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MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENTS

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Dedicated to the memory of Professor Antal Iványi

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Abstract. Let $k \in \{1, 2, 3\}$. For a polynomial $P(x) = a_0 + a_1 x + \cdots + a_k x^k \in \mathbb{C}[x]$ let $P(E)f(n) = a_0 f(n) + a_1 f(n+1) + \cdots + a_k f(n+k)$. We give all multiplicative functions f which satisfy the relation

$$\sum_{n \leq x} \frac{|P(E)f(n)|}{n} = O(\log x).$$

In the case $P(x) = (E^B - I)^k$, we also give all completely multiplicative function with the conditions $|f(n)| = 1$ if $(n, B) = 1$ and $f(n) = 0$ if $(n, B) > 1$ which satisfy

$$\sum_{n \leq x} \frac{|P(E)f(n)|}{n} = o(\log x).$$

where B is a positive integer.

1. Introduction

Let Ω be the set of arithmetical functions having complex values. Sometimes a function $f \in \Omega$ is considered as an infinite dimensional vector, the n'th coordinate of which is $f(n)$. We write: $f = (f(1), f(2), \cdots)$. Let $\underline{x} = (x_1, x_2, \cdots)$

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be a general element of Ω. The operators I, E, Δ, Δ_B $(\Omega \to \Omega)$ are defined according to the following rules: the n'th coordinate of $I\underline{x}$, $E\underline{x}$, $\Delta\underline{x}$, $\Delta_B\underline{x}$ are x_n, x_{n+1}, $x_{n+1}-x_n$, $x_{n+B}-x_n$, respectively. Let $\Delta^k=(E-I)^k$, $\Delta_B^k=(E^B-I)^k$. If $P(x)=a_0+a_1x+\cdots+a_kx^k\in\mathbb{C}[x]$, then the n'th coordinate of $P(E)\underline{x}$ equals

$$a_0x_n+a_1x_{n+1}+\cdots+a_kx_{n+k}.$$

Let $\mathcal{M}$ $(\mathcal{M}^*)$ be the set of complex-valued multiplicative (completely multiplicative) functions. In paper [3], K-H. Indlekofer and I. Kátai proved a theorem which is more general than the following.

Theorem A. *If $f\in\mathcal{M}$, $P\in\mathbb{C}[x]$, $P\neq 0$ with $k=\deg P$ and*

$$\sum_{n\le x}|P(E)f(n)|=O(x)\quad (x\to\infty), \tag{1.1}$$

then, either

$$\sum_{n\le x}|f(n)|=O(x), \tag{1.2}$$

or there are $s\in\mathbb{C}$ and $F\in\mathcal{M}$ with $0<\Re s\le k$ such that

$$f(n)=n^sF(n)\quad \text{and}\quad P(E)F(n)=0 \tag{1.3}$$

are satisfied for every positive integer n.

For other results we refer to works [1], [2], [4], [5] and [9].

We shall prove

Theorem 1. *Let $f\in\mathcal{M}$, $P\in\mathbb{C}[x]$, $P\neq 0$ with $k=\deg P\le 3$. Assume that*

$$\sum_{n\le x}\frac{|P(E)f(n)|}{n}=O(\log x)\quad (x\to\infty), \tag{1.4}$$

then, either

$$\sum_{n\le x}\frac{|f(n)|}{n}=O(\log x), \tag{1.5}$$

or there are $s\in\mathbb{C}$ and $F\in\mathcal{M}$ with $0<\Re s\le k$ such that

$$f(n)=n^sF(n)\quad \text{and}\quad P(E)F(n)=0 \tag{1.6}$$

are satisfied for every positive integer n.

We think that Theorem 1 is true for every $k \in \mathbb{N}$. Now we can only prove the following result.

Theorem 2. *Let $k \in \mathbb{N}$, $k \geq 1$, $P \in \mathbb{C}[x]$ be the smallest degree monic polynomial for which* (1.4) *holds. Then*

$$P(x) = (x^B - 1)^k, \quad B \quad \text{is a suitable natural number}$$

and

$$|f(n)| = n^a, \quad a \quad \text{is a positive constant.}$$

The proof of Theorem 1 is based on a simple generalization of a famous theorem of O. Klurman [8], which we state as follows.

Theorem B. *Let $f \in \mathcal{M}^*$, $B \in \mathbb{N}$,*

$$|f(n)| = \begin{cases} 1 & \text{if } (n, B) = 1 \\ 0 & \text{if } (n, B) > 1 \end{cases}$$

and assume that

$$\sum_{n \leq x} \frac{|\Delta_B f(n)|}{n} = o(\log x) \quad (x \to \infty). \tag{1.7}$$

Then

$$f(n) = n^{i\tau} \chi_B(n), \text{ where } \chi_B \text{ is a Dirichlet character} \pmod{B}.$$

Theorem 3. *If $f \in \mathcal{M}^*$, $B \in \mathbb{N}$,*

$$|f(n)| = \begin{cases} 1 & \text{if } (n, B) = 1 \\ 0 & \text{if } (n, B) > 1, \end{cases}$$

$k = 2$ or $k = 3$, and

$$\sum_{n \leq x} \frac{|\Delta_B^k f(n)|}{n} = o(\log x) \quad (x \to \infty), \tag{1.8}$$

then (1.7) *is true, and so*

$$f(n) = n^{i\tau} \chi_B(n), \text{ where } \chi_B \text{ is a Dirichlet character} \pmod{B}.$$

2. Proof of Theorem 3

The case $k = 2$.

Let $e(x) := e^{2\pi ix}$, $\arg f(n) = 2\pi u(n)$, $u(n) \pmod 1$ is additive, $\Delta_B u(n) = = u(n+B) - u(n)$.

Assume that $\Delta_B^2 f(n) = \epsilon \ (< 1/4)$. Since

$$\Delta_B^2 f(n) = \Big|\Big(\frac{f(n+2B)}{f(n+B)} - 1\Big) + \Big(\frac{f(n)}{f(n+B)} - 1\Big)\Big| < \epsilon,$$

therefore $\cos \Delta_B u(n+B) > 1 - \epsilon$, $\cos(-\Delta_B u(n)) > 1 - \epsilon$, whence

$$\Big|\frac{f(n+B)}{f(n)} - 1\Big| < c\sqrt{\epsilon},$$

consequently (1.8) implies (1.7).

The case $k = 3$.

Let $\xi_1 := \frac{f(n+B)}{f(n)}$, $\xi_2 := \frac{f(n+2B)}{f(n+B)}$, $\xi_3 := \frac{f(n+3B)}{f(n+2B)}$.

We have

$$|\Delta_B^3 f(n)| = |\xi_1\xi_2\xi_3 - 3\xi_1\xi_2 + 3\xi_1 - 1|.$$

Assume that $|\Delta_B^3 f(n)| < \epsilon \ (< 1/8)$. Then

$$|\xi_2(\xi_3 - 3) - (\overline{\xi}_1 - 3| < \epsilon,$$

$$\Big|\xi_2 - \frac{\overline{\xi}_1 - 3}{\xi_3 - 3}\Big| < 2|\Delta_B^3 f(n)|,$$

$$\Big|1 - \Big|\frac{\overline{\xi}_1 - 3}{\xi_3 - 3}\Big|\Big| < 2|\Delta_B^3 f(n)|,$$

$$\Big||\xi_3 - 3| - |\overline{\xi}_1 - 3|\Big| < 8|\Delta_B^3 f(n)|,$$

$$\Big||\xi_3 - 3|^2 - |\overline{\xi}_1 - 3|^2\Big| < 64|\Delta_B^3 f(n)|,$$

$$\Big|6\cos \Delta_B u(u+2B) - 6\cos(-\Delta_B u(n))\Big| < 64|\Delta_B^3 f(n)|.$$

It implies that

$$\|\Delta_B u(n+2B) - \Delta_B u((n) \pmod 1)\| < c|\Delta_B^3 f(n)|^{\frac{1}{2}}.$$

Thus

$$|\xi_2 - 1| < c|\Delta_B^3 f(n)|^{\frac{1}{2}}.$$

The condition (1.8) is equivalent to the assertion:

$$\frac{1}{\log x} \sum_{\substack{|\Delta_B^3 f(n)|>\epsilon \\ n\le x}} \frac{1}{n} \to 0 \qquad \text{for every } \epsilon > 0. \tag{2.1}$$

Thus our assertion is true, since we proved if $|\Delta_B^3 f(n)| \le \epsilon$, in which case $|\Delta_B f(n+B)| \le c\epsilon^{1/2}$.

3. Proof of Theorem 1 and Theorem 2

The proof is similar to the proof of Theorem A, therefore we can shorten the argument.

Let f be given, $\mathcal{A}$ be the set of those $P \in \mathbb{C}[x]$ for which (1.4) holds. If there is a polynomial P with $\deg P = 0$, then (1.5) clearly holds. It is clear that $\mathcal{A}$ is an ideal (see page 122 in [3]).

Let $p(n)$ be the smallest prime factor of n, and for some prime divisor p of n let $\ell_p(n)$ be that exponent for which $p^{\ell_p(n)} \| n$.

Let P be the generator element of $\mathcal{A}$, $k = \deg P$. If $k = 0$, then (1.5) holds. Let $k > 0$. If $P(0) = 0$, then $P(x) = xQ(x)$, $P(E)f(n) = Q(E)f(n+1)$, and so (1.4) holds with Q instead of P. Thus $Q \in \mathcal{A}$. This cannot occur. Repeating the argument used in pages 122–124 of [3], we obtain the following.

Lemma 1. *Assume that P is the minimal degree monic polynomial for which (1.4) holds, and that $k \ge 1$. Then $f(nm) = f(n)f(m)$ whenever $p(n) > 2k+2$ or $p(m) > 2k+2$.*

Arguing as in pages 124–126 (see [3]) we obtain Lemma 2 and Lemma 3.

Lemma 2. *Let $P \in \mathbb{C}[x]$ be the minimal degree monic polynomial for which (1.4) holds. Let $k = \deg P \ge 1$. Then $P(x)$ is a divisor of $(x^B - 1)^k$, B is a suitable integer. Consequently*

$$\sum_{n\le x} \frac{|(x^B - 1)^k f(n)|}{n} = O(\log x). \tag{3.1}$$

Lemma 3. *If there exists an integer D such that*

$$\sum_{\substack{n\le x\\(n,D)=1}} \frac{|f(n)|}{n} = O(\log x), \tag{3.2}$$

then (1.5) *holds.*

We note that if (3.1) is true with B, then it remains valid with Br instead of B with $r = 1,2,3,\cdots$. We may assume that all the primes up to $2k+2$ divide B.

Assume this. Let

$$f^*(n) := \chi_{0,B}(n) f(n), \tag{3.3}$$

where $\chi_{0,B}(n)$ is the principal character $\pmod B$.

Then $f^* \in \mathcal{M}^*$, and

$$\sum_{n\le x} \frac{|(x^B-1)^k f^*(n)|}{n} = O(\log x). \tag{3.4}$$

From Lemma 3 we obtain that

$$\limsup_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|f^*(n)|}{n} = \infty. \tag{3.5}$$

Let q be coprime to B, $q>1$. Let

$$H(n) := (x^B-1)^{k-1} f^*(n).$$

Let K be arbitrary large fixed positive integer. From (3.4) we obtain that

$$\sum_{\substack{n\le x\\(n,B)=1}} \frac{1}{n} \max_{0\le \ell\le K} |H(n+\ell B) - H(n)| = O(\log x). \tag{3.6}$$

The constant on the right hand side of (3.6) may depend on K. Let $h = (q-1)(k-1)$, and let $\beta_0,\cdots,\beta_h$ be the coefficients of $\left(\frac{x^q-1}{x-1}\right)^{k-1}$. Therefore

$$(1+x+\cdots+x^{q-1})^{k-1} = \beta_0+\cdots+\beta_h x^h, \quad q^{k-1} = \beta_0+\cdots+\beta_h.$$

We have

$$\begin{aligned} &(E^{Bq}-I)^{k-1} f^*(qn) = \\ &= (I+E^B+\cdots+E^{B(q-1)})^{k-1}(E^B-I)^{k-1} f^*(qn) = \\ &= \sum_{j=0}^{h} \beta_j H(qn+jB). \end{aligned} \tag{3.7}$$

Let $(n,B)=1$. The left hand side of (3.7) is $f^*(q)H(n)$. Let K be a large constant, ℓ_n any integer, $0\le \ell_n\le K$. From (3.6) we obtain that

$$H(qn+\ell_n B)=\frac{f^*(q)}{q^{k-1}}H(n)+\epsilon_{n,\ell_n}, \tag{3.8}$$

where

$$\sum_{n\le x}\frac{|\epsilon_{n,\ell_n}|}{n}=O(\log x). \tag{3.9}$$

Let

$$E(x)=\sum_{N\le x}\frac{|H(N)|}{N}. \tag{3.10}$$

For an integer N let $a(N)\in\{0,\cdots,q-1\}$ be the integer for which $q|N-a(n)B$, and let

$$N_1=\frac{N-a(N)B}{q}.$$

Some fixed integer M plays the role of N_1 for q distinct values of N, namely for $qM+\ell N \quad (\ell=0,\cdots,q-1)$.

From (3.8) we obtain that (for $N\ge qB, (N,B)=1$)

$$H(N)=\frac{f^*(q)}{q^{k-1}}H(N_1)+\epsilon_{N_1,a(N)}. \tag{3.11}$$

Let $\theta=\theta_q=\frac{|f^*(q)|}{q^{k-1}}$. Then

$$|H(N)|=\theta_q|H(N_1)|+\varrho_{N_1,a(N)} \tag{3.12}$$

$$|\varrho_{N_1,a(N)}|\le|\epsilon_{N_1,a(N)}|.$$

Since

$$\frac{N}{q}-B\le N_1\le\frac{N}{q},$$

therefore

$$E(x)=\theta_q(1+\delta(x))E\Big(\frac{x}{q}\Big)+O(\log x), \tag{3.13}$$

where $|\delta(x)|\to 0$ as $x\to\infty$.

If $E(x)=O(\log x)$, then k can be reduced to $k-1$.

Assume that $E(x)\ne O(\log x)$.

Let $q_1, q_2 \in \mathbb{N}$, $(q_1, q_2) = 1$, $q_1, q_2 > 1$. There exist infinitely many pairs h_1, h_2 for which

$$0 < \frac{\log q_1}{\log q_2} - \frac{h_1}{h_2} < \frac{1}{h_2},$$

and for which

$$-\frac{1}{h_2} < \frac{\log q_1}{\log q_2} - \frac{h_1}{h_2} < 0.$$

From (3.13) we obtain that

$$E(xq^h) = \theta_q^h(1 + \delta(xq^h))E(x) + O(\log x) \tag{3.14}$$

for every fixed q^h. Since $E(x)$ is monotonic, we obtain that

$$\text{if} \quad q_1^{h_1} > q_2^{h_2}, \quad \text{then} \quad \theta_{q_1}^{h_1} > \theta_{q_2}^{h_2},$$

and this may hold only in the case

$$\frac{\log |f^*(q)|}{\log q} = \mathit{constant} = A.$$

If $\theta_q < 1$, then

$$\sup E(x) < \infty,$$

which contradicts our assumption. Thus $A \geq k - 1$.

Theorem 2 is thus proved. ■

Now we complete the proof of Theorem 1.

Assume that $k \leq 3$.

Let us write $f^*(n) = n^A t(n) \chi_{0,B}(n)$, $|t(n)| = 1$ $(n \in \mathbb{N})$. We have

$$\Delta_B^k f^*(n) = n^A \Delta_B^k t(n) + O(n^{A-1}),$$

and so

$$|\Delta_B^k t(n)| \leq \frac{1}{n^A} |\Delta_B^k f^*(n)| + O(\frac{1}{n}).$$

If $k = 1$, $A = 0$, then $|f^*(n)| = 1$ if $(n, B) = 1$, and so (1.5) holds. If $k = 1$, $A > 0$, then

$$\sum_{n \in \mathbb{N}} \frac{|\Delta_B t(n)|}{n} < \infty.$$

In [6] and [7] we proved that $t(n) = n^{i\tau}$ in this case.

Let $k = 2$. If $A = 1$, then

$$\sum_{n \leq x} |\Delta_B^2 t(n)| \leq \sum_{n \leq x} \frac{|\Delta_B^2 f^*(n)|}{n} + c \sum_{n \leq x} \frac{1}{n^2},$$

and so

$$\sum_{n\le x} |\Delta_B^2 t(n)| = O(\log x).$$

If $k=2, A>1$ or if $k=3$, then

$$\sum_{n\le x} |\Delta_B^k t(n)| = O(1).$$

In these cases, for every $\epsilon > 0$,

$$\frac{1}{x}\sharp\Big\{ n\in \big[\frac{x}{2}, x\big] \quad \Big| \quad |\Delta_B^k t(n)| > \epsilon \Big\} \to 0,$$

consequently the Theorem 3 can be applied.

Theorem 1 is thus proved. ■

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