SUPPLEMENT TO PAPER "ON SOME PROBLEMS OF EXPANSIONS INVESTIGATED BY P. ERDŐS ET AL"

Gábor Kallós (Győr, Hungary)

Dedicated to the memory of Professor Antal Iványi

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Abstract. Based on a seminal theorem of P. Erdős et al and using the theory of interval-filling sequences, in this paper we present a generalized result, which ensures the existence of continuum many different fraction expansions, for numbers $x \in (0, L)$, with digit set $\{0, 1\}$.

1. The original theorem, notations

In this short paper we generalize an important theorem presented by P. Erdős, I. Joó and V. Komornik.

We will use the following notations.

Let β be a base of a system, with the assumption β in $(1, 2)$; so, the set of usable digits is $\{0, 1\}$.

For arbitrary such base β let us focus our attention on fraction expansions of a number $x \in [0, L]$, i.e. expansions in the form

$$x = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \frac{\varepsilon_3}{\beta^3} + \cdots \tag{1.1}$$

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where for the digits $\varepsilon_i \in \{0,1\}$ hold. Here L is the largest element in the set of fractions, its value is $1/(\beta-1)$.

Number $1/\beta$ will be denoted by Θ, as usual. Let G be the value of golden ratio, namely $G=(1+\sqrt{5})/2$, and set $g=1/G$.

In the paper we will use the greedy/regular (i), the quasi-greedy/quasi-regular (ii) and the lazy expansions (iii), respectively – see the formal definitions e.g. in [1], [3] and [7]. These expansions are defined so, that by determining digits ε_i-s in (1.1) (i) we always choose the largest possible digit; (ii) we always choose the largest possible digit but with the assumption that for the actual sub-sum equality is not allowed (i.e. the sum is infinite); (iii) we always choose the smallest possible digit so, that the remainder has to be still legally expanded.

The original result was presented in 1991 (paper [4]), and states exactly as follows (several historical relations are listed in [6]):

Theorem ([4]/3)**.** *If $1<\beta<G$ and $0<x<L$, then number x has $2^{\aleph_0}$ (continuum many) different (fraction) expansions.*

The base-pillar of the proof – similarly as in the similar theorem of P. Erdős, M. Horváth and I. Joó, which states the same for 1-expansions (see [3], Theorem 1./a)) – is the following observation.

For base G we have $1=g+g^2$, thus $g=g^2+g^3$. From this it follows $g=2g^3+g^4=g^3+2g^4+g^5,\ldots$, so finally $g=g^3+g^4+g^5+\ldots$ (which is the lazy expansion, as well). If $1<\beta<G$, then $1<\Theta+\Theta^2$, and so, using the above mentioned facts we have

$$\Theta<\Theta^3+\Theta^4+\Theta^5+\ldots, \tag{1.2}$$

and from this it follows, too, that we can find an index m, for which $\Theta < \Theta^3+\Theta^4+\cdots+\Theta^m$ holds. Using this, we are able to thin out cleverly the original index-sequence used for the expansion.

Here we present a modified, new, elegant proof of the original theorem ([4]/3) published by V. Komornik in 2011 ([7]).

Proof. (based on [7])

Since we have $0<x<L$, thus

$$0<x<\Theta+\Theta^2+\Theta^3+\cdots, \tag{1.3}$$

(here on the right-hand side every digit is maximal) and

$$1<\Theta^2+\Theta^3+\Theta^4+\cdots, \tag{1.4}$$

where we have exploited, that $\beta < G$ (for base G the lazy expansion is $1 = 0.0(1)^\infty$). Using inequality (1.3) we are able to fix a large number k so, that

$$\Theta^k + \Theta^{2k} + \Theta^{3k} + \cdots \le x \le \sum_{k \nmid j} \Theta^j \tag{1.5}$$

(the multiples of k are the "scarce" indices, and on the right-hand side index j runs through on those positive integers, which are not multiples of k), and

$$1 \le \Theta^2 + \Theta^3 + \cdots + \Theta^k \tag{1.6}$$

holds, too. Since digits $a_k, a_{2k}, a_{3k}, \dots \in \{0,1\}$ can be chosen in continuum many different ways, the proof will be completed, if we show, that for every such choice there can be found appropriate digits $a_j \in \{0,1\}$ for indices $k \nmid j$ so, that

$$x - (a_k\Theta^k + a_{2k}\Theta^{2k} + a_{3k}\Theta^{3k} + \cdots) = \sum_{k \nmid j} a_j \Theta^j \tag{1.7}$$

should hold. However, this follows from the theorem of S. Kakeya *presented below*, choosing $(\lambda_i) = (\Theta^j)_{k \nmid j}$. The conditions are satisfied, since we have $\lambda_n \to 0$ (and the series is convergent), inequality $\lambda_n \le \lambda_{n+1} + \lambda_{n+2} + \cdots$ $(n = 1, 2, \dots)$ follows from (1.6), and

$$x - (a_k\Theta^k + a_{2k}\Theta^{2k} + a_{3k}\Theta^{3k} + \cdots) \le \lambda_1 + \lambda_2 + \cdots$$

holds because of (1.5).

(So, in (1.7) we are able to supplement the terms with scarce indices left out with the terms with dense indices so, that we would get an expansion of x, and the scarce and dense indices are independent from each other.) ■

2. Interval-filling sequences

The "Erdős et. al."-proof presented above is based on the interval-filling sequences, and to the construction of the more general results we will use these sequences. Thus, in the following we present briefly the original definition and the most important related results.

Theorem (S. Kakeya, 1914).[1] *Let* (λ_n) *be a sequence of positive real numbers, such that series*

$$\sum_{n=1}^{\infty} \lambda_n = L \tag{2.1}$$

is convergent[2] *with sum* L*; moreover inequalities*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \tag{2.2}$$

are fulfilled. Thus, every number $x \in [0, L]$ *may be written in the form*

$$x = \sum_{n=1}^{\infty} e_n \lambda_n, \quad e_n \in \{0, 1\} \tag{2.3}$$

if and only if

$$\lambda_n \leq \lambda_{n+1} + \lambda_{n+2} + \cdots \tag{2.4}$$

holds for all $n = 1, 2, \ldots$.

Definition ([2], [5])**.** A sequence $\{\lambda_i\}$ satisfying conditions (2.1) and (2.2) above is said to be interval-filling (in $[0, L]$), if every number $x \in [0, L]$ can be written in the form (2.3).

Example 2.1. Sequence $\{1/2^i\}$ is interval-filling, with $L = 1$.

The idea of the proof is (the details can be found e.g. in [2]; Z. Daróczy, A. Járai and I. Kátai) that:

a) If the assertion of the theorem does not hold, then we can find a number x, which cannot be written in the form (2.3);

b) If the assertion of the theorem holds, then for every $x \in [0, L]$ we are able to construct a generator sequence $\{e_i\}$.

3. Generalization

To the assertion and proof of the main theorem of this paper, let us fix several conditions, based on the interval-filling sequences.

[1] Based on [5].
[2] From this follows clearly $\lambda_n \to 0$.

Let $\{\lambda_n\}$ be a null-sequence, and for all tail-sum let $L_m = \lambda_{m+1} + \lambda_{m+2} + \cdots$ (notation). Let us assume that $\lambda_n < L_n$ (strict inequality; c.f. (2.4)), and $L = L_0 = \sum_i \lambda_i < \infty$.

Similarly as in Kakeya's theorem, we will examine expansions of numbers $x \in (0, L)$ based on the λ_i's (we can say: "reciprocal values of base-number-powers") i.e. expansions

$$x = \varepsilon_1 \lambda_1 + \varepsilon_2 \lambda_2 + \cdots \tag{3.1}$$

where for the digits $\varepsilon_i \in \{0, 1\}$ hold. The usual greedy (regular) and the quasi-greedy (quasiregular) expansions can be introduced even in this environment, we omit the details here.

Lemma 3.1. *Let be $x \in (0, L)$, and let its quasiregular expansion be $x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n$. Then, in the sequence $\varepsilon_1, \varepsilon_2, \ldots$ there are infinitely many digits* 1 *and* 0, *respectively.*

Proof. According to the definition, the quasiregular expansion cannot be finite, so, the number of digits 1s included in it is clearly infinite.

We claim that this holds even for digits 0s, too.

Let us assume *indirectly* that in the expansion of x we have $\varepsilon_{m+1} = \varepsilon_{m+2} = \cdots = 1$. Since $x < L_0$, therefore here $m \geq 1$ holds. Let us choose m as small as possible. Then

$$x = \varepsilon_1 \lambda_1 + \varepsilon_2 \lambda_2 + \cdots + \varepsilon_m \lambda_m + \sum_{\nu=m+1}^{\infty} \lambda_\nu,$$

and the value of the right-hand side sum is L_m (see it above), and $\varepsilon_m = 0$. Since here $\lambda_m < L_m$, then

$$x = \varepsilon_1 \lambda_1 + \varepsilon_2 \lambda_2 + \cdots + 1 \cdot \lambda_m + (L_m - \lambda_m),$$

where the last term is positive (following the condition above) and smaller than L_{m+1}. From this it follows, however, that in the quasiregular expansion of x it cannot be $\varepsilon_m = 0$, so, our indirect assumption was wrong. ■

Theorem 3.1. *Let $\{\lambda_n\}$ be a null-sequence, and let for all tail-sums be $L_m = \lambda_{m+1} + \lambda_{m+2} + \ldots$. Let us assume that*

a) $\lambda_n < L_n$, *for all n, and*

b) $\lambda_n < L_{n+1}$, *if $n \geq N$ (here N is an arbitrary, fixed positive integer).*

Then, for all $x \in (0, L)$ we have continuum many different expansions in form (3.1).

Proof. Case 1. Let be $N = 1$, which means that condition b) holds for all $n \geq 1$ (e.g. already $\lambda_1 < L_2 = \lambda_3 + \lambda_4 + \cdots$; i.e. λ_2 falls out, c.f. with inequality (1.2) above). Let be $x \in (0, L)$. Then we can find a small positive number ξ for which $\xi < x < L - \xi$. Let us choose from $\{i\}$ a sub-index sequence $m_1, m_2, \ldots$ in the way that $\sum \lambda_{m_i} < \xi$ would hold. Let be $\mathcal{R} = \{m_1, m_2, \ldots\}$, where $m_1 < m_2 < \cdots$ (scarce indices), and $\mathcal{S} = \{n \mid n \notin \mathcal{R}\}$ (dense indices).

Then sequence $\{\lambda_n \mid n \in \mathcal{S}\}$ remains interval-filling, since the elimination of the terms with scarce indices can be replaced from the tail-sum.

Let be $\kappa = \sum_j \delta_j \lambda_{m_j}$, where $\delta_j \in \{0, 1\}$, and δ_j is an arbitrary sequence of values $0 - 1$ (these can be chosen in continuum many different ways). Number $x - \kappa$ can be written using the dense indices: $x - \kappa = \sum_{\nu \in \mathcal{S}} \varepsilon_\nu \lambda_\nu$, $\varepsilon_\nu \in \{0, 1\}$. Then

$$x = \sum_{\nu \in \mathcal{S}} \varepsilon_\nu \lambda_\nu + \sum_j \delta_j \lambda_{m_j},$$

so number x has continuum many different expansions.

Case 2. $N > 1$.

Let the quasiregular expansion of x be

$$x = \sum_{i=1}^{\infty} \alpha_i \lambda_i = \sum_{i=1}^{N} \alpha_i \lambda_i + \vartheta,$$

where in the sequence $\alpha_{N+1}, \alpha_{N+2}, \ldots$ digits 0 and 1 occur infinitely many times (which follows form the Lemma), so $0 < \vartheta < L_N = \lambda_{N+1} + \lambda_{N+2} + \cdots$ (on the right-hand side every coefficient is chosen to be 1).

Let us consider now sequence $\{\lambda_{N+1}, \lambda_{N+2}, \ldots\}$, and let us apply to it the scenario of Case 1, i.e. let us make the scarce and dense indices to it. Thus, besides the continuum many different expansions of a number κ_N we are able to construct a legal expansion to the "remainder" $x_N - \kappa_N$, so, finally we get continuum many different expansions to number x_N.

With this, the proof of the theorem is completed. ■

We note, that if in Theorem 1 numbers $\lambda_1, \lambda_2, \ldots$ are chosen as reciprocal values of base number powers, then we get exactly the case of the Erdős–Joó–Komornik theorem, since e.g. for $N = 1$ rewriting $\lambda_1 < L_2 = \lambda_3 + \lambda_4 + \cdots$ we get

$$\Theta < \Theta^3 + \Theta^4 + \cdots = \frac{\Theta^3}{1 - \Theta},$$

i.e. $\Theta(1-\Theta) < \Theta^3$, from which $(1-\Theta) < \Theta^2$, so finally $1 < \Theta + \Theta^2$. This clearly holds only if the value of Θ is greater than g, i.e. for $1 < \beta < G$. However, the result of Theorem 1 can be applied to more general constructions, too.

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G. Kallós
Széchenyi István University
Dept. of Mathematics and Computer Science
H-9026 Győr
Egyetem tér 1. Hungary
`kallos@sze.hu`