ON THE EQUATION $f(n^2) = g^2(n)$
FOR $q$-ADDITIVE FUNCTIONS $f$ AND $g$

Imre Kátai and Bui Minh Phong
(Budapest, Hungary)

Dedicated to the memory of Professor Antal Iványi

Communicated by Jean-Marie De Koninck
(Received May 20, 2017; accepted June 25, 2017)

Abstract. We prove that if the $q$-additive functions $f$ and $g$ satisfy the equation $f(n^2) = g^2(n)$ for every $n \in \mathbb{N}$ and $\{a \in \{0, 1, \ldots, q - 1\}, k \in \mathbb{N} \mid g(aq^k) \neq 0\}$ is an infinite set, then there is a non-zero complex number $c$ such that $g(n) = cn$ and $f(n^2) = c^2n^2$ for every $n \in \mathbb{N}$.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{C}$ be the set of positive integers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers. Let $\mathcal{M}^*$ be the class of completely multiplicative functions.

For some integer $q \geq 2$ let $\mathcal{A}_q$ be the set of $q$-additive functions. Let

$\mathcal{A}_q := \{0, 1, \ldots, q - 1\}.$

Every $n \in \mathbb{N}_0$ can be uniquely represented in the form

$n = \sum_{r=0}^{\infty} a_r(n)q^r$ with $a_r(n) \in \mathcal{A}_q$

Key words and phrases: Completely additive, completely multiplicative, $q$-additive function.

2010 Mathematics Subject Classification: 11K65, 11N37, 11N64.
and $a_r(n) = 0$ if $q^r > n$. We say that $f \in \mathcal{A}_q$, if $f : \mathbb{N}_0 \to \mathbb{C}$,

$$f(0) = 0 \quad \text{and} \quad f(n) = \sum_{r=0}^{\infty} f(a_r(n)q^r) \quad \text{holds for every} \quad n \in \mathbb{N}_0.$$ 

Recently we proved in [2] the following assertion:

**Theorem A.** Assume that $f \in \mathcal{M}^*$, $g \in \mathcal{A}_q$ satisfy the condition

$$f(n) = g^2(n) \quad \text{for every} \quad n \in \mathbb{N}.$$ 

Then either

$$f(q_0) = 0, \quad q_0 | q, \quad \text{and} \quad f(n) = \chi_{q_0}(n), g^2(n) = \chi_{q_0}(n),$$

where $\chi_{q_0}$ is a Dirichlet character (mod $q_0$), or $f(n) = n^2$, and either $g(n) = n$, or $g(n) = -n$ for all $n \in \mathbb{N}$.

Let

$$K(g) := \{(a, \ell) \in (\mathcal{A}_q, \mathbb{N}) \mid g(aq^\ell) \neq 0\}.$$ 

In this paper, we prove the following

**Theorem 1.** Assume that $f, g \in \mathcal{A}_q$ satisfy

(1.1) 

$$f(n^2) = g^2(n) \quad \text{for every} \quad n \in \mathbb{N}.$$ 

If

(1.2) 

$$|K(g)| = \infty,$$

then there is a non-zero complex number $c$ such that $g(n) = cn$ and $f(n^2) = c^2n^2$ for every $n \in \mathbb{N}$.

If

$$|K(g)| < \infty,$$

then

$$|K(f)| < \infty.$$ 

We are unable to give all solutions of (1.1) if $K(g)$ is finite. We prove

**Theorem 2.** Assume that $f \in \mathcal{A}_q$ satisfies

$$f(n^2) = f^2(n) \quad \text{for every} \quad n \in \mathbb{N}.$$
On the equation $f(n^2) = g^2(n)$ for $q$-additive functions $f$ and $g$

If $K(f) = \{(a, \ell) \in (\mathbb{A}_q, \mathbb{N}) \mid f(aq^\ell) \neq 0\}$ is a finite set, then there are integers $0 = m_0 < \cdots < m_k$ such that
\[
\mathbb{A}_q = S(m_0) \cup \cdots \cup S(m_k)
\]
and either $f(m_\nu) = 0$ or $f(m_\nu)$ is a root of unity, for which
\[
f(n) = \varepsilon \cdot \left( f(m_\nu) \right)^{2^\nu} \quad \text{if} \quad n \in S(m_\nu) \quad (\nu \in \{0, \cdots, k\}),
\]
where $\varepsilon \in \mathbb{N}_0$ and $\varepsilon$ is a root of unity.

2. Proof of Theorem 1

We shall use the following two lemmas.

**Lemma 1.** If $h \in \mathbb{A}_q \cap \mathcal{M}^*$, $h(1) = 1$ and $h(q) \neq 0$, then $h(n) = n$ for every $n \in \mathbb{N}$.

**Proof.** This lemma is a consequence of Theorem 2 in [1]. \hfill \blacksquare

**Lemma 2.** Assume that $f, g \in \mathbb{A}_q$ satisfy (1.1). Then $g(1) \neq 0$ and the function $G(n) := \frac{g(n)}{g(1)}$ is an element of $\mathcal{M}^*$.

**Proof.** Since $K(g)$ is an infinite set and $g \in \mathbb{A}_q$, then there exists an infinite sequence $k_1 < k_2 < \cdots$ of positive integers and $A \in \mathbb{A}_q$ such that
\[
g(Aq^{k_i}) \neq 0 \quad \text{for every} \quad i \in \mathbb{N}.
\]

Let $n, m \in \mathbb{N}$. Then the above relation shows that there exists $K \in \{k_1, k_2, \cdots\}$ such that
\[
g^K > \max(2nm, (nm)^2, (Aq^{k_i})^2) \quad \text{and} \quad g(Aq^K) \neq 0.
\]

Since $f, g \in \mathbb{A}_q$, we can assume that $f, g \in \mathbb{A}_q^K$.

Then, we infer from (2.1) that
\[
f\left( (Aq^K n + m)^2 \right) = f\left( A^2q^{2K}n^2 + 2Aq^Knm + m^2 \right) = f\left( A^2q^{2K}n^2 \right) + f\left( 2Aq^Knm \right) + f(m^2)
\]
and
\[
(g(Aq^K n + m))^2 = (g(Aq^K n) + g(m))^2 = \\
g^2(Aq^K n) + 2g(Aq^K n)g(m) + g^2(m).
\]

From (1.1), we have
\[
f(A^2 q^{2K} n^2) = g^2(Aq^K n), \quad f(m^2) = g^2(m),
\]
consequently
(2.2) \[ f(2Aq^K nm) = 2g(Aq^K n)g(m). \]

By taking \( n = 1 \) into (2.2), we have
\[
f(2Aq^K m) = 2g(Aq^K )g(m),
\]
which gives
(2.3) \[ f(2Aq^K nm) = 2g(Aq^K )g(nm). \]

It is clear from (2.2) and (2.3) that
(2.4) \[ g(Aq^K n)g(m) = g(Aq^K )g(nm), \]
which with \( m = 1 \) implies
\[
g(Aq^K n)g(1) = g(Aq^K )g(n).
\]

This relation with \( n = Aq^{k1} \) shows that \( g(1) \neq 0 \). Therefore
\[
g(Aq^K n) = \frac{g(Aq^K )}{g(1)} g(n).
\]

Finally, we obtain from (2.4) and the fact \( g(Aq^K ) \neq 0 \) that
(2.5) \[ g(Aq^K )g(nm) = g(Aq^K n)g(m) = \frac{g(Aq^K )}{g(1)} g(n)g(m), \]
which implies
\[
\frac{g(nm)}{g(1)} = \frac{g(n)}{g(1)} \frac{g(m)}{g(1)}
\]
and so
\[
G(nm) = G(n)G(m) \quad \text{for every} \quad n, m \in \mathbb{N}.
\]

Lemma 2 is thus proved. \( \blacksquare \)
On the equation $f(n^2) = g^2(n)$ for $g$-additive functions $f$ and $g$

**Proof of Theorem 1.** Assume that $f, g \in \mathcal{A}_q$ satisfy (1.1) and (1.2). Since $G(n) = \frac{g(n)}{g(1)}$, we have $G \in \mathcal{A}_q$, consequently

(2.6) \hspace{1cm} G \in \mathcal{A}_q \cap M^*$.

We shall prove that $g(q) \neq 0$. Assume that $g(q) = 0$. Then we obtain from (2.6) that

$$g(mq^e) = g(1)G(mq^e) = g(1)G(m)G(q)^e = g(1)\left(\frac{g(q)}{g(1)}\right)^e = 0 \quad \text{for every} \quad m, e \in \mathbb{N},$$

which contradicts the assumption (1.2).

Assume now that $g(q) \neq 0$. Then $G(1) = 1$, $G(q) \neq 0$ and we infer from Lemma 1 that

$$G(n) = \frac{g(n)}{g(1)} = n, g(n) = g(1)n, \quad \text{and} \quad f(n^2) = g(n)^2 = g(1)^2n^2,$$

consequently Theorem 1 is proved for $c = g(1) \neq 0$.

Now we prove the second assertion of Theorem 1. Assume that $|K(g)| < \infty$. Then there is a number $K \in \mathbb{N}, K \geq 3$ such that $g(mq^k) = 0$ for every $m \in \mathbb{N}$ and $k \geq K$. Then $g(n) = g(\nu)$ if $n \equiv \nu \pmod{q^K}$. Let $\nu, s \in \mathbb{A}_q$, $n = \nu + sq^k$.

Then $n^2 = \nu^2 + 2\nu sq^k + s^2q^{2k}$ and in the case $k \geq K$, we have

$$2\nu s < 2q^2 < q^3 < q^K < q^k \quad \text{and} \quad \nu^2 < q^K < q^k,$$

consequently

$$g^2(n) = \left(g(\nu + sq^k)^2 = \left(g(\nu) + g(sq^k)\right)^2 = g^2(\nu) + 2g(\nu)g(sq^k) + g^2(sq^k)\right.$$

and

$$g^2(n) = f(\nu^2 + f(2\nu sq^k) + f(s^2q^{2k}) = g^2(\nu) + f(2\nu sq^k) + g^2(sq^k).$$

Thus

(2.7) \hspace{1cm} f(2\nu sq^k) = 0 \quad \text{if} \quad k \geq K \quad \text{and} \quad \nu, s \in \mathbb{A}_q.$$

Assume first that $q$ is even, $q = 2Q$. Let $s = Q$. Then $2\nu sq^k = \nu q^{k+1}$, and so $f(\nu q^{k+1}) = 0$ for every $\nu \in \mathbb{A}_q$ if $k \geq K$. Therefore, we have $|K(f)| < \infty$.

Assume now that $q$ is odd. If $\nu \in \mathbb{A}_q, \nu$ is even, then $\nu/2 \in \mathbb{A}_q$ and we infer from (2.7) with $s = 1$ that $f(\nu q^k) = 0$ if $k \geq K$. 

We note from (2.7) that
\[ f(q^k) + f(q^{k+1}) = f((q + 1)q^k) = 0 \quad \text{if} \quad k \geq K \]
and
\[ f(q^{2t}) = g^2(q^t) = 0 \quad \text{for every} \quad t \geq K, \]
consequently
\[ f(q^\ell) = 0 \quad \text{if} \quad \ell \geq 2K. \] (2.8)

Let now \( \nu \in \mathbb{N}_q \) and \( \nu \) is odd. Then \( \frac{q^{k+\nu}}{2} \in \mathbb{N}_q \) and we obtain from (2.7) that
\[ f((q + \nu)q^k) = f\left(2q + \nu \frac{q^k}{2}\right) = 0 \]
and so we obtain from (2.8) that
\[ 0 = f((q + \nu)q^k) = f\left(q^{k+1}\right) + f(\nu q^k) = f(\nu q^k) \quad \text{if} \quad k \geq 2K. \]

Consequently \( |K(f)| < \infty \) in the case \( q \) is odd.

Theorem 1 is thus proved. \( \blacksquare \)

3. Proof of Theorem 2

Assume that \( f \in \mathcal{A}_q \) satisfies \( |K(f)| < \infty \) and
\[ f(n^2) = f^2(n) \quad \text{for every} \quad n \in \mathbb{N}. \] (3.1)
If
\[ f(qm) \neq 0 \quad \text{for some} \quad m \in \mathbb{N}, \]
then we infer from (3.1) that
\[ f\left((qm)^{2^n}\right) = \left(f(qm)\right)^{2^n} \neq 0 \quad \text{for every} \quad \alpha \in \mathbb{N}, \]
which is impossible. Thus we proved that
\[ f(qm) = 0 \quad \text{for every} \quad m \in \mathbb{N}, \] (3.2)
and so \( f \in \mathcal{A}_q \) implies that
\[ f(qm + a) = f(a) \quad \text{for every} \quad a \in \mathbb{A}_q, \quad m \in \mathbb{N}. \]
On the equation $f(n^2) = g^2(n)$ for $q$-additive functions $f$ and $g$

It is clear that $f$ is a solution of (3.1) under the condition (3.2) if and only if

(3.3) $f^2(\ell) = f(\ell^2 \pmod{q})$ for every $\ell \in \mathbb{A}_q$.

Let us define the directed graph $\ell \to \ell^2 \pmod{q}$ over $\mathbb{A}_q$. We shall classify the elements of $\mathbb{A}_q$, saying that $a \sim b$ if there is a path from $a$ to $b$, or from $b$ to $a$. Let $U_0, U_1, \ldots, U_k$ be the classes we obtain. Let $m_i := \min\{t \in U_i\}$ and $S(m_i) := U_i$.

These are the connected components of this graph. Each $S(m_i)$ contains a directed circle (loop is allowed):

$$h_0 \to h_1 \to \cdots \to h_{t-1} (\to h_0).$$

Then $h_1 \equiv h_0^2 \pmod{q}$, $h_2 \equiv h_0^{2^2} \pmod{q}$, $\ldots$, $h_{t-1} \equiv h_0^{2^{t-1}}$, $h_0 \equiv h_0^{2^t} \pmod{q}$,

and so, if $f$ is a solution, then

$$f(h_j) = f^{2^j}(h_0) \quad \text{and} \quad f(h_0) = f^{2^t}(h_0),$$

consequently $f(h_0)$ is a root of unity of rank $2^t - 1$, or $f(h_0) = 0$.

The values $f(h_j)$ are determined by $f(h_0)$. Let $m \in S(m_\ell)$ which is not on the circle. Let $m \to t_1 \to \cdots \to t_{s-1} \to h_\ell$

be the path from $m$ to the circle. Then $f(h_\ell) = f(m)^{2^s}$, and $f(m) = f(h_\ell)^{2^{-s}}$.

If we do this for every element of $S(m_j)$ and for every $j$, then we choose a solution of (3.1) satisfying $f(mq) = 0$ for every $m \in \mathbb{N}$.

Examples. 1. $q = 24$.

$$S(0) = \{0, 6, 12, 18\}, \quad S(1) = \{1, 5, 7, 11, 13, 17, 19, 23\},$$

$$S(2) = \{2, 4, 8, 10, 14, 16, 20, 22\} \quad \text{and} \quad S(3) = \{3, 9, 15, 21\}.$$

It is clear that $f(1) \in \{0, 1, i\}$ and

$$f(n) = \begin{cases} 
0 & \text{if } n \in S(0), \\
\pm f(1) & \text{if } n \in S(1).
\end{cases}$$

We have $f(2)^4 = f(2)^8$, consequently $f(2) \in \{0, \pm 1, \pm i\}$. It is easy to check that $f(4) = f(2)^2$, $f(8) = \pm f(2)^2$, $f(10) = \pm f(2)$, $f(14) = \pm f(2)$, $f(16) = f(2)^4$, $f(20) = \pm f(2)^2$, $f(22) = \pm f(2)$. 

In $S(3)$ it is obvious that $S(3) \in \{0, \pm 1\}$, furthermore $f(9) = f(3)^2$, $f(15) = \pm f(3)$, $f(21) = \pm f(3)$.

2. $q = 40$.

$S(0) = \{0, 10, 20, 30\}$,

$S(1) = \{1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39\}$,

$S(2) = \{2, 4, 6, 8, 12, 14, 16, 18, 22, 24, 26, 28, 32, 34, 36, 38\}$

and

$S(5) = \{5, 15, 25, 35\}$.

It is easy to check that $f(1) \in \{0, 1\}$ and

$$f(n) = \begin{cases} 0 & \text{if } n \in S(0), \\ \pm f(1) & \text{if } n \in \{1, 9, 11, 19, 21, 29, 31, 39\}, \\ \pm f(1), \pm if(1) & \text{if } n \in \{3, 7, 13, 17, 23, 27, 33, 37\}. \end{cases}$$

In $S(2)$, we have $f(2) \in \{0, \pm 1, \pm i\}$, furthermore $f(4) = f(2)^2$, $f(16) = f^4(2)$ and

$$f(n) = \begin{cases} \pm f(2) & \text{if } n \in \{18, 22, 38\}, \\ \pm f^2(2) & \text{if } n \in \{24, 36\}, \\ \pm f(2), \pm if(2) & \text{if } n \in \{6, 8, 12, 14, 18, 26, 28, 32, 34\}. \end{cases}$$

Finally, we have $f(5) \in \{0, 1\}$,

$$f(15) = \pm f(5), \ f(25) = f(5)^2 \text{ and } f(35) = \pm f(5).$$

References


I. Kátai and B. M. Phong
Department of Computer Algebra
Eötvös Loránd University
H-1117 Budapest
Pázmány Péter Sétány 1/C
Hungary
katai@inf.elte.hu
bui@inf.elte.hu