Annales Univ. Sci. Budapest., Sect. Comp. 46 (2017) 115–122

ON THE EQUATION $f(n^2) = g^2(n)$ FOR q-ADDITIVE FUNCTIONS f AND g

Imre Kátai and Bui Minh Phong
(Budapest, Hungary)

Dedicated to the memory of Professor Antal Iványi

Communicated by Jean-Marie De Koninck

(Received May 20, 2017; accepted June 25, 2017)

Abstract. We prove that if the q-additive functions f and g satisfy the equation $f(n^2) = g^2(n)$ for every $n \in \mathbb{N}$ and $\{a \in \{0, 1, \ldots, q-1\}, k \in \mathbb{N} \mid g(aq^k) \neq 0\}$ is an infinite set, then there is a non-zero complex number c such that $g(n) = cn$ and $f(n^2) = c^2 n^2$ for every $n \in \mathbb{N}$.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{C}$ be the set of positive integers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers. Let $\mathcal{M}^*$ be the class of completely multiplicative functions.

For some integer $q \geq 2$ let $\mathcal{A}_q$ be the set of q-additive functions. Let

$$\mathbb{A}_q := \{0, 1, \cdots, q-1\}.$$

Every $n \in \mathbb{N}_0$ can be uniquely represented in the form

$$n = \sum_{r=0}^{\infty} a_r(n) q^r \quad \text{with} \quad a_r(n) \in \mathbb{A}_q$$

Key words and phrases: Completely additive, completely multiplicative, q-additive function.
2010 Mathematics Subject Classification: 11K65, 11N37, 11N64.

and $a_r(n) = 0$ if $q^r > n$. We say that $f \in \mathcal{A}_q$, if $f : \mathbb{N}_0 \to \mathbb{C}$,

$$f(0) = 0 \quad \text{and} \quad f(n) = \sum_{r=0}^{\infty} f\Big(a_r(n)q^r\Big) \quad \text{holds for every} \quad n \in \mathbb{N}_0.$$

Recently we proved in [2] the following assertion:

Theorem A. *Assume that $f \in \mathcal{M}^*$, $g \in \mathcal{A}_q$ satisfy the condition*

$$f(n) = g^2(n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

Then either

$$f(q_0) = 0,\ q_0|q, \quad \textit{and} \quad f(n) = \chi_{q_0}(n), g^2(n) = \chi_{q_0}(n),$$

where χ_{q_0} is a Dirichlet character $\pmod{q_0}$, *or $f(n) = n^2$, and either $g(n) = n$, or $g(n) = -n$ for all $n \in \mathbb{N}$.*

Let

$$K(g) := \{(a, \ell) \in (\mathbb{A}_q, \mathbb{N}) \ \mid g(aq^\ell) \neq 0\}.$$

In this paper, we prove the following

Theorem 1. *Assume that $f, g \in \mathcal{A}_q$ satisfy*

$$f(n^2) = g^2(n) \quad \textit{for every} \quad n \in \mathbb{N}. \tag{1.1}$$

If

$$|K(g)| = \infty, \tag{1.2}$$

then there is a non-zero complex number c such that $g(n) = cn$ and $f(n^2) = c^2n^2$ for every $n \in \mathbb{N}$.

If

$$|K(g)| < \infty,$$

then

$$|K(f)| < \infty.$$

We are unable to give all solutions of (1.1) if $K(g)$ is finite. We prove

Theorem 2. *Assume that $f \in \mathcal{A}_q$ satisfies*

$$f(n^2) = f^2(n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

If $K(f) = \{(a, \ell) \in (\mathbb{A}_q, \mathbb{N}) \mid f(aq^\ell) \neq 0\}$ *is a finite set, then there are integers* $0 = m_0 < \cdots < m_k$ *such that*

$$\mathbb{A}_q = S(m_0) \cup \cdots \cup S(m_k)$$

and either $f(m_\nu) = 0$ *or* $f(m_\nu)$ *is a root of unity, for which*

$$f(n) = \varepsilon \cdot \Big(f(m_\nu)\Big)^{2^e} \quad \textit{if} \quad n \in S(m_\nu) \quad (\nu \in \{0, \cdots, k\}),$$

where $e \in \mathbb{N}_0$ *and* ε *is a root of unity.*

2. Proof of Theorem 1

We shall use the following two lemmas.

Lemma 1. *If* $h \in \mathcal{A}_q \cap \mathcal{M}^*$, $h(1) = 1$ *and* $h(q) \neq 0$, *then* $h(n) = n$ *for every* $n \in \mathbb{N}$.

Proof. This lemma is a consequence of Theorem 2 in [1]. ■

Lemma 2. *Assume that* $f, g \in \mathcal{A}_q$ *satisfy* (1.1). *Then* $g(1) \neq 0$ *and the function* $G(n) := \frac{g(n)}{g(1)}$ *is an element of* $\mathcal{M}^*$.

Proof. Since $K(g)$ is an infinite set and $g \in \mathcal{A}_q$, then there exists an infinite sequence $k_1 < k_2 < \cdots$ of positive integers and $A \in \mathbb{A}_q$ such that

$$g(Aq^{k_i}) \neq 0 \quad \text{for every} \quad i \in \mathbb{N}.$$

Let $n, m \in \mathbb{N}$. Then the above relation shows that there exists $K \in \in \{k_1, k_2, \cdots\}$ such that

$$q^K > \max(2nm, (nm)^2, (Aq^{k_1})^2) \quad \text{and} \quad g(Aq^K) \neq 0. \tag{2.1}$$

Since $f, g \in \mathcal{A}_q$, we can assume that $f, g \in \mathcal{A}_{q^K}$.

Then, we infer from (2.1) that

$$f\Big((Aq^K n + m)^2\Big) = f\Big(A^2 q^{2K} n^2 + 2Aq^K nm + m^2\Big) =$$
$$= f\Big(A^2 q^{2K} n^2\Big) + f\Big(2Aq^K nm\Big) + f(m^2)$$

and

$$\Big(g(Aq^K n+m)\Big)^2 = \Big(g(Aq^K n)+g(m)\Big)^2 = \\ = g^2(Aq^K n) + 2g(Aq^K n)g(m) + g^2(m).$$

From (1.1), we have

$$f\Big(A^2 q^{2K} n^2\Big) = g^2\big(Aq^K n\big), \quad f(m^2) = g^2(m),$$

consequently

$$f\Big(2Aq^K nm\Big) = 2g(Aq^K n)g(m). \tag{2.2}$$

By taking $n=1$ into (2.2), we have

$$f\Big(2Aq^K m\Big) = 2g(Aq^K)g(m),$$

which gives

$$f\Big(2Aq^K nm\Big) = 2g(Aq^K)g(nm). \tag{2.3}$$

It is clear from (2.2) and (2.3) that

$$g(Aq^K n)g(m) = g(Aq^K)g(nm), \tag{2.4}$$

which with $m=1$ implies

$$g(Aq^K n)g(1) = g(Aq^K)g(n).$$

This relation with $n = Aq^{k_1}$ shows that $g(1) \neq 0$. Therefore

$$g(Aq^K n) = \frac{g(Aq^K)}{g(1)} g(n).$$

Finally, we obtain from (2.4) and the fact $g(Aq^K) \neq 0$ that

$$g(Aq^K)g(nm) = g(Aq^K n)g(m) = \frac{g(Aq^K)}{g(1)} g(n)g(m), \tag{2.5}$$

which implies

$$\frac{g(nm)}{g(1)} = \frac{g(n)}{g(1)} \frac{g(m)}{g(1)}$$

and so

$$G(nm) = G(n)G(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Lemma 2 is thus proved. ■

Proof of Theorem 1. Assume that $f,g\in\mathcal{A}_q$ satisfy (1.1) and (1.2). Since $G(n)=\frac{g(n)}{g(1)}$, we have $G\in\mathcal{A}_q$, consequently

$$G\in\mathcal{A}_q\cap\mathcal{M}^*. \tag{2.6}$$

We shall prove that $g(q)\neq 0$. Assume that $g(q)=0$. Then we obtain from (2.6) that

$$\begin{aligned}g(mq^e)&=g(1)G(mq^e)=g(1)G(m)G(q)^e=\\&=g(1)\frac{g(m)}{g(1)}\Big(\frac{g(q)}{g(1)}\Big)^e=0\quad\text{for every}\quad m,e\in\mathbb{N},\end{aligned}$$

which contradicts the assumption (1.2).

Assume now that $g(q)\neq 0$. Then $G(1)=1,\ G(q)\neq 0$ and we infer from Lemma 1 that

$$G(n)=\frac{g(n)}{g(1)}=n,g(n)=g(1)n,\quad\text{and}\quad f(n^2)=g(n)^2=g(1)^2n^2,$$

consequently Theorem 1 is proved for $c=g(1)\neq 0$.

Now we prove the second assertion of Theorem 1. Assume that $|K(g)|<\infty$. Then there is a number $K\in\mathbb{N},K\geq 3$ such that $g(mq^k)=0$ for every $m\in\mathbb{N}$ and $k\geq K$. Then $g(n)=g(\nu)$ if $n\equiv\nu\pmod{q^K}$. Let $\nu,s\in\mathbb{A}_q$, $n=\nu+sq^k$. Then $n^2=\nu^2+2\nu sq^k+s^2q^{2k}$ and in the case $k\geq K$, we have

$$2\nu s<2q^2\leq q^3\leq q^K<q^k\quad\text{and}\quad\nu^2<q^K\leq q^k,$$

consequently

$$g^2(n)=\Big(g(\nu+sq^k)\Big)^2=\Big(g(\nu)+g(sq^k)\Big)^2=g^2(\nu)+2g(\nu)g(sq^k)+g^2(sq^k)$$

and

$$\begin{aligned}g^2(n)=f\Big((\nu+sq^k)^2\Big)&=f(\nu^2)+f(2\nu sq^k)+f(s^2q^{2k})=\\&=g^2(\nu)+f(2\nu sq^k)+g^2(sq^k).\end{aligned}$$

Thus

$$f(2\nu sq^k)=0\quad\text{if}\quad k\geq K\quad\text{and}\quad\nu,s\in\mathbb{A}_q. \tag{2.7}$$

Assume first that q is even, $q=2Q$. Let $s=Q$. Then $2\nu sq^k=\nu q^{k+1}$, and so $f(\nu q^{k+1})=0$ for every $\nu\in\mathbb{A}_q$ if $k\geq K$. Therefore, we have $|K(f)|<\infty$.

Assume now that q is odd. If $\nu\in\mathbb{A}_q$, ν is even, then $\nu/2\in\mathbb{A}_q$ and we infer from (2.7) with $s=1$ that $f(\nu q^k)=0$ if $k\geq K$.

We note from (2.7) that

$$f(q^k)+f(q^{k+1})=f\Big((q+1)q^k\Big)=0 \quad \text{if} \quad k\geq K$$

and

$$f(q^{2t})=g^2(q^t)=0 \quad \text{for every} \quad t\geq K,$$

consequently

$$f(q^\ell)=0 \quad \text{if} \quad \ell\geq 2K. \tag{2.8}$$

Let now $\nu\in\mathbb{A}_q$ and ν is odd. Then $\frac{q+\nu}{2}\in\mathbb{A}_q$ and we obtain from (2.7) that

$$f\Big((q+\nu)q^k\Big)=f\Big(2\frac{q+\nu}{2}q^k\Big)=0$$

and so we obtain from (2.8) that

$$0=f\Big((q+\nu)q^k\Big)=f\Big(q^{k+1}\Big)+f\Big(\nu q^k\Big)=f\Big(\nu q^k\Big) \quad \text{if} \quad k\geq 2K.$$

Consequently $|K(f)|<\infty$ in the case q is odd.

Theorem 1 is thus proved. ∎

3. Proof of Theorem 2

Assume that $f\in\mathcal{A}_q$ satisfies $|K(f)|<\infty$ and

$$f(n^2)=f^2(n) \quad \text{for every} \quad n\in\mathbb{N}. \tag{3.1}$$

If

$$f(qm)\neq 0 \quad \text{for some} \quad m\in\mathbb{N},$$

then we infer from (3.1) that

$$f\Big((qm)^{2^\alpha}\Big)=\Big(f(qm)\Big)^{2^\alpha}\neq 0 \quad \text{for every} \quad \alpha\in\mathbb{N},$$

which is impossible. Thus we proved that

$$f(qm)=0 \quad \text{for every} \quad m\in\mathbb{N}, \tag{3.2}$$

and so $f\in\mathcal{A}_q$ implies that

$$f(qm+a)=f(a) \quad \text{for every} \quad a\in\mathbb{A}_q,\ m\in\mathbb{N}.$$

It is clear that f is a solution of (3.1) under the condition (3.2) if and only if

$$f^2(\ell)=f\big(\ell^2 \pmod q\big) \quad \text{for every} \quad \ell\in\mathbb{A}_q. \tag{3.3}$$

Let us define the directed graph $\ell\to\ell^2 \pmod q$ over $\mathbb{A}_q$. We shall classify the elements of $\mathbb{A}_q$, saying that $a\sim b$ if there is a path from a to b, or from b to a. Let $U_0,U_1,\dots,U_k$ be the classes we obtain. Let

$$m_i:=min\{t\in U_i\} \quad \text{and} \quad S(m_i):=U_i.$$

These are the connected components of this graph. Each $S(m_i)$ contains a directed circle (loop is allowed):

$$h_0\to h_1\to\cdots\to h_{t-1}(\to h_0).$$

Then

$$h_1{\equiv}h_0^2 \pmod q,\; h_2{\equiv}h_0^{2^2} \pmod q,\;\dots,\; h_{t-1}{\equiv}h_0^{2^{t-1}},\; h_0{\equiv}h_0^{2^t} \pmod q,$$

and so, if f is a solution, then

$$f(h_j)=f^{2^j}(h_0) \quad \text{and} \quad f(h_0)=f^{2^t}(h_0),$$

consequently $f(h_0)$ is a root of unity of rank 2^t-1, or $f(h_0)=0$.

The values $f(h_j)$ are determined by $f(h_0)$. Let $m\in S(m_\ell)$ which is not on the circle. Let

$$m\to t_1\to\cdots\to t_{s-1}\to h_\ell$$

be the path from m to the circle. Then $f(h_\ell)=f(m)^{2^s}$, and $f(m)=f(h_\ell)^{2^{-s}}$.

If we do this for every element of $S(m_j)$ and for every j, then we choose a solution of (3.1) satisfying $f(mq)=0$ for every $m\in\mathbb{N}$.

Examples. 1. $q=24$.

$$S(0)=\{0,6,12,18\},\quad S(1)=\{1,5,7,11,13,17,19,23\},$$

$$S(2)=\{2,4,8,10,14,16,20,22\} \quad \text{and} \quad S(3)=\{3,9,15,21\}.$$

It is clear that $f(1)\in\{0,1\}$ and

$$f(n)=\begin{cases}0 & \text{if } n\in S(0),\\ \pm f(1) & \text{if } n\in S(1).\end{cases}$$

We have $f(2)^4=f(2)^8$, consequently $f(2)\in\{0,\pm1,\pm i\}$. It is easy to check that $f(4)=f(2)^2$, $f(8)=\pm f(2)^2$, $f(10)=\pm f(2)$, $f(14)=\pm f(2)$, $f(16)= = f(2)^4$, $f(20)=\pm f(2)^2$, $f(22)=\pm f(2)$.

In $S(3)$ it is obvious that $S(3) \in \{0, \pm 1\}$, furthermore $f(9) = f(3)^2$, $f(15) = \pm f(3)$, $f(21) = \pm f(3)$.

2. $q = 40$.

$$S(0) = \{0, 10, 20, 30\},$$
$$S(1) = \{1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39\},$$
$$S(2) = \{2, 4, 6, 8, 12, 14, 16, 18, 22, 24, 26, 28, 32, 34, 36, 38\}$$

and

$$S(5) = \{5, 15, 25, 35\}.$$

It is easy to check that $f(1) \in \{0, 1\}$ and

$$f(n) = \begin{cases} 0 & \text{if } \ n \in S(0), \\ \pm f(1) & \text{if } \ n \in \{1, 9, 11, 19, 21, 29, 31, 39\}, \\ \pm f(1), \pm i f(1) & \text{if } \ n \in \{3, 7, 13, 17, 23, 27, 33, 37\}. \end{cases}$$

In $S(2)$, we have $f(2) \in \{0, \pm 1, \pm i\}$, furthermore $f(4) = f(2)^2$, $f(16) = f^4(2)$ and

$$f(n) = \begin{cases} \pm f(2) & \text{if } \ n \in \{18, 22, 38\}, \\ \pm f^2(2) & \text{if } \ n \in \{24, 36\}, \\ \pm f(2), \pm i f(2) & \text{if } \ n \in \{6, 8, 12, 14, 18, 26, 28, 32, 34\}. \end{cases}$$

Finally, we have $f(5) \in \{0, \pm 1\}$,

$$f(15) = \pm f(5), \ f(25) = f(5)^2 \quad \text{and} \quad f(35) = \pm f(5).$$

References

[1] **Kátai, I. and B.M. Phong,** Some relations among arithmetical functions, *Ann. Univ. Sci. Budapest., Sect. Comp.*, **44** (2015), 49–57.

[2] **Kátai, I. and B.M. Phong,** Some relations among multiplicative and q-additive functions, *Lithuanian Mathematical Journal*, **57(2)** (2017), 30–37.

I. Kátai and B. M. Phong
Department of Computer Algebra
Eötvös Loránd University
H-1117 Budapest
Pázmány Péter Sétány 1/C
Hungary
katai@inf.elte.hu
bui@inf.elte.hu