Annales Univ. Sci. Budapest., Sect. Comp. 45 (2016) 277–290

SOME GENERALIZATIONS OF LEBESGUE'S THEOREM FOR TWO-DIMENSIONAL FUNCTIONS

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Communicated by Ferenc Schipp

(Received July 29, 2016; accepted September 1, 2016)

Abstract. The classical Lebesgue's theorem about the convergence of the Fejér means at Lebesgue points is generalized for five different summability methods and for two-dimensional functions from the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d)$.

1. Introduction

For the Fejér means [11] of an integrable function f, the classical theorem of Lebesgue [20] says that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T s_t f(x)\, dt = f(x)$$

at each Lebesgue point of f, thus almost everywhere, where $s_t f$ denotes the tth Dirichlet integral of the one-dimensional function f. In the present survey paper five different summability methods and five generalizations of this result will be given for two-dimensional functions. All the five summability methods are investigated exhaustively in the literature.

Key words and phrases: Fourier transforms, Fejér summability, θ-summability, Lebesgue points.
2010 Mathematics Subject Classification: Primary 42B08, Secondary 42A38, 42A24, 42B25.
This research was supported by the Hungarian Scientific Research Funds (OTKA) No K115804.

A general method of summation, the so called θ-summation method, which is generated by a single function θ and which includes the well known Fejér, Riesz, Weierstrass, Abel, etc. summability methods, is studied intensively in the literature (see e.g. Butzer and Nessel [6], Trigub and Belinsky [2, 29, 30], Liflyand [21], Gát [12, 13, 14], Goginava [15, 16, 17], Simon [26], Persson, Tephnadze and Wall [24] and Weisz [33, 34]). Lebesgue points of multi-dimensional functions are investigated in Belinsky, Liflyand and Trigub [3, 1] and in Feichtinger and Weisz [9, 10].

For two-dimensional functions the θ-summability can be defined by

$$\sigma_T^\theta f(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta\left(\frac{\|(u,v)\|_q}{T}\right)\widehat{f}(u,v)e^{\imath(xu+yv)}\,du\,dv \qquad (T>0)$$

or by

$$\sigma_T^\theta f(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta_1\left(\frac{|u|}{T_1}\right)\theta_2\left(\frac{|v|}{T_2}\right)\widehat{f}(u,v)e^{\imath(xu+yv)}\,du\,dv \qquad (T_1,T_2>0).$$

The second type of summation was considered e.g. in Zygmund [38], Gát [12] and Weisz [33, 34]. In the first definition the cases $q=1,2,\infty$ are investigated exhaustively in the literature, the case $q=2$ in Stein and Weiss [26], Davis and Chang [8] and Grafakos [18], the case $q=1$ in Berens [4, 5], Szili and Vértesi [28] and the case $q=\infty$ in Marcinkiewicz [22], Zhizhiashvili [37] and Weisz [33, 34].

In this paper we introduce a new concept of Lebesgue points for each summability just mentioned. We generalize Lebesgue's theorem for these new Lebesgue points and for the different summability methods and for two-dimensional functions from the Wiener amalgam space $W(L_1,\ell_\infty)(\mathbb{R}^d) \supset L_1(\mathbb{R}^d)$. All the results of this paper hold e.g. for the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations.

This paper was the base of my talk given at the 11th Joint Conference on Mathematics and Computer Science, May 2016, in Eger (Hungary).

2. Wiener amalgam spaces

The space $L_p(\mathbb{R}^d)$ is equipped with the norm

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}^2} |f|^p\,d\lambda\right)^{1/p}, & 0<p<\infty; \\ \sup_{\mathbb{R}^2} |f|, & p=\infty, \end{cases}$$

where the dimension d is 1 or 2. A measurable function f belongs to the *Wiener amalgam space* $W(L_p,\ell_q)(\mathbb{R}^d)$ $(1 \leq p, q \leq \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|^q_{L_p[0,1)^d} \right)^{1/q} < \infty,$$

with the obvious modification for $q = \infty$.

It is easy to see that $W(L_p,\ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and

$$W(L_\infty,\ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1,\ell_\infty)(\mathbb{R}^d) \qquad (1 \leq p \leq \infty).$$

3. The one-dimensional θ-summability

The *Fourier transform* of a one-dimensional function $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-\imath xu}\, du \qquad (x \in \mathbb{R}),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R})$ for some $1 \leq p \leq 2$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(u) e^{\imath xu}\, du \qquad (x \in \mathbb{R}, \widehat{f} \in L_1(\mathbb{R})). \tag{3.1}$$

The integrability condition of $\widehat{f}$ is a very strong condition. If this not holds, we may consider the Dirichlet integral $s_T f$:

$$s_T f(x) \quad := \quad \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{f}(u) e^{\imath xu}\, du,$$

which is well defined. It is known that for $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T\to\infty} s_T f = f \qquad \text{in the } L_p(\mathbb{R})\text{-norm and a.e.}$$

The norm convergence is due to Riesz [25] and the almost everywhere convergence is the famous Carleson's theorem (see Carleson [7] and Hunt [19] or recently Grafakos [18]).

This convergence does not hold for $p = 1$. However, using a summability method, we can generalize these results. We may take a general summability method, the so called θ-summation defined by a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$\theta(0) = 1$. This summation contains all well known summability methods, such as the well known Weierstrass, Abel, Picar, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations. This means that we multiply the integrand by a suitable function θ in the Fourier inversion formula (3.1). More precisely, in the one-dimensional case let

$$\sigma_T^\theta f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \theta\left(\frac{|u|}{T}\right) \widehat{f}(u) e^{\imath x u}\, du. \tag{3.2}$$

This definition can easily be extended to all $f \in W(L_1, \ell_\infty)(\mathbb{R})$.

4. One-dimensional Lebesgue's theorem

A point $x \in \mathbb{R}$ is called a *Lebesgue point* of $f \in W(L_1, \ell_\infty)(\mathbb{R})$ if

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} |f(x-s) - f(x)|\, ds = 0.$$

It is known that almost every point $x \in \mathbb{R}$ is a Lebesgue point of f.

For $\theta(t) = \max((1-|t|), 0)$, we obtain the Fejér means:

$$\begin{aligned} \sigma_T f(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \frac{|u|}{T}\right) \widehat{f}(u) e^{\imath x u}\, du \\ &= \frac{1}{T} \int_0^T s_t f(x)\, dt. \end{aligned}$$

The following well known theorem is due to Lebesgue [20] (see also [11]).

Theorem 4.1. *For all Lebesgue points of $f \in L_1(\mathbb{R})$,*

$$\lim_{T\to\infty} \sigma_T f(x) = f(x).$$

5. The two-dimensional θ-summability

Let us turn to the two-dimensional functions. The first question is, how can we generalize the definition (3.2) for two-dimensional functions? There are some generalizations, which are investigated exhaustively in the literature. In the first natural generalization, instead of $|u|$ we write the q-norm $\|(u,v)\|_q$ of the two-dimensional vector (u,v) and the function θ remains a one-dimensional function. More exactly,

$$\sigma_T^\theta f(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta\left(\frac{\|(u,v)\|_q}{T}\right) \widehat{f}(u,v) e^{\imath(xu+yv)}\, du\, dv.$$

Here T is a positive real number. In this paper and also in the literature the cases $q=1,2,\infty$ are investigated. These summations are called triangular, circular and cubic summability, respectively.

In the other natural generalization, instead of the function θ we write two one-dimensional functions θ_1 and θ_2:

$$\sigma_T^\theta f(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta_1\left(\frac{|u|}{T_1}\right)\theta_2\left(\frac{|v|}{T_2}\right) \widehat{f}(u,v) e^{\imath(xu+yv)}\, du\, dv.$$

In this definition $T=(T_1,T_2)\in\mathbb{R}_+^2$. We suppose that $\theta(0)=\theta_i(0)=1$, $i=1,2$. Two subcases of this summability will be investigated, the restricted (when $T\in\mathbb{R}_+^2$ is in a cone) and the unrestricted (when $T\in\mathbb{R}_+^2$) summability. For each five generalization we need a new concept of Lebesgue points. The proofs are strongly different for different cases.

6. Circular summability

First we consider the circular summability, when $q=2$. Suppose that

$$\theta_0(u,v) := \theta(\sqrt{u^2+v^2}), \qquad \theta_0\in L_1(\mathbb{R}^2), \qquad \widehat{\theta_0}\in L_1(\mathbb{R}^2).$$

Then the θ-means can be written in the form

$$\sigma_T^\theta f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta_0\left(\frac{u}{T},\frac{v}{T}\right) \widehat{f}(u,v) e^{\imath(xu+yv)}\, du\, dv.$$

We denote by $B(c,h)$ ($c\in\mathbb{R}^d, h>0$) the ball $\{x\in\mathbb{R}^d : \|x-c\|_2<h\}$. Let the dyadic coronas be defined by

$$Q_k := B(0,2^k)\setminus B(0,2^{k-1}) \quad (k>0), \qquad Q_0 := B(0,1).$$

The Herz space $E_q(\mathbb{R}^d)$ contains all measurable functions f for which

$$\|f\|_{E_q} := \sum_{k=0}^{\infty} 2^{dk(1-1/q)} \left\|f\mathbf{1}_{Q_k}\right\|_q < \infty.$$

Then obviously

$$L_1(\mathbb{R}^d) = E_1(\mathbb{R}^d) \supset E_q(\mathbb{R}^d) \supset E_{q'}(\mathbb{R}^d) \supset E_\infty(\mathbb{R}^d), \qquad 1 < q < q' < \infty.$$

By Lebesgue's differentiation theorem,

$$\lim_{h\to 0} \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x-s, y-t)\, ds\, dt = f(x,y)$$

for almost every $(x,y) \in \mathbb{R}^2$. Then

$$\lim_{h\to 0} \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x-s, y-t) - f(x,y)\, ds\, dt = 0.$$

If we can write the absolute value in the integrand, which is a stronger condition, then (x,y) is a Lebesgue point. More exactly, a point $(x,y) \in \mathbb{R}^2$ is called a *p-Lebesgue point* of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |f(x-s, y-t) - f(x,y)|^p\, ds\, dt \right)^{1/p} = 0.$$

By Hlder's inequality, all r-Lebesgue points are p-Lebesgue points, whenever $p < r$. The following theorem can be found in Butzer and Nessel [6], Stein and Weiss [27] or Feichtinger and Weisz [9, 10].

Theorem 6.1. *Almost every point $(x,y) \in \mathbb{R}^2$ is a p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ if $1 \le p < \infty$.*

The first generalization of Lebesgue's theorem reads as follows.

Theorem 6.2. *Let $\theta_0 \in L_1(\mathbb{R}^2)$, $1 \le p < \infty$ and $1/p + 1/q = 1$. If $\widehat{\theta}_0 \in E_q(\mathbb{R}^2)$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x,y) = f(x,y)$$

for all p-Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$.

The theorem is due to Feichtinger and Weisz [10]. Originally, it was proved for Riesz summation, for $p = 1$ and for integrable functions without using the Herz spaces in Stein and Weiss [27] or Butzer and Nessel [6]. We proved in [10] that the converse of the theorem holds also.

Theorem 6.3. *Suppose that $\theta_0 \in L_1(\mathbb{R}^2)$, $\widehat{\theta}_0 \in L_1(\mathbb{R}^2)$, $1 \le p < \infty$ and $1/p + 1/q = 1$. If*

$$\lim_{T\to\infty} \sigma_T^\theta f(x,y) = f(x,y)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{R}^2)$, then $\widehat{\theta}_0 \in E_q(\mathbb{R}^2)$.

Note that $W(L_p, \ell_\infty)(\mathbb{R}^2) \supset L_p(\mathbb{R}^2)$.

7. Rectangular summability

The Lebesgue points of the rectangular summability are similar to the previous ones. Suppose that $\theta_i \in L_1(\mathbb{R}^2)$, $\widehat{\theta}_i \in L_1(\mathbb{R}^2)$ $(i = 1, 2)$ and $T = (T_1, T_2)$. Recall that

$$\sigma_T^\theta f(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta_1 \left(\frac{|u|}{T_1} \right) \theta_2 \left(\frac{|v|}{T_2} \right) \widehat{f}(u, v) e^{\imath(xu+yv)} \, du \, dv.$$

The Wiener amalgam space $W(L_p, \ell_\infty)(\mathbb{R}^2)$ was defined with the norm

$$\|f\|_{W(L_p,\ell_\infty)} := \left(\sup_{n,m\in\mathbb{Z}} \int_n^{n+1} \int_m^{m+1} |f(x, y)|^p \, dydx \right)^{1/p}.$$

If we change one integral and one supremum in this expression, then we obtain the definition of the iterated Wiener amalgam spaces. In other words, a function f is in the iterated Wiener amalgam spaces $W_I(L_p, \ell_\infty)(\mathbb{R}^2)$ or $W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$ $(1 \le p \le \infty)$ if

$$\|f\|_{W_I(L_p,\ell_\infty)} := \left(\sup_{n\in\mathbb{Z}} \int_n^{n+1} \sup_{m\in\mathbb{Z}} \int_m^{m+1} |f(x, y)|^p \, dydx \right)^{1/p} < \infty$$

or

$$\|f\|_{W_I(L_p \log L,\ell_\infty)} :=$$

$$:= \left(\sup_{n\in\mathbb{Z}} \int_n^{n+1} \sup_{m\in\mathbb{Z}} \int_m^{m+1} |f(x, y)|^p \log^+ |f(x, y)| \, dydx \right)^{1/p} < \infty.$$

Moreover, f is in the set $L_p \log L(\mathbb{R}^2)$ $(1 \le p < \infty)$ if

$$\|f\|_{L_p \log L} := \left(\int_{\mathbb{R}^2} |f|^p \log^+ |f| \, d\lambda \right)^{1/p} < \infty,$$

where $\log^+ u := \max(0, \log u)$. It is easy to see that

$$W(L_p, \ell_\infty)(\mathbb{R}^2) \supset W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2) \supset L_p \log L(\mathbb{R}^2), L_r(\mathbb{R}^2)$$

for all $1 \le p < r \le \infty$. Note that the space $W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$ does not contain $L_p(\mathbb{R}^2)$.

A point $(x, y) \in \mathbb{R}^2$ is called a *strong p-Lebesgue point* of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{4h_1h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} |f(x - s, y - t) - f(x, y)|^p \, ds \, dt \right)^{1/p} = 0.$$

We suppose also that the supremum of the last expression over all $h \in \mathbb{R}_+^2$ is finite. Here $h \to 0$ means that both $h_1 \to 0$ and $h_2 \to 0$.

The following two theorems are due to the author [35].

Theorem 7.1. *Almost every point $(x,y) \in \mathbb{R}^2$ is a strong p-Lebesgue point of $f \in W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$ if $1 \le p < \infty$.*

Theorem 7.2. *Let $\theta_i \in L_1(\mathbb{R})$, $1 \le p < \infty$ and $1/p + 1/q = 1$. If $\widehat{\theta}_i \in E_q(\mathbb{R})$ $(i = 1, 2)$, then*

$$\lim_{T \to \infty} \sigma_T^\theta f(x,y) = f(x,y)$$

for all strong p-Lebesgue points of $f \in W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$.

Here $T \to \infty$ means again that $T_1 \to \infty$ and $T_2 \to \infty$. The iterated Wiener amalgam space $W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$ is the largest function space for which these two theorems hold, even if we consider another type of Lebesgue points or only almost everywhere convergence. So they are not true either for $W(L_p, \ell_\infty)(\mathbb{R}^2)$ or for $L_p(\mathbb{R}^2)$ (see Gát [12]).

8. Restricted rectangular summability

The third generalization is almost the same as the second one, the difference is that we assume here that T is in a cone. So the definition of the θ-means are the same as before:

$$\sigma_T^\theta f(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta_1\left(\frac{|u|}{T}\right) \theta_2\left(\frac{|v|}{T}\right) \widehat{f}(u,v) e^{\imath(xu+yv)} \, du \, dv.$$

Suppose that $\theta_i \in L_1(\mathbb{R}^2)$, $\widehat{\theta}_i \in L_1(\mathbb{R}^2)$ $(i = 1, 2)$, $\tau \ge 0$ and T is in a cone, i.e.,

$$T = (T_1, T_2) \in \mathbb{R}_\tau^2 := \{x \in \mathbb{R}_+^2 : 2^{-\tau} \le x_1/x_2 \le 2^\tau\}.$$

Instead of the Herz spaces we use here a weighted version of these spaces. The weighted Herz space $E_q^\mu(\mathbb{R})$ $(\mu \ge 0)$ contains all measurable functions f for which

$$\|f\|_{E_q^\mu} := \sum_{k=0}^\infty 2^{k(\mu+1-1/q)} \left\| f \mathbf{1}_{Q_k} \right\|_q < \infty.$$

Obviously,

$$E_q(\mathbb{R}) = E_q^0(\mathbb{R}) \supset E_q^\mu(\mathbb{R}) \qquad 0 \le \mu < \infty$$

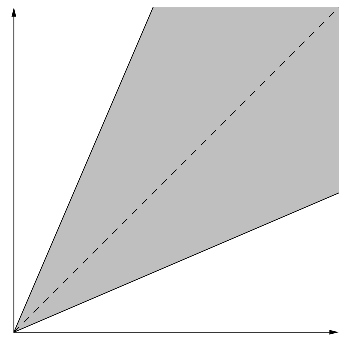

Figure 1. The cone for $d = 2$.

and

$$L_1(\mathbb{R}) \supset E_1^\mu(\mathbb{R}) \supset E_q^\mu(\mathbb{R}) \supset E_{q'}^\mu(\mathbb{R}) \supset E_\infty^\mu(\mathbb{R}), \qquad 1 < q < q' < \infty.$$

Recall the definition of the Lebesgue and strong Lebesgue points given before:

$$\lim_{h\to 0}\left(\frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}|f(x-s,y-t)-f(x,y)|^p\,ds\,dt\right)^{1/p} = 0$$

and

$$\lim_{h\to 0}\left(\frac{1}{4h_1h_2}\int_{-h_1}^{h_1}\int_{-h_2}^{h_2}|f(x-s,y-t)-f(x,y)|^p\,ds\,dt\right)^{1/p} = 0.$$

In the first definition, we integrate over squares and we can prove the convergence result for the large Wiener amalgam space $W(L_p,\ell_\infty)(\mathbb{R}^2)$. In the second one, we integrate over rectangles with sides parallel to the axes and the result holds for the smaller iterated Wiener amalgam space $W_I(L_p\log L,\ell_\infty)(\mathbb{R}^2)$. The reason of this is that e.g. h_1 can be small and h_2 large. It is easy to see that the second definition can also be rewritten as

$$\lim_{r\to 0}\sup_{0<h_1,h_2<r}\left(\frac{1}{4h_1h_2}\int_{-h_1}^{h_1}\int_{-h_2}^{h_2}|f(x-s,y-t)-f(x,y)|^p\,ds\,dt\right)^{1/p} = 0.$$

In the next definition, we will integrate again over rectangles and one side of the rectangle can be small and the other large. However, now we multiply by a weight function. A point $(x,y)\in\mathbb{R}^2$ is a *modified p-Lebesgue point* of f if for all $\mu > 0$,

$$\lim_{r\to 0} \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r,2^j h<r}} 2^{-\mu(i+j)}$$

$$\left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{-2^j h}^{2^j h} |f(x-s,y-t)-f(x,y)|^p\,ds\,dt\right)^{1/p} = 0. \tag{8.1}$$

We assume also that the supremum over all $r > 0$ is finite.

The next two theorems are due to the author [31]. A first version of Theorem 8.2 was shown by Marcinkiewicz and Zygmund [23] in 1939. Later Gát [12] and the author [33, 34] proved the almost everywhere convergence.

Theorem 8.1. *Almost every point $(x, y) \in \mathbb{R}^2$ is a modified p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ if $1 \le p < \infty$.*

Theorem 8.2. *Let $\theta_i \in L_1(\mathbb{R})$, $1 \le p < \infty$ and $1/p + 1/q = 1$ and $\mu > 0$. If $\widehat{\theta}_i \in E_q^\mu(\mathbb{R})$ $(i = 1, 2)$, then*

$$\lim_{T\to\infty, T\in\mathbb{R}^2_\tau} \sigma_T^\theta f(x, y) = f(x, y)$$

for all modified p-Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$.

9. Cubic summability

In the fourth generalization, when $q = \infty$, we do not use Herz spaces. Instead, we suppose that θ is continuous on $\mathbb{R}_+$, the support of θ is $[0, c]$ for some $0 < c \le \infty$ and θ is differentiable on $(0, c)$. Suppose further that

$$\int_0^\infty (t \vee 1)^2 |\theta'(t)|\,dt < \infty, \qquad \lim_{t\to\infty} t^2\theta(t) = 0,$$

where $\vee$ denotes the maximum and $\wedge$ the minimum. Assume also that

$$\left|\int_0^\infty \theta'(t)\cos(tu)\,dt\right| \le Cu^{-\alpha}, \qquad \left|\int_0^\infty \theta'(t)\,t\,\sin(tu)\,dt\right| \le Cu^{-\alpha}$$

for all $u > 0$ and for some $0 < \alpha < \infty$. Recall that the cubic summability means are defined by

$$\sigma_T^\theta f(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta\left(\frac{|u| \vee |v|}{T}\right) \widehat{f}(u, v) e^{\imath(xu+yv)}\,du\,dv.$$

If in addition to the definition of the modified p-Lebesgue point, we suppose that for all $\mu > 0$,

$$\lim_{r\to 0} \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r,2^j h<r}} 2^{-\mu(i+j)} \left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{s-2^j h}^{s+2^j h} |f(x-s,y-t)-f(x,y)|^p \,dt\,ds\right)^{1/p} = 0,$$

then we say that $(x,y)\in\mathbb{R}^2$ is a *modified strong p-Lebesgue point*. We assume also that the supremum over all $r>0$ is finite. Here we integrate over parallelograms with sides parallel to one of the axes and to one of the diagonals of the square $[0,1]^2$.

The next theorems of this section were proved in [36].

Theorem 9.1. *Almost every point $(x,y)\in\mathbb{R}^2$ is a modified strong p-Lebesgue point of $f\in W(L_p,\ell_\infty)(\mathbb{R}^2)$ if $1\le p<\infty$.*

If $p>1$ and $f\in W(L_p,\ell_\infty)(\mathbb{R}^2)$, then we do not need the concept of modified strong p-Lebesgue points just introduced, it is enough to consider modified p-Lebesgue points.

Theorem 9.2. *If $1<p<\infty$ and $f\in W(L_p,\ell_\infty)(\mathbb{R}^2)$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x,y) = f(x,y)$$

for all modified p-Lebesgue points of f.

This result does not hold for $p=1$. In this case, we need the concept of modified strong p-Lebesgue points.

Theorem 9.3. *If $f\in W(L_1,\ell_\infty)(\mathbb{R}^2)$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x,y) = f(x,y)$$

for all modified strong 1-Lebesgue points of f.

Using the theorems of this section, we have given simple proofs for the classical strong summability results in [36].

10. Triangular summability

Finally, in the last generalization let $q=1$. Then the triangular θ-means are defined by

$$\sigma_T^\theta f(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta\left(\frac{|u|+|v|}{T}\right)\widehat{f}(u,v)e^{\imath(xu+yv)}\,du\,dv.$$

The convergence results are similar to those for cubic summability and are proved in [32].

Theorem 10.1. *If* $1<p<\infty$ *and* $f \in W(L_p,\ell_\infty)(\mathbb{R}^2)$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x,y) = f(x,y)$$

for all modified p*-Lebesgue points of* f.

Theorem 10.2. *If* $f \in W(L_1,\ell_\infty)(\mathbb{R}^2)$, *then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x,y) = f(x,y)$$

for all modified strong 1*-Lebesgue points of* f.

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