \mathbb{C}$ for which

\[ f(n^2 + m^2) = f^2(n) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}. \]

Our purpose in this paper is to prove a conjecture of Kátai and Phong [3].

**Theorem.** Assume that the number $D \in \mathbb{N}$ and the arithmetical function $f: \mathbb{N} \to \mathbb{C}$ satisfy the equation

\[ f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}. \]

Then one of the following assertions holds:

a) $f(n) = 0$ for every $n \in \mathbb{N}$,

b) $f(n) = \frac{\epsilon(n)}{D + 1}$ for every $n \in \mathbb{N}$,

c) $f(n) = \epsilon(n)n$ for every $n \in \mathbb{N}$,

where $E := \{n^2 + Dm^2 \mid n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$.

It is proved earlier for the case $D = 2, 3$ in [4] and for $D = 4, 5$ in [14] (see also [10]).

**Corollary 1.** Assume that the number $D \in \mathbb{N}$ and a multiplicative function $f: \mathbb{N} \to \mathbb{C}$ satisfy the equation (1.1). Then

\[ f(n) = \epsilon(n)n \quad \text{for every} \quad n \in \mathbb{N}, \]

where $E := \{n^2 + Dm^2 \mid n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$.

**Corollary 2.** Assume that the number $k \in \mathbb{N}, k \geq 2$ and an arithmetic function $f: \mathbb{N} \to \mathbb{C}$ satisfy the relation

\[ f(n_1^2 + n_2^2 + \cdots + n_k^2) = f^2(n_1) + f^2(n_2) + \cdots + f^2(n_k) \]
for all \( n_1, n_2, \ldots, n_k \in \mathbb{N} \). Then one of the following assertions holds:

a) \( f(n) = 0 \) for every \( n \in \mathbb{N} \),

b) \( f(n) = \frac{\epsilon(n)}{k} \) for every \( n \in \mathbb{N} \),

c) \( f(n) = \epsilon(n)n \) for every \( n \in \mathbb{N} \),

where \( F := \{ n_1^2 + n_2^2 + \cdots + n_k^2 \mid n_1, n_2, \ldots, n_k \in \mathbb{N} \} \), \( \epsilon(n) = 1 \) if \( n \in F \) and \( \epsilon(n) \in \{-1, 1\} \) if \( n \in \mathbb{N} \setminus F \).

We note that Corollary 2 is proved by Park in [8] and [9] for multiplicative function and recently by Lee in [7] for general arithmetic functions.

2. Lemmas

In this section we assume that the function \( F, G : \mathbb{N} \to \mathbb{C} \) and the numbers \( D \in \mathbb{N}, U \in \mathbb{C}, U \neq 0 \) satisfy the relation

\[
F(n^2 + Dm^2) = G(n) + UG(m) \quad \text{for every } n, m \in \mathbb{N}.
\]

**Lemma 1.** Assume that the function \( F, G : \mathbb{N} \to \mathbb{C} \) and the numbers \( D \in \mathbb{N}, U \in \mathbb{C}, U \neq 0 \) satisfy (2.1). Then

\[
G(\ell + 12m) = G(\ell + 9m) + G(\ell + 8m) + G(\ell + 7m) - G(\ell + 5m) - G(\ell + 4m) - G(\ell + 3m) + G(\ell)
\]

holds for every \( \ell, m \in \mathbb{N} \) and

\[
\begin{align*}
G(7) &= 2G(5) - G(1) \\
G(8) &= 2G(5) + G(4) - 2G(1) \\
G(9) &= G(6) + 2G(5) - G(2) - G(1) \\
G(10) &= G(6) + 3G(5) - G(3) - 2G(1) \\
G(11) &= G(6) + 4G(5) - G(3) - G(2) - 2G(1) \\
G(12) &= G(6) + 4G(5) + G(4) - G(2) - 4G(1)
\end{align*}
\]

**Proof.** We note from (2.1) that

\[
F(x^2 + Dy^2) = G(|x|) + UG(|y|) \quad \text{for every } x, y \in \mathbb{Z} \setminus \{0\}.
\]

First we prove the following assertion:

\[
G(n + 2m) - G(|n - 2m|) = G(2n + m) - G(|2n - m|)
\]

for every \( n, m \in \mathbb{N}, n \neq 2m, m/2 \).
Assume that the numbers \(n, m \in \mathbb{N}\) satisfy the conditions \(n \neq 2m, n \neq m/2\). If \(Dn - 2m \neq 0\), then we infer from (2.4) and the next relations

\[
(Dn + 2m)^2 + D(n - 2m)^2 = (Dn - 2m)^2 + D(n + 2m)^2
\]

and

\[
(Dn + 2m)^2 + D(2n - m)^2 = (Dn - 2m)^2 + D(2n + m)^2
\]

that

\[
G(Dn + 2m) + UG\left(|n - 2m|\right) = G(|Dn - 2m|) + UG\left(n + 2m\right)
\]

and

\[
G(Dn + 2m) + UG\left(2n - m\right) = G(|Dn - 2m|) + UG\left(2n + m\right).
\]

These prove (2.5) in the case \(Dn - 2m \neq 0\).

If \(Dn - 2m = 0\), then \(2Dn - m \neq 0\). In this case, we infer from (2.4) and the next relations

\[
(2Dn + m)^2 + D(n - 2m)^2 = (2Dn - m)^2 + D(n + 2m)^2
\]

and

\[
(2Dn + m)^2 + D(2n - m)^2 = (2Dn - m)^2 + D(2n + m)^2
\]

that

\[
G(2Dn + m) + UG\left(|n - 2m|\right) = G(|2Dn - m|) + UG\left(n + 2m\right)
\]

and

\[
G(2Dn + m) + UG\left(2n - m\right) = G(|2Dn - m|) + UG\left(2n + m\right).
\]

These prove (2.5) in the case \(Dn - 2m = 0\), and so (2.5) is proved.

Applications of (2.5) in the cases \((n, m) \in \{(1, 3); (2, 3); (1, 4); (1, 5); (3, 4); (2, 5)\}\) prove that (2.3) holds for \(G(7), G(8), G(9), G(11), G(10)\) and \(G(12)\). Thus, (2.3) is proved.

Now we prove (2.2).

By applying (2.5) with \(n = \ell + 2m\), we have

\[
G(2\ell + 5m) - G(2\ell + 3m) = G(\ell + 4m) - G(\ell) \quad \text{for every } \ell, m \in \mathbb{N}.
\]

This shows that

\[
G(\ell + 12m) - G(\ell) = G(2\ell + 15m) - G(2\ell + 9m)
\]
and
\[ G(2\ell + 15m) - G(2\ell + 9m) = \left[ G\left(2\ell + 5m + 5m\right) - G\left(2\ell + 5m + 3m\right)\right] + \\
+ \left[ G\left(2\ell + 4m + 5m\right) - G\left(2\ell + 4m + 3m\right)\right] + \\
+ \left[ G\left(2\ell + 3m + 5m\right) - G\left(2\ell + 3m + 3m\right)\right] = \\
= \left[ G\left(\ell + 9m\right) - G\left(\ell + 5m\right)\right] + \\
+ \left[ G\left(\ell + 8m\right) - G\left(\ell + 4m\right)\right] + \\
+ \left[ G\left(\ell + 7m\right) - G\left(\ell + 3m\right)\right], \]

which prove (2.2).

Lemma 1 is proved. \[ \qed \]

In the proof of the next lemma we shall follow a method in part similar to the one used in the proof of Lemma 2 of the paper [13].

**Lemma 2.** Assume that the function \( F, G : \mathbb{N} \to \mathbb{C} \) and the numbers \( D \in \mathbb{N}, U \in \mathbb{C}, U \neq 0 \) satisfy (2.1). Let
\[
A := \frac{1}{120} \left( G(6) + 4G(5) - G(3) - G(2) - 3G(1) \right),
\]
\[
\Gamma_2 := -\frac{1}{8} \left( G(6) - 4G(5) + 4G(4) - G(3) + 3G(2) - 3G(1) \right),
\]
\[
\Gamma_3 := -\frac{1}{3} \left( G(6) - 2G(5) + 2G(3) - G(2) \right),
\]
\[
\Gamma_4 := \frac{1}{4} \left( G(6) - 2G(4) - G(3) + G(2) + G(1) \right),
\]
\[
\Gamma_5 := \frac{1}{5} \left( G(6) - G(5) - G(3) - G(2) + 2G(1) \right),
\]
\[
\Gamma := \frac{1}{4} \left( G(6) - 4G(5) + 2G(4) + 3G(3) + G(2) + G(1) \right),
\]
\[
B_k := \Gamma_2 \chi_2(k) + \Gamma_3 \chi_3(k) + \Gamma_4 \chi_4(k - 1) + \Gamma_5 \chi_5(k) + \Gamma,
\]

where \( \chi_2(k) \) (mod 2), \( \chi_3(k) \) (mod 3) are the principal Dirichlet characters and \( \chi_4(k) \) (mod 4), \( \chi_5(k) \) (mod 5) are the real, non-principal Dirichlet characters, i.e.
\[
\chi_2(0) = 0, \chi_2(1) = 1, \chi_3(0) = 0, \chi_3(1) = \chi_3(2) = 1,
\]
\[
\chi_4(0) = \chi_4(2) = 0, \chi_4(1) = 1, \chi_4(3) = -1,
\]
\[
\chi_5(2) = \chi_5(3) = -1, \chi_5(1) = \chi_5(4) = 1.
\]

Then we have
\[
G(\ell) = A\ell^2 + B_\ell \quad \text{for every} \quad \ell \in \mathbb{N}.
\]
**Proof.** By computation, we proved that (2.6) holds for 1 \leq k \leq 12.

Assume that \(G(k) = Ak^2 + B_k\) holds for \(\ell \leq k \leq \ell + 11\), where \(\ell \geq 1\). By applying (2.2) with \(m = 1\), we have

\[
G(\ell + 12) = G(\ell + 9) + G(\ell + 8) + G(\ell + 7) - G(\ell + 5) - G(\ell + 4) - G(\ell + 3) + G(\ell) = A[(\ell + 9)^2 + (\ell + 8)^2 + (\ell + 7)^2 - (\ell + 5)^2 - (\ell + 4)^2 - (\ell + 3)^2 + \ell^2] + \left[ B_{\ell+9} + B_{\ell+8} + B_{\ell+7} - B_{\ell+5} - B_{\ell+4} - B_{\ell+3} + B_{\ell} \right] = A(\ell + 12)^2 + B_{\ell+12},
\]

which proves that (2.6) holds for \(\ell + 12\) and so it is true for all \(\ell\). In the last relation we have used

\[
B_{\ell+9} + B_{\ell+8} + B_{\ell+7} - B_{\ell+5} - B_{\ell+4} - B_{\ell+3} + B_{\ell} = \Gamma_2 \left[ \sum_{k=\ell+6}^{\ell+9} \chi_2(k) - \sum_{k=\ell+3}^{\ell+6} \chi_2(k) + \chi_2(\ell) \right] + \Gamma_3 \left[ \sum_{k=\ell+3}^{\ell+9} \chi_3(k) - \sum_{k=\ell+3}^{\ell+7} \chi_3(k) + \chi_3(\ell) \right] + \Gamma_4 \left[ \sum_{k=\ell+3}^{\ell+9} \chi_4(k-1) - \sum_{k=\ell+3}^{\ell+6} \chi_4(k-1) + \chi_4(\ell - 1) \right] + \Gamma_5 \left[ \sum_{k=\ell+6}^{\ell+10} \chi_5(k) - \sum_{k=\ell+2}^{\ell+6} \chi_5(k) - \chi_5(\ell + 10) + \chi_5(\ell + 2) + \chi_5(\ell) \right] = \Gamma_2 \chi_2(\ell) + \Gamma_3 \chi_3(\ell) + \Gamma_4 \chi_4(\ell - 1) + \Gamma_5 \chi_5(\ell + 2) + \Gamma = \Gamma_2 \chi_2(\ell + 12) + \Gamma_3 \chi_3(\ell + 12) + \Gamma_4 \chi_4(\ell + 11) + \Gamma_5 \chi_5(\ell + 12) + \Gamma = B_{\ell+12}.
\]

Lemma 2 is proved.

3. **Proof of the Theorem**

Assume that the numbers \(D \in \mathbb{N}\) and the arithmetical function \(f : \mathbb{N} \to \mathbb{C}\) satisfy the equation (1.1), that is

\[
f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.
\]
Let $G(n) := f^2(n)$ for every $n \in \mathbb{N}$ and $U = D$. We shall use the notations of Lemmas 1-2. From (2.6) we have
\[ G(\ell) = f^2(\ell) = A\ell^2 + B\ell \quad \text{for every} \quad \ell \in \mathbb{N}, \]
where
\[ B_\ell := \Gamma_2\chi_2(\ell) + \Gamma_3\chi_3(\ell) + \Gamma_4\chi_4(\ell - 1) + \Gamma_5\chi_5(\ell) + \Gamma. \]
Consequently, we obtain from (1.1) that
\[ G(n^2 + Dm^2) = f^2(n^2 + Dm^2) = \left(G(n) + DG(m)\right)^2, \]
and so (2.6) implies
\[ (3.1) \quad A(n^2 + Dm^2)^2 + Bn^2 + Dm^2 = \left(A(n^2 + Dm^2) + Bn + DB_m\right)^2 \]
for every $n, m \in \mathbb{N}$. Since
\[ |B_\ell| \leq |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma| \quad \text{for every} \quad \ell \in \mathbb{N} \]
and
\[ n^2 + Dm^2 \to \infty \quad \text{as} \quad n, m \to \infty, \]
we infer from (3.1) that
\[
A = \lim_{n,m \to \infty} \left[A + \frac{Bn^2 + Dm^2}{(n^2 + Dm^2)^2}\right] = \\
= \lim_{n,m \to \infty} \left(A + \frac{Bn + DB_m}{n^2 + Dm^2}\right)^2 = A^2.
\]
Therefore, we have $A \in \{0, 1\}$.

**Case I. $A = 1$.** From (3.1) we obtain that
\[ (n^2 + Dm^2)^2 + Bn^2 + Dm^2 - \left((n^2 + Dm^2) + Bn + DB_m\right)^2 = \\
= (-2B_n - 2DB_m)n^2 + W(n, m) = 0,\]
holds for every $n, m \in \mathbb{N}$, where
\[ W(n, m) := Bn^2 + Dm^2 - B_n^2 - D^2B_m^2 - \\
-2D^2m^2B_m - 2Dm^2B_n - 2DB_nB_m.\]

Now let $m \in \mathbb{N}$ be fixed, $n \in \mathbb{N}$, $n \equiv a \pmod{60}$ with some $a \in \mathbb{N}$, $0 \leq a < 60$. Then $B_n = B_a$ and
\[ |W(n, m)| < \infty. \]
and so we obtain from (3.2) that
\[ B_n + DB_m = \lim_{n \equiv a \pmod{60}} \frac{-W(n,m)}{2n^2} = 0, \]
consequently
\[ B_m = \frac{-B_a}{D} = c \quad \text{for every} \quad m \in \mathbb{N}, \tag{3.4} \]
where \( c \in \mathbb{C} \) is some fixed constant. This shows that
\[ c(D+1) = 0 \quad \text{and} \quad c = 0. \tag{3.5} \]
This proves
\[ B_m = c = 0 \quad \text{and} \quad G(m) = m^2 \quad \text{for every} \quad m \in \mathbb{N} \]
and in the case \( A = 1 \), we proved that
\[ f^2(m) = m^2 \quad \text{and} \quad f(n^2 + Dm^2) = f^2(n) + Df^2(m) = n^2 + Dm^2 \]
for every \( n, m \in \mathbb{N} \).

The part (c) of the theorem is proved.

**Case II.** \( A = 0 \). In this case, we have
\[ T(n,m) := B_{n^2+Dm^2} - \left(B_n + DB_m\right)^2 = 0 \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{3.6} \]
We prove now that
\[ \Gamma_4 = 0. \tag{3.7} \]
Assume that \( \Gamma_4 \neq 0 \). By applying (3.6) with \( T(2, 2) \) and \( T(30, 30) \), we obtain that
\[ B_{4+4D} = (\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma + D(\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma))^2 \]
and
\[ B_0 = (\Gamma_4 + \Gamma + D(\Gamma_4 + \Gamma))^2. \]
Consequently
\[ T(8, 8) = 4(D+1)^2\Gamma_4(-\Gamma_5 + \Gamma + \Gamma_3) = 0 \]
and
\[ T(60, 60) = 4\Gamma_4\Gamma(D+1)^2 = 0, \]
which with \((D + 1)^2 \Gamma_4 \neq 0\) imply that

\[
\Gamma = 0 \quad \text{and} \quad \Gamma_5 - \Gamma_3 = 0.
\]

Finally, we obtain (3.6) that

\[
0 = T(2, 8) = B_{2^2 + 8} - \left( B_2 + DB_8 \right)^2 = B_{4 + 4D} - \left( B_2 + DB_8 \right)^2 = \\
\left( \Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma + D(\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma) \right)^2 - \\
\left( \Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma + D(\Gamma_3 - \Gamma_4 - \Gamma_5 + \Gamma) \right)^2 = \\
\left( \Gamma_4(D + 1) \right)^2 - \left( \Gamma_4(D - 1) \right)^2 = 4D \Gamma_4^2.
\]

This is impossible, because \(\Gamma_4 \neq 0\). Thus, (3.4) is proved.

In the next part, we assume that \(\Gamma_4 = 0\), and so

\[
B_k := \Gamma_2 \chi_2(k) + \Gamma_3 \chi_3(k) + \Gamma_5 \chi_5(k) + \Gamma \quad \text{for every} \quad k \in \mathbb{N},
\]

furthermore

\[
B_k = B_{k + 30} \quad \text{for every} \quad k \in \mathbb{N}.
\]

Since

\[
T(30, 30) = \Gamma - (\Gamma + D\Gamma)^2 = 0,
\]

consequently

\[
\Gamma \in \{0, \frac{1}{(D + 1)^2}\}.
\]

**Lemma 3.** Assume that (3.6) and (3.8) hold. If \(\Gamma = 0\), then \(B_n = 0\) for every \(n \in \mathbb{N}\).

**Proof.** We deduce from \(\Gamma = 0\), (3.6) and (3.8) that

\[
B_{30} = \Gamma_2 \chi_2(30) + \Gamma_3 \chi_3(30) + \Gamma_5 \chi_5(30) + \Gamma = 0
\]

and

\[
B_{n^2} - B_n^2 = B_{n^2 + 30^2} - \left( B_n + D B_{30} \right)^2 = T(n, 30) = 0 \quad \text{for every} \quad n \in \mathbb{N}.
\]

Since

\[
B_{n^2} = \Gamma_2 \chi_2(n^2) + \Gamma_3 \chi_3(n^2) + \Gamma_5 \chi_5(n^2) = \Gamma_2 \chi_2(n) + \Gamma_3 \chi_3(n) + \Gamma_5 \chi_5(n),
\]
we have
\[ \Gamma_2 \chi_2^2(n) + \Gamma_3 \chi_3^2(n) + \Gamma_5 \chi_5^2(n) = \left( \Gamma_2 \chi_2(n) + \Gamma_3 \chi_3(n) + \Gamma_5 \chi_5(n) \right)^2 \]
holds for every \( n \in \mathbb{N} \). This with \( n = 2, 3, 5, 6, 10, 15 \) gives the following equations
\[
\begin{cases}
\Gamma_3 + \Gamma_5 = (\Gamma_3 - \Gamma_5)^2 \\
\Gamma_2 + \Gamma_5 = (\Gamma_2 - \Gamma_5)^2 \\
\Gamma_2 + \Gamma_3 = (\Gamma_2 + \Gamma_3)^2 \\
\Gamma_2^2 = \Gamma_5 \\
\Gamma_3^2 = \Gamma_3 \\
\Gamma_5^2 = \Gamma_2.
\end{cases}
\]
Solve this systems of equations, the solutions \((\Gamma_2, \Gamma_3, \Gamma_5)\) are
\[
(\Gamma_2, \Gamma_3, \Gamma_5) \in \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}.
\]
Now we prove that \((\Gamma_2, \Gamma_3, \Gamma_5) = (0, 0, 0)\). Assume that \((\Gamma_2, \Gamma_3, \Gamma_5) \neq (0, 0, 0)\).

Then
\[ B_k := \chi_i(k) \quad (i = 2, 3, 5), \]
and by applying (3.6) for the case \( n = m = 1 \), we have
\[ B_{D+1} = (B_1 + DB_1)^2, \]
which implies
\[ \chi_i(D + 1) = (D + 1)^2. \]
This is impossible, because \( 1 \geq |\chi_i(D + 1)| = (D + 1)^2 \geq 4. \)

Lemma 3 is proved.

Thus we proved the part (a) of the theorem.

\[ \square \]

**Lemma 4.** Assume that (3.6) and (3.8) hold. If
\[ \Gamma = \frac{1}{(D + 1)^2}, \]
then \( B_n = \Gamma = \frac{1}{(D + 1)^2} \) for every \( n \in \mathbb{N} \).

**Proof.** We shall prove that \( \Gamma_2 = \Gamma_3 = \Gamma_5 = 0. \)

We infer from (3.6) that
\[ T(6, 30) = B_{6^2 + D.30^2} - \left( B_6 + D.B_{30} \right)^2 = B_6 - \left( B_5 + \frac{D}{(D + 1)^2} \right)^2 = 0 \]
and 
\[ T(12, 30) = B_{12^2 + D, 30^2} - \left( B_{12} + D.B_{30} \right)^2 = B_{24} - \left( B_{12} + \frac{D}{(D + 1)^2} \right)^2 = 0. \]

Since

\[ B_6 = B_{24}, \]

\[ B_6 = \Gamma_5 + \frac{1}{(D + 1)^2}, B_6 + \frac{D}{(D + 1)^2} = \Gamma_5 + \frac{1}{D + 1} \]

and

\[ B_{12} = -\Gamma_5 + \frac{1}{(D + 1)^2}, B_{12} + \frac{D}{(D + 1)^2} = -\Gamma_5 + \frac{1}{D + 1}, \]

we obtain that

\[ (D\Gamma_5 - D + \Gamma_5 + 1)\Gamma_5 = 0, \quad (D\Gamma_5 - D + \Gamma_5 - 3)\Gamma_5 = 0. \]

These relations show that \( \Gamma_5 = 0. \) Thus, we have

\[ B_k = \Gamma_2\chi_2(k) + \Gamma_3\chi_3(k) + \frac{1}{(D + 1)^2} \]

and

\[ B_{k+6} = B_k \]

hold for \( k \in \mathbb{N}. \) By using (3.6) for \( (n, m) = (1, 6), (2, 6), (3, 6), \) we have

\[ T(1, 6) = B_{1^2 + D, 6^2} - \left( B_1 + DB_6 \right)^2 = B_1 - \left( B_1 + \frac{D}{(D + 1)^2} \right)^2 = \]

\[ = -\frac{(\Gamma_2 + \Gamma_3)(D\Gamma_2 + D\Gamma_3 - D + \Gamma_2 + \Gamma_3 + 1)}{D + 1} = 0, \]

\[ T(2, 6) = B_{2^2 + D, 6^2} - \left( B_2 + DB_6 \right)^2 = B_4 - \left( B_2 + \frac{D}{(D + 1)^2} \right)^2 = \]

\[ = -\frac{(D\Gamma_3 - D + \Gamma_3 + 1)\Gamma_3}{D + 1} = 0 \]

and

\[ T(3, 6) = B_{3^2 + D, 6^2} - \left( B_3 + DB_6 \right)^2 = B_4 - \left( B_1 + \frac{D}{(D + 1)^2} \right)^2 = \]

\[ = -\frac{(D\Gamma_2 - D + \Gamma_2 + 1)\Gamma_2}{D + 1} = 0. \]

Solving the above system of equations, we obtain

\[ (\Gamma_2, \Gamma_3) \in \{(0, 0), \left( \frac{D - 1}{D + 1}, 0 \right), \left( 0, \frac{D - 1}{D + 1} \right) \}. \]
Let \( H := \frac{D-1}{D+1} \). If \((\Gamma_2, \Gamma_3) \neq (0, 0)\), then \( B_k = H\chi_i(k) + \frac{1}{(D+1)^2} \quad (i = 2, 3) \).

Therefore

\[
T(1, 1) = B_{i+1}^2 + D_{i+1}^2 - \left( B_1 + DB_1 \right)^2 =
\]

\[
= H\chi_i(D + 1) + \frac{1}{(D+1)^2} - (D+1)^2(H + \frac{1}{(D+1)^2})^2 =
\]

\[
= H\chi_i(D + 1) + \frac{1}{(D+1)^2} - \frac{D^4}{(D+1)^2} =
\]

\[
= H\left( \chi_i(D + 1) - (D^2 + 1) \right) = 0.
\]

This is impossible, because \( H \neq 0 \) and \( |\chi_i(D + 1)| \leq 1 < D^2 + 1 \).

Lemma 4 is proved. \( \blacksquare \)

Thus we proved the part (b) of the theorem, and so the proof of the theorem is completes.

4. Proof of Corollaries

Corollary 1 follows from the theorem, because if \( f \) is multiplicative, then \( f(n) \neq 0 \) for some \( n \in \mathbb{N} \) and \( f^2(m) \neq \frac{1}{(D+1)^2} \) for some \( m \in \mathbb{N} \).

Corollary 2 is a consequence of the theorem by applying \( x_2 = \cdots = x_k \) and \( D = k - 1 \).

References


[2] Chung, P. V., Multiplicative functions satisfying the equation \( f(m^2 + n^2) = f(m^2) + f(n^2) \), Math. Slovaca, 46 (1996), 165–171.


On conjecture concerning the functional equation


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