Annales Univ. Sci. Budapest., Sect. Comp. 46 (2017) 15–25

NORMAL NUMBERS IN GENERALIZED NUMBER SYSTEMS IN EUCLIDEAN SPACES

Jean-Marie De Koninck (Québec, Canada)

Imre Kátai (Budapest, Hungary)

Dedicated to Professor Antal Iványi on the occasion of his 75-th birthday

Communicated by Bui Minh Phong

(Received November 20, 2016; accepted December 29, 2016)

Abstract. We introduce the notion of normal numbers for generalized number systems in Euclidean spaces and then explore the relevance of certain conjectures to normality.

1. Generalized number systems in Euclidean spaces

Given a positive integer k, let $\mathbb{R}_k$ and $\mathbb{Z}_k$ stand respectively for the k-dimensional real Euclidean space and the ring of k-dimensional vectors with integer entries. Fix k and let M be a $k\times k$ matrix with integer elements. Assume that M has k distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ such that $|\lambda_1| > |\lambda_2| > \dots >$ $> |\lambda_k| > 1$. Let $\mathcal{L} := M\,\mathbb{Z}_k$. Then, $\mathcal{L}$ is a subgroup of $\mathbb{Z}_k$ and let t stand for the order of $\mathbb{Z}_k/\mathcal{L}$, so that $t = |\det M|$. Further let $A_0, A_1, \dots, A_{t-1}$ stand for the residue classes mod $\mathcal{L}$ and let $A_0 = \mathcal{L}$. For each $j \in \{0, 1, \dots, t-1\}$, choose an arbitrary element $\underline{a}_j \in A_j$ such that the vector $\underline{a}_0$ is the zero vector $\underline{0} = (0, 0, \dots, 0)$, and then write

$$\mathcal{A} := \{\underline{a}_0, \underline{a}_1, \dots, \underline{a}_{t-1}\}.$$

If the norm $\|\underline{n}\|$ of $\underline{n} = (n_1, \dots, n_k)$ is $\|\underline{n}\| = \max_{1\le i\le k} |n_i|$ or $\|\underline{n}\| = \sum_{1\le i\le k} |n_i|$, then the operator norm $\|\cdot\|$ of M^{-1} is $1/|\lambda_k|$ while that of M is $|\lambda_1|$.

Key words and phrases: Number systems, normal numbers.
2010 Mathematics Subject Classification: 11K16, 11A67.

Let us now introduce the function $J : \mathbb{Z}_k \to \mathbb{Z}_k$ as follows. Since for each $\underline{n} \in \mathbb{Z}_k$, there exist a unique $\underline{b}_0 \in \mathcal{A}$ for which $\underline{n} - \underline{b}_0 \in \mathcal{L}$ and a unique $\underline{n}_1 \in \mathbb{Z}_k$ for which $\underline{n} = \underline{b}_0 + M\,\underline{n}_1$, that is, $\underline{n}_1 = M^{-1}(\underline{n} - \underline{b}_0)$, we define $J : \mathbb{Z}_k \to \mathbb{Z}_k$ by $J(\underline{n}) = \underline{n}_1$.

We further define the real numbers K, ξ and L by

$$K = \max_{b \in \mathcal{A}} \|b\|, \qquad \xi = \frac{1}{\min_{1 \le j \le k} |\lambda_j|} = \left|M^{-1}\right|, \qquad L = \frac{K\xi}{1-\xi}.$$

In [3], the following result was proved.

Lemma 1.

(a) *If $\|\underline{n}\| > L$, then $\|J(\underline{n})\| < \|\underline{n}\|$.*

(b) *If $\|\underline{n}\| \le L$, then $\|J(\underline{n})\| \le L$.*

Since the disks contain only a finite number of elements of $\mathbb{Z}_k$, it follows that the path

$$\underline{n}, \quad J(\underline{n}), \quad J^2(\underline{n}), \ \ldots$$

is ultimately periodic.

Now, let $\mathcal{P}$ stand for the set of periodic elements. Then, $\underline{n} \in \mathcal{P}$ if there is an integer $j \ge 1$ such that $J^j(\underline{n}) = \underline{n}$. The directed graph (over $\mathcal{P}$) is defined by $\underline{n} \to J(\underline{n})$ $(\underline{n} \in \mathcal{P})$. It is clear that $\underline{n} \in \mathcal{P}$ implies that $J(\underline{n}) \in \mathcal{P}$ and that the directed graph $J\mathcal{P} \to \mathcal{P}$, which we denote by $G(\mathcal{P})$, is the union of disjoint directed circles (allowing for loops). Moreover, $\underline{0}\,(\to \underline{0}) \in \mathcal{P}$, and if $\pi \in \mathcal{P}$, then $\|\pi\| \le L$.

Now, for each $\underline{n} \in \mathbb{Z}_k$ and integer $h \ge 1$, we have

$$\begin{aligned} \underline{n} &= \underline{b}_0 + M\,\underline{b}_1 + \cdots + M^{h-1}\underline{b}_{h-1} + M^h\underline{n}_h, \\ \underline{n}_h &= J^h(\underline{n}_0), \qquad \underline{b}_\nu \in \mathcal{A}. \end{aligned}$$

Further define

$$\ell(\underline{n}) := \begin{cases} 0 & \text{if} \quad \underline{n} \in \mathcal{P}, \\ h & \text{if} \quad \underline{n} \notin \mathcal{P}, \end{cases}$$

where h is the smallest integer for which $\underline{n}_h \in \mathcal{P}$. For this reason, we will say and write that the standard expansion of $\underline{n}$ is $(\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_{h-1}|\pi)$, where $\pi = \underline{n}_h$. In the special case where $\underline{n} = \pi \in \mathcal{P}$, the expansion is written as $(*|\pi)$.

We say that $(\mathcal{A}, M)$ is a *number system* (written for short as NS) in $\mathbb{Z}_k$ if each $\underline{n} \in \mathbb{Z}_k$ can be written as

$$\underline{n} = \underline{b}_0 + M\,\underline{b}_1 + \cdots + M^{h-1}\underline{b}_{h-1}.$$

In other words, $(\mathcal{A}, M)$ is a number system in $\mathbb{Z}_k$ if and only if $\mathcal{P} = \{\underline{0}\}$.

Let H be the set of those $\underline{z} \in \mathbb{R}_k$ which can be expanded as

$$\underline{z} = \sum_{\nu=1}^{\infty} M^{-\nu} \underline{b}_\nu, \qquad \underline{b}_\nu \in \mathcal{A}.$$

The set H is called the *fundamental region* with respect to $(\mathcal{A}, M)$.

For each integer $h \geq 0$, let

$$\Gamma_h := \left\{ \underline{n} : \underline{n} = \sum_{j=0}^{h} M^j \underline{b}_j, \quad \underline{b}_j \in \mathcal{A} \right\},$$

so that in particular $\Gamma_h \subseteq \Gamma_{h+1}$. Letting $\Gamma = \bigcup_{h=0}^{\infty} \Gamma_h$, we have that $\Gamma \subseteq \mathbb{Z}_k$ and one can easily see that $\Gamma = \mathbb{Z}_k$ if and only $(\mathcal{A}, M)$ is a number system.

Since we can write the fundamental region H as

$$H = \bigcup_{\underline{a} \in \mathcal{A}} \left(M^{-1}\underline{a} + M^{-1}H \right),$$

it is easily seen that H is a compact set.

The following result was proved in [3].

Theorem A. *Let λ stand for the Lebesgue measure in $\mathbb{R}_k$.*

(a) *We have* $\bigcup_{\underline{n} \in \mathbb{Z}_k} (H + \underline{n}) = \mathbb{R}_k$.

(b) *If $\underline{n}_1, \underline{n}_2 \in \Gamma$, $\underline{n}_1 \neq \underline{n}_2$, then*

$$\lambda(H + \underline{n}_1 \cap H + \underline{n}_2) = 0.$$

(c) *If $\Gamma = \mathbb{Z}_k$, that is if $(\mathcal{A}, M)$ is a number system, then*

$$\lambda(H + \underline{n}_1 \cap H + \underline{n}_2) = 0$$

for every $\underline{n}_1, \underline{n}_2 \in \mathbb{Z}_k$ with $\underline{n}_1 \neq \underline{n}_2$.

2. Just touching covering system

We now introduce the concept of just touching covering system. We say that $(\mathcal{A}, M)$ is a *just touching covering system* (for short JTCS) if $\lambda(H + \underline{n}_1 \cap$ $\cap H + \underline{n}_2) = 0$ for every $\underline{n}_1, \underline{n}_2 \in \mathbb{Z}_k$ with $\underline{n}_1 \neq \underline{n}_2$.

Interestingly, if $(\mathcal{A}, M)$ is a JTCS, then

$$\lambda(M^{-h}\underline{n}_1 + M^{-h}H \cap M^{-h}\underline{n}_2 + M^{-h}H) = 0$$

for every $\underline{n}_1, \underline{n}_2 \in \mathbb{Z}_k$ with $\underline{n}_1 \neq \underline{n}_2$.

The next two results reveal interesting properties regarding JTCS.

Theorem B. ([4]) *The number system $(\mathcal{A}, M)$ is a JTCS if $\Gamma - \Gamma = \mathbb{Z}_k$, that is if every $\underline{n} \in \mathbb{Z}_k$ can be written as $\underline{n}_1 - \underline{n}_2$, where $\underline{n}_1, \underline{n}_2 \in \mathbb{Z}_k$.*

Theorem C. ([6]) *Given $D \in \mathbb{Z} \setminus \{0\}$, let $A = \{a_0, a_1, \dots, a_{|D|-1}\}$ (where $a_0 = 0$) be a complete residue system mod D. Then, $(\mathcal{A}, D)$ is a JTCS if and only if $\gcd(a_1, \dots, a_{|D|-1}) = 1$.*

Let $(\mathcal{A}, M)$ be a JTCS and let

$$\xi = \sum_{\ell=-r}^{\infty} M^{-\ell}\underline{c}_\ell \qquad (\underline{c}_\ell \in \mathcal{A}).$$

We write the "integer part" and "fractional part" of ξ as follows:

$$\begin{aligned} \lfloor \xi \rfloor &= \sum_{\ell=-r}^{0} M^{-\ell}\underline{c}_\ell \qquad (\in \mathbb{Z}_k), \\ \{\xi\} &= \sum_{\ell=1}^{\infty} M^{-\ell}\underline{c}_\ell \qquad (\in H). \end{aligned}$$

Observe that it is clear that

$$\{M^u \xi\} = \sum_{\ell=1}^{\infty} M^{-\ell}\underline{c}_{u+\ell} \qquad (\in H).$$

Moreover, letting $\beta = \underline{b}_1\underline{b}_2 \dots \underline{b}_k$, let us define

$$H_\beta := \left\{ \eta : \eta = \sum_{\ell=1}^{\infty} M^{-\ell}\underline{c}_\ell : \underline{c}_\ell = \underline{b}_\ell \text{ for } \ell = 1, 2, \dots, k \right\}.$$

It is clear that, for a fixed k, any two H_{β_1} and H_{β_2} will be isomorphic since

$$H = \sum_{\ell=1}^{k} M^{-\ell}\underline{b}_\ell + M^{-k}H,$$

and

(i) $H = \bigcup_{\beta \in A^k} H_\beta$,

(ii) $\lambda(H_{\beta_1} \cap H_{\beta_2}) = 0$,

(iii) $\lambda(H_{\beta_1}) = \lambda(H_{\beta_2})$,

(iv) $\lambda(H_\beta) t^k = \lambda(H)$.

3. Normal sequences and normal numbers in $\mathbb{R}$

Let $A = \{a_1, \ldots, a_N\}$ be a finite set of letters. Let A^* be the set of finite words over A. Given a word $\alpha \in A^*$, we write $\lambda(\alpha)$ to denote its length (that is, the number of letters in the word α). We let Λ stand for the empty word and write $\lambda(\Lambda) = 0$. The operation $(\alpha, \beta) \to \alpha\beta$ is called *concatenation*. The expression $A^{\mathbb{N}}$ stands for the set of infinite sequences over A, that is, $\beta \in A^{\mathbb{N}}$ if it can be written as $\beta = b_1 b_2 b_3 \ldots$, where each $b_i \in A$. Moreover, given $\beta \in A^{\mathbb{N}}$ and a positive integer T, we set $\beta^T := b_1 b_2 \ldots b_T$. Given $\gamma, \delta \in A^*$, we let $S(\delta|\gamma)$ stand for $\#\{\epsilon_1, \epsilon_2 \in A^* : \gamma = \epsilon_1 \delta \epsilon_2\}$, that is, the number of occurrences of δ as a subword in γ.

Definition. Let $\beta \in A^{\mathbb{N}}$. We say that β is a *normal sequence* (over A) if

$$\lim_{T \to \infty} \frac{S(\alpha|\beta^T)}{T} = \frac{1}{N^{\lambda(\alpha)}}$$

for every $\alpha \in A^*$.

4. Normal sequences and normal numbers in $\mathbb{R}_k$

Definition. Let $(\mathcal{A}, M)$ be a number system and let $\eta = \sum_{\ell=1}^{\infty} M^{-\ell} \underline{b}_\ell$, with each $\underline{b}_\ell \in \mathcal{A}$. We say that η is a *normal number in* $\mathbb{R}_k$ with respect to $(\mathcal{A}, M)$ if, for every $\beta \in A^*$,

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \le N : \{M^n \eta\} \in H_\beta\} = \frac{1}{t^{\lambda(\beta)}},$$

where $t = |\det M|$.

The following two assertions are obvious.

- (I) η is a normal number in $\mathbb{R}_k$ with respect to $(\mathcal{A}, M)$ if and only if $\beta = \underline{b}_1\underline{b}_2\dots$ is a normal sequence over A.

- (II) Let $E = \{e_1, \dots, e_k\}$, $D = \{d_1, \dots, d_k\}$, $\varphi : E \to D$ defined by $\varphi(e_j) = d_j$, $\beta = b_1 b_2 \dots \in E^{\mathbb{N}}$, $\varphi(\beta) = \varphi(b_1)\varphi(b_2)\dots$ $(\in D^{\mathbb{N}})$. Then, β is a normal sequence in $E^{\mathbb{N}}$ if and only if $\varphi(\beta)$ is a normal sequence in $D^{\mathbb{N}}$.

In light of these assertions, one can easily prove the following theorem.

Theorem 1. *Let $(\mathcal{A}, M)$ be a JTCS with $\mathcal{A} = \{\underline{a}_0 = \underline{0}, \underline{a}_1, \dots, \underline{a}_{t-1}\}$, where $t = |\det M|$. Moreover, let $E = \{0, 1, \dots, t-1\}$ and let $\eta = 0.\epsilon_1\epsilon_2\dots$ be an arbitrary t-ary normal number. Then, $\psi = \sum_{\ell=1}^{\infty} M^{-\ell}\underline{a}_{\epsilon_\ell}$ is a normal number in $\mathbb{R}_k$ with respect to $(\mathcal{A}, M)$.*

5. Construction of base Q normal numbers

Fix an integer $Q \geq 2$. Let $\mathcal{A}_Q = \{0, 1, \dots, Q-1\}$ and let $\mathcal{A}_Q^*$ stand for the set of words over $\mathcal{A}_Q$. For each integer $N \geq 1$, let $J_N = [Q^{N-1}, Q^N - 1]$. Given an integer $n \in J_N$, write it as $n = \sum_{\nu=0}^{N-1} \epsilon_\nu(n) Q^\nu$ and define $\overline{n} := \epsilon_1(n)\epsilon_1(n)\dots\epsilon_{N-1}(n) \in \mathcal{A}_Q^*$. Finally, we let $\lambda(\overline{n}) = N$ stand for the length of $\overline{n}$.

For each integer $N \geq 3$, consider a subset S_N of $\{1, 2, \dots, N-1\}$, writing it as $S_N = \{\ell_1^{(N)}, \dots, \ell_{r_N}^{(N)}\}$, where the $\ell_i^{(N)}$'s are in increasing order. Assume that $r_N \geq 1$ and that $(r_1 + \dots + r_{N-1})/r_N \to \infty$ as $N \to \infty$.

To each prime $p \in J_N$, let us associate the number

$$\kappa(p) = \epsilon_{\ell_1^{(N)}}(p) \dots \epsilon_{\ell_{r_N}^{(N)}}(p).$$

Let $p_1 < \dots < p_{\pi(J_N)}$ be all the primes included in J_N. Moreover, let σ_N be an arbitrary permutation of $\{1, \dots, \pi(J_N)\}$. Further define

$$\eta_N := \kappa(p_{\sigma_N(1)}) \dots \kappa(p_{\sigma_N(\pi(J_N))}).$$

Finally, consider the number

$$\alpha = 0.\eta_1\eta_2\dots$$

Theorem 2. *The number α is a normal number in base Q.*

Proof. This is an easy consequence of an earlier result obtained by Harman and Kátai [8] and according to which, given r integers $(1 \le)\ j_1 < \cdots < j_r\ (\le N-1)$, setting

$$\Pi\left(J_N \,\middle|\, \begin{matrix} j_1,\dots,j_r \\ b_1,\dots,b_r \end{matrix}\right) := \#\{p \in J_N : a_{j_\ell}(p) = b_{j_\ell} \text{ for } \ell = 1,\dots,r\},$$

we have

$$\max_{\substack{1\le j_1<\cdots<j_r\le N-1 \\ b_1,\dots,b_r}} \left| \frac{Q^r \Pi\left(J_N \,\middle|\, \begin{matrix} j_1,\dots,j_r \\ b_1,\dots,b_r \end{matrix}\right)}{\pi(J_N)} - 1 \right| \to 0 \qquad (N \to \infty)$$

for every fixed integer $r \ge 1$. ■

Theorem 3. *If $S_N = \{1,\dots,N-1\}$, then Theorem 2 holds without the condition $(r_1 + \cdots + r_{N-1})/r_N \to \infty$ as $N \to \infty$.*

Theorem 4. *Let $\wp_N$ be the set of primes in J_N. Given a prime $p \in J_N$, write its Q-ary expansion as*

$$\overline{p} = \varepsilon_0(p)\epsilon_1(p)\dots\epsilon_{N-1}(p).$$

Then, set

$$\gamma_N = \mathit{Concat}(\overline{p} : p \in \wp_N).$$

Fix an integer $D \in \mathbb{N}$ and consider the real number

$$\alpha = 0.\gamma_D\gamma_{2D}\dots = 0.a_1a_2\dots,$$

say. Further consider the number

$$\alpha^{(\ell)} = 0.\mathit{Concat}(a_m : m \equiv \ell \pmod D) = 0.a_\ell a_{D+\ell}a_{D+2\ell}\dots,$$

say. Let $\ell_1,\dots,\ell_h$ be a set of distinct residues mod D and consider the real number

$$\delta = 0.\mathit{Concat}(a_m : m \equiv \ell \pmod D \text{ for some } \ell \in \{\ell_1,\dots,\ell_h\}).$$

Then the numbers α, $\alpha^{(\ell)}$ for each $\ell = 0,1,\dots,D-1$, δ for each $\ell \in \{\ell_1,\dots,\ell_h\}$, are all Q-normal numbers.

Proof. The proof can be obtained along the same lines as that of Theorem 2. ■

6. The relevance of certain conjectures to normality

6.1. On the conjecture of Chowla and its generalisations

Let $\Omega(1) = 0$ and, for each integer $n \geq 2$, let $\Omega(n) := \sum_{p^a \| n} a$. Then, the Liouville function λ is defined on positive integers n by $\lambda(n) = (-1)^{\Omega(n)}$. An old conjecture of Chowla states that, for any given positive integers $a_1 < a_2 < \cdots < a_k$,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \lambda(n)\lambda(n+a_1)\cdots\lambda(n+a_k) = 0. \tag{6.1}$$

If the Chowla conjecture were true, then, given any predetermined vector $(\delta_0, \delta_1, \ldots, \delta_k)$, where each $\delta_j \in \{-1, 1\}$, it would follow that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : \lambda(n+j) = \delta_j \text{ for } j = 0, 1, \ldots, k\} = \frac{1}{2^{k+1}},$$

in which case, by setting $\epsilon_n = (\lambda(n)+1)/2$, it would also follow that the number

$$\alpha = 0.\epsilon_1\epsilon_2\ldots \tag{6.2}$$

is a binary normal number.

Recently, Terence Tao [9] obtained an important result in this direction, namely by proving that, given any fixed positive integer a,

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\leq x} \frac{\lambda(n)\lambda(n+a)}{n} = 0. \tag{6.3}$$

From this, setting $b_n = (\lambda(n)+1)/2$ and

$$\gamma = 0.b_1b_2\ldots, \tag{6.4}$$

it follows that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{n\leq x \\ b_n=\epsilon_1,\ b_{n+1}=\epsilon_2}} \frac{1}{n} = \frac{1}{4}$$

for every choice of $(\epsilon_1, \epsilon_2) \in \{0, 1\}^2$.

If the Chowla conjecture is true (in the form given by (6.1)), one can prove that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\leq x} \frac{\lambda(n)\lambda(n+a_1)\cdots\lambda(n+a_k)}{n} = 0. \tag{6.5}$$

Perhaps (6.5) is easier to prove that the original conjecture (6.1).

In any event, from conjecture (6.5), it would follow that the real number γ (in (6.4)) is a binary normal number with "weight $1/n$", meaning that if for each positive integer n, we set $\gamma_n := 0.b_{n+1}b_{n+2}\ldots$ and, for any given interval $E = [a, b) \subseteq [0, 1)$, we consider the characteristic function

$$\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \in E, \\ 0 & \text{if} \quad x \notin E \end{cases}$$

along with the corresponding function $S_N(E) = \sum_{n=1}^{N} \frac{1}{n}\chi_E(\gamma_n)$, then

$$\lim_{N\to\infty} \frac{S_N(E)}{\log N} = b - a,$$

namely the length of the interval E.

6.2. A conjecture of Elliott

The following conjecture was stated by Elliott [7] in 1994.

Conjecture 1. (Elliott) *Let $g_1, \ldots, g_k$ be multiplicative functions such that $|g_j(n)| \le 1$ for all integers $n \ge 1$, for each $j \in \{1, 2, \ldots, k\}$. Moreover, for each $j = 1, 2, \ldots, k$, let $a_j \in \mathbb{N}$ and $b_j \in \mathbb{Z}$ be such that $a_r b_t - a_t b_r \neq 0$ when ever $1 \le r < t \le k$. Then, there exist constants $A, \alpha \in \mathbb{R}$ and a slowly oscillating function $L(u)$ such that $|L(u)| = 1$ for all $u \in \mathbb{R}$, such that, as $x \to \infty$,*

$$s(x) := \frac{1}{x}\sum_{n\le x} g_1(a_1 n + b_1)\cdots g_k(a_k n + b_k) = A\, x^{i\alpha} L(\log x) + o(1).$$

If $\limsup_{x\to\infty} |s(x)| = |A| > 0$, then there are Dirichlet characters χ_j and real numbers τ_j for which the series

$$\Re\left(\sum_p \frac{1 - g_j(p)\chi_j(p)p^{-i\tau_j}}{p}\right)$$

converges.

It is clear that the Chowla conjecture would follow from the Elliott conjecture. Another interesting consequence of Conjecture 1 is the following yet unproven result.

Conjecture 2. Let g be a multiplicative function such that $|g(n)| = 1$ for all $n \in \mathbb{N}$ and assume that, for every $\tau \in \mathbb{R}$ and Dirichlet character χ,

$$\sum_p \frac{\Re(1 - g(p)\chi(p)p^{i\tau})}{p} = \infty.$$

Then, given arbitrary positive integers $a_1 < a_2 < \cdots < a_k$,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} g(n)g(n + a_1)\cdots g(n + a_k) = 0. \tag{6.6}$$

As a special case of Conjecture 2, one has the following. Fix an integer $Q \ge 2$ and assume that $g(n)^Q = 1$ for all integers $n \ge 1$. Hence the range of $g(\mathbb{N})$ is $\{\xi^\ell : \ell = 0, 1, \ldots, Q - 1\}$ for some root of unity ξ, namely $\xi = e^{2\pi i/Q}$. We can therefore write $g(n)$ as $g(n) = \xi^{\epsilon_n}$, where each $\epsilon_n \in \mathcal{A}_Q$. With this set up, let us introduce the real number

$$\alpha = 0.\epsilon_1\epsilon_2\ldots \tag{6.7}$$

If (6.6) were true, then this would imply that α is a normal number in base Q.

Observe that the multiplicative function g could have been chosen differently. Here are some appropriate choices for Q and g:

(I) $Q = 2$ and $g(n) = (-1)^{\Omega(n)}$.

(II) $Q = 2$ and $g(n) = (-1)^{\omega(n)}$.

(III) $Q \ge 2$, $\xi = e^{2\pi i\ell/Q}$ with $(\ell, Q) = 1$ and then choose $g(p) = \xi$ for each prime p and, for each $k \ge 2$, choose choose $g(p^k)$ in an arbitrary way as long as $|g(p^k)| = 1$.

(IV) $Q \ge 2$, $\xi = e^{2\pi i/Q}$ and then, if $p \equiv \ell \pmod{K}$ for any given ℓ and K with $(\ell, K) = 1$ and $(e_\ell, Q) = 1$, choose $g(p) = \xi^{e_\ell}$ for each prime p, and $g(p) = 1$ if $p \mid K$, while choosing $g(p^k)$ in an arbitrary way for each $k \ge 2$ as long as $|g(p^k)| = 1$.

Acknowledgement. Research of the first author was supported in part by a grant from NSERC.

References

[1] **Kátai, I.,** *Generalized Number Systems and Fractal Geometry*, Lecture Notes Janus Pannonius Universitas, Pécs, 1985, pp. 40.

[2] **Kátai, I.,** Generalized number systems in Euclidean spaces, *Mathematical and Computer Modelling*, **98** (2003), 883–892.

[3] **Indlekofer, K.-H., I. Kátai and P. Racskó,** Some remarks on generalized number systems, *Acta Sci. Math.* (*Szeged*), **57(1–4)** (1993), 543–553.

[4] **Indlekofer, K.-H., I. Kátai and P. Racskó,** Number systems and fractal geometry, in: *Probability Theory and its Applications* (Editors J. Galambos and I. Kátai). Kluwer Ac. Publ. Dordrecht, 1992, 319–334.

[5] **Michalek, G.E.,** Base three just touching covering systems, *Publ. Math. Debrecen*, **51(3–4)** (1997), 241–263.

[6] **Michalek, G.E.,** Base N just touching covering systems, *Publ. Math. Debrecen*, **58** (2001), 2003–2016.

[7] **Elliott, P.D.T.A.,** On the correlations of multiplicative and the sum of additive arithmetic functions, *Mem. Amer. Math. Soc.*, **112** (1994), no. 538, viii+88 pp.

[8] **Harman, G. and I. Kátai,** Primes with preassigned digits II, *Acta Arith.*, **133(2)** (2008), 171–184.

[9] **Tao, T.,** The logarithmically averaged Chowla and Elliott conjectures for two-point correlations, `https://arxiv.org/pdf/1509.05422v4`.

J.-M. De Koninck
Dép. math. et statistique
Université Laval
Québec
Québec G1V 0A6
Canada
`jmdk@mat.ulaval.ca`

I. Kátai
Computer Algebra Department
Eötvös Loránd University
1117 Budapest
Pázmány Péter Sétány 1/C
Hungary
`katai@inf.elte.hu`