MULTIPlicative FUNCTIONS WITH SMALL INCREMENTS

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Dedicated to the memory of Professor Antal Iványi

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Abstract. Let $k \in \{1, 2, 3\}$. For a polynomial $P(x) = a_0 + a_1 x + \cdots + a_k x^k \in \mathbb{C}[x]$ let $P(E)f(n) = a_0 f(n) + a_1 f(n+1) + \cdots + a_k f(n+k)$. We give all multiplicative functions $f$ which satisfy the relation

$$\sum_{n \leq x} \frac{|P(E)f(n)|}{n} = O(\log x).$$

In the case $P(x) = (E^B - I)^k$, we also give all completely multiplicative function with the conditions $|f(n)| = 1$ if $(n, B) = 1$ and $f(n) = 0$ if $(n, B) > 1$ which satisfy

$$\sum_{n \leq x} \frac{|P(E)f(n)|}{n} = o(\log x).$$

where $B$ is a positive integer.

1. Introduction

Let $\Omega$ be the set of arithmetical functions having complex values. Sometimes a function $f \in \Omega$ is considered as an infinite dimensional vector, the $n$'th coordinate of which is $f(n)$. We write: $f = (f(1), f(2), \cdots)$. Let $\mathbf{x} = (x_1, x_2, \cdots)$
be a general element of $\Omega$. The operators $I$, $E$, $\Delta$, $\Delta_B$ ($\Omega \to \Omega$) are defined according to the following rules: the $n'$th coordinate of $I_x$, $E_x$, $\Delta_x$, $\Delta_B x$ are $x_n$, $x_{n+1}$, $x_{n+1} - x_n$, $x_{n+B} - x_n$, respectively. Let $\Delta^k = (E - I)^k$, $\Delta_B^k = (E_B - I)^k$. If $P(x) = a_0 + a_1x + \cdots + a_kx^k \in \mathbb{C}[x]$, then the $n'$th coordinate of $P(E)x$ equals

$$a_0x_n + a_1x_{n+1} + \cdots + a_kx_{n+k}.$$ 

Let $\mathcal{M}$ ($\mathcal{M}^*$) be the set of complex-valued multiplicative (completely multiplicative) functions. In paper [3], K-H. Indlekofer and I. Kátai proved a theorem which is more general than the following.

**Theorem A.** If $f \in \mathcal{M}$, $P \in \mathbb{C}[x]$, $P \neq 0$ with $k = \deg P$ and

$$\sum_{n \leq x} |P(E)f(n)| = O(x) \quad (x \to \infty),$$

then, either

$$\sum_{n \leq x} |f(n)| = O(x),$$

or there are $s \in \mathbb{C}$ and $F \in \mathcal{M}$ with $0 < \Re s \leq k$ such that

$$f(n) = n^sF(n) \quad \text{and} \quad P(E)F(n) = 0$$

are satisfied for every positive integer $n$.

For other results we refer to works [1], [2], [4], [5] and [9].

We shall prove

**Theorem 1.** Let $f \in \mathcal{M}$, $P \in \mathbb{C}[x]$, $P \neq 0$ with $k = \deg P \leq 3$. Assume that

$$\sum_{n \leq x} \frac{|P(E)f(n)|}{n} = O(\log x) \quad (x \to \infty),$$

then, either

$$\sum_{n \leq x} \frac{|f(n)|}{n} = O(\log x),$$

or there are $s \in \mathbb{C}$ and $F \in \mathcal{M}$ with $0 < \Re s \leq k$ such that

$$f(n) = n^sF(n) \quad \text{and} \quad P(E)F(n) = 0$$

are satisfied for every positive integer $n$. 
We think that Theorem 1 is true for every $k \in \mathbb{N}$. Now we can only prove the following result.

**Theorem 2.** Let $k \in \mathbb{N}$, $k \geq 1$, $P \in \mathbb{C}[x]$ be the smallest degree monic polynomial for which (1.4) holds. Then

$$P(x) = (x^B - 1)^k, \quad B \text{ is a suitable natural number}$$

and

$$|f(n)| = n^a, \quad a \text{ is a positive constant.}$$

The proof of Theorem 1 is based on a simple generalization of a famous theorem of O. Klurman [3], which we state as follows.

**Theorem B.** Let $f \in \mathcal{M}^*$, $B \in \mathbb{N}$,

$$|f(n)| = \begin{cases} 1 & \text{if } (n, B) = 1 \\ 0 & \text{if } (n, B) > 1 \end{cases}$$

and assume that

(1.7) \[
\sum_{n \leq x} \frac{|\Delta_B f(n)|}{n} = o(\log x) \quad (x \to \infty).
\]

Then

$$f(n) = n^i \chi_B(n), \quad \text{where } \chi_B \text{ is a Dirichlet character } \pmod{B}.$$
Proof of Theorem 3

The case $k = 2$.
Let $e(x) := e^{2\pi i x}$, arg$f(n) = 2\pi u(n)$, $u(n) \pmod{1}$ is additive, $\Delta_B u(n) = u(n+B) - u(n)$.
Assume that $\Delta_B^2 f(n) = \epsilon \ (\epsilon < 1/4)$. Since
$$
\Delta_B^2 f(n) = \left| \left( \frac{f(n+2B)}{f(n+B)} - 1 \right) + \left( \frac{f(n)}{f(n+B)} - 1 \right) \right| < \epsilon,
$$
therefore $\cos \Delta_B u(n+B) > 1 - \epsilon$, $\cos(-\Delta_B u(n)) > 1 - \epsilon$, whence
$$
\left| \frac{f(n+B)}{f(n)} - 1 \right| < c\sqrt{\epsilon},
$$
consequently (1.8) implies (1.7).

The case $k = 3$.
Let $\xi_1 := \frac{f(n+B)}{f(n)}$, $\xi_2 := \frac{f(n+2B)}{f(n+B)}$, $\xi_3 := \frac{f(n+3B)}{f(n+2B)}$.
We have
$$
|\Delta_B^3 f(n)| = |\xi_1 \xi_3 - 3| \xi_1 \xi_2 + 3| \xi_1 - 1|.
$$
Assume that $|\Delta_B^3 f(n)| < \epsilon \ (\epsilon < 1/8)$. Then
$$
|\xi_2 (\xi_3 - 3) - (\xi_1 - 3)| < \epsilon,
$$
$$
|\xi_2 - \frac{\xi_1 - 3}{\xi_3 - 3}| < 2|\Delta_B^3 f(n)|,
$$
$$
|1 - \left| \frac{\xi_1 - 3}{\xi_3 - 3} \right| < 2|\Delta_B^3 f(n)|,
$$
$$
|\xi_3 - 3| - |\xi_1 - 3| < 8|\Delta_B^3 f(n)|,
$$
$$
|\xi_3 - 3|^2 - |\xi_1 - 3|^2 < 64|\Delta_B^3 f(n)|,
$$
$$
\left| 6 \cos \Delta_B u(n+2B) - 6 \cos(-\Delta_B u(n)) \right| < 64|\Delta_B^3 f(n)|.
$$
It implies that
$$
\left| \Delta_B u(n+2B) - \Delta_B u(n) \pmod{1} \right| < c|\Delta_B^3 f(n)|^{1/2}.
$$
Thus
\[ |\xi_2 - 1| < c|\Delta_B^3 f(n)|^{1/2}. \]

The condition (1.8) is equivalent to the assertion:
\[
(2.1) \quad \frac{1}{\log x} \sum_{n \leq x, \, |\Delta_B^3 f(n)| > n} \frac{1}{n} \to 0 \quad \text{for every } \epsilon > 0.
\]

Thus our assertion is true, since we proved if \( |\Delta_B^3 f(n)| \leq \epsilon \), in which case \( |\Delta_B f(n + B)| \leq c\epsilon^{1/2} \).

3. Proof of Theorem 1 and Theorem 2

The proof is similar to the proof of Theorem A, therefore we can shorten the argument.

Let \( f \) be given, \( \mathcal{A} \) be the set of those \( P \in \mathbb{C}[x] \) for which (1.4) holds. If there is a polynomial \( P \) with \( \deg P = 0 \), then (1.5) clearly holds. It is clear that \( \mathcal{A} \) is an ideal (see page 122 in [3]).

Let \( p(n) \) be the smallest prime factor of \( n \), and for some prime divisor \( p \) of \( n \) let \( \ell_p(n) \) be that exponent for which \( p^{\ell_p(n)} \) divides \( n \).

Let \( P \) be the generator element of \( \mathcal{A} \), \( k = \deg P \). If \( k = 0 \), then (1.5) holds. Let \( k > 0 \). If \( P(0) = 0 \), then \( P(x) = xQ(x) \), \( P(E)f(n) = Q(E)f(n+1) \), and so (1.4) holds with \( Q \) instead of \( P \). Thus \( Q \in \mathcal{A} \). This cannot occur. Repeating the argument used in pages 122–124 of [3], we obtain the following.

**Lemma 1.** Assume that \( P \) is the minimal degree monic polynomial for which (1.4) holds, and that \( k \geq 1 \). Then \( f(nm) = f(n)f(m) \) whenever \( p(n) > 2k + 2 \) or \( p(m) > 2k + 2 \).

Arguing as in pages 124–126 (see [3]) we obtain Lemma 2 and Lemma 3.

**Lemma 2.** Let \( P \in \mathbb{C}[x] \) be the minimal degree monic polynomial for which (1.4) holds. Let \( k = \deg P \geq 1 \). Then \( P(x) \) is a divisor of \( (x^B - 1)^k \), \( B \) is a suitable integer. Consequently
\[
(3.1) \quad \sum_{n \leq x} \frac{|(x^B - 1)^k f(n)|}{n} = O(\log x).
\]
Lemma 3. If there exists an integer $D$ such that
\[(3.2) \sum_{\substack{n \leq x \\ (n, D) = 1}} \frac{|f(n)|}{n} = O(\log x),\]
then (1.5) holds.

We note that if (3.1) is true with $B$, then it remains valid with $Br$ instead of $B$ with $r = 1, 2, 3, \ldots$. We may assume that all the primes up to $2k + 2$ divide $B$.

Assume this. Let
\[(3.3) f^*(n) := \chi_{0, B}(n)f(n),\]
where $\chi_{0, B}(n)$ is the principal character (mod $B$).

Then $f^* \in \mathcal{M}^*$, and
\[(3.4) \sum_{n \leq x} \frac{|(x^B - 1)^k f^*(n)|}{n} = O(\log x).\]

From Lemma 3 we obtain that
\[(3.5) \limsup_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|f^*(n)|}{n} = \infty.\]

Let $q$ be coprime to $B$, $q > 1$. Let
\[H(n) := (x^B - 1)^{k-1} f^*(n).\]

Let $K$ be arbitrary large fixed positive integer. From (3.4) we obtain that
\[(3.6) \sum_{n \leq x} \frac{1}{n} \max_{0 \leq \ell \leq K} |H(n + \ell B) - H(n)| = O(\log x).\]

The constant on the right hand side of (3.6) may depend on $K$. Let $h = (q - 1)(k - 1)$, and let $\beta_0, \ldots, \beta_h$ be the coefficients of $\left(\frac{x^q - 1}{x - 1}\right)^{k-1}$. Therefore
\[(1 + x + \cdots + x^{q-1})^{k-1} = \beta_0 + \cdots + \beta_h x^h, \quad q^{k-1} = \beta_0 + \cdots + \beta_h.\]

We have
\[(3.7) (E^{Bq} - I)^{k-1} f^*(qn) = (I + E^B + \cdots + E^{B(q-1)})^{k-1} (E^B - I)^{k-1} f^*(qn) = \sum_{j=0}^{h} \beta_j H(qn + jB).\]
Let \((n, B) = 1\). The left hand side of (3.7) is \(f^*(q)H(n)\). Let \(K\) be a large constant, \(\ell_n\) any integer, \(0 \leq \ell_n \leq K\). From (3.6) we obtain that

\[
H(qn + \ell_n B) = \frac{f^*(q)}{q^{k-1}}H(n) + \epsilon_{n, \ell_n},
\]

where

\[
\sum_{n \leq x} \left| \frac{\epsilon_{n, \ell_n}}{n} \right| = O(\log x).
\]

Let

\[
E(x) = \sum_{N \leq x} \frac{|H(N)|}{N}.
\]

For an integer \(N\) let \(a(N) \in \{0, \cdots, q-1\}\) be the integer for which \(q|N - a(n)B\), and let

\[
N_1 = \frac{N - a(N)B}{q}.
\]

Some fixed integer \(M\) plays the role of \(N_1\) for \(q\) distinct values of \(N\), namely for \(qM + \ell N\) \((\ell = 0, \cdots, q-1)\).

From (3.8) we obtain that (for \(N \geq qB, (N, B) = 1\))

\[
H(N) = \frac{f^*(q)}{q^{k-1}}H(N_1) + \epsilon_{N_1, a(N)}.
\]

Let \(\theta = \theta_q = \frac{|f^*(q)|}{q^{k-1}}\). Then

\[
|H(N)| = \theta |H(N_1)| + \varrho_{N_1, a(N)}
\]

\[
|\varrho_{N_1, a(N)}| \leq |\epsilon_{N_1, a(N)}|.
\]

Since

\[
\frac{N}{q} - B \leq N_1 \leq \frac{N}{q},
\]

therefore

\[
E(x) = \theta_q (1 + \delta(x))E\left(\frac{x}{q}\right) + O(\log x),
\]

where \(|\delta(x)| \to 0\) as \(x \to \infty\).

If \(E(x) = O(\log x)\), then \(k\) can be reduced to \(k - 1\).

Assume that \(E(x) \neq O(\log x)\).
Let \( q_1, q_2 \in \mathbb{N}, (q_1, q_2) = 1, q_1, q_2 > 1 \). There exist infinitely many pairs \( h_1, h_2 \) for which
\[
0 < \frac{\log q_1}{\log q_2} - \frac{h_1}{h_2} < \frac{1}{h_2},
\]
and for which
\[
-\frac{1}{h_2} < \frac{\log q_1}{\log q_2} - \frac{h_1}{h_2} < 0.
\]
From (3.13) we obtain that
\[
E(xq^h) = \theta_q^h (1 + \delta(xq^h)) E(x) + O(\log x)
\]
for every fixed \( q^h \). Since \( E(x) \) is monotonic, we obtain that
\[
\text{if } q_1^{h_1} > q_2^{h_2}, \text{ then } \theta_{q_1}^{h_1} > \theta_{q_2}^{h_2},
\]
and this may hold only in the case
\[
\frac{\log |f^*(q)|}{\log q} = \text{constant} = A.
\]
If \( \theta_q < 1 \), then
\[
\sup E(x) < \infty,
\]
which contradicts our assumption. Thus \( A \geq k - 1 \).

Theorem 2 is thus proved. \[\blacksquare\]

Now we complete the proof of Theorem 1.

Assume that \( k \leq 3 \).

Let us write \( f^*(n) = n^A t(n) \chi_{0, B}(n), |t(n)| = 1 \ (n \in \mathbb{N}) \). We have
\[
\Delta_B^k f^*(n) = n^A \Delta_B^k t(n) + O(n^{A-1}),
\]
and so
\[
|\Delta_B^k t(n)| \leq \frac{1}{n^A} |\Delta_B^k f^*(n)| + O(\frac{1}{n}).
\]
If \( k = 1 \), \( A = 0 \), then \( |f^*(n)| = 1 \) if \( (n, B) = 1 \), and so (1.5) holds. If \( k = 1 \), \( A > 0 \), then
\[
\sum_{n \in \mathbb{N}} \frac{|\Delta_B t(n)|}{n} < \infty.
\]

In \[\ref{1}\] and \[\ref{2}\] we proved that \( t(n) = n^{i^*} \) in this case.

Let \( k = 2 \). If \( A = 1 \), then
\[
\sum_{n \leq x} |\Delta_B^2 t(n)| \leq \sum_{n \leq x} \frac{|\Delta_B^2 f^*(n)|}{n} + c \sum_{n \leq x} \frac{1}{n^2},
\]
and
and so
\[ \sum_{n \leq x} |\Delta^2_B t(n)| = O(\log x). \]

If \( k = 2, A > 1 \) or if \( k = 3 \), then
\[ \sum_{n \leq x} |\Delta^k_B t(n)| = O(1). \]

In these cases, for every \( \epsilon > 0 \),
\[ \frac{1}{x} \# \left\{ n \in \left[ \frac{x}{2}, x \right] \mid |\Delta^k_B t(n)| > \epsilon \right\} \to 0, \]

consequently the Theorem 3 can be applied.

Theorem 1 is thus proved. \( \blacksquare \)

References

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